ACOUSTIC RADIATION PRESSURE FORCES AND TORQUES FROM ELASTIC SCATTERING

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Abstract

A generalized theory of radiation pressure forces for arbitrary non-dissipative acoustic systems is applied to the calculation of the force and torque exerted on a body elastically scattering an incident plane wave. The theory leads to the Westervelt and Maidanik formulae for force and torque respectively. Alternative forms for the Maidanik formula are generated by application of the generalized optical theorem.

I. INTRODUCTION

When sound waves are scattered by an object, steady forces and torques are exerted on the object as a manifestation of the radiation pressure. The radiation pressure force arising from the scattering of an incident plane wave has been investigated theoretically by numerous authors (e.g. Westervelt 1951, 1957; Olsen, Romberg, and Wergeland 1957, 1958; Olsen, Wergeland, and Westervelt 1958), Westervelt's formula, or a closely related result, usually being obtained. Correspondingly, the torque exerted on the scattering body has been derived by Maidanik (1958). Westervelt's and Maidanik's results are obtained by integrating respectively the stresses and the moments of the stresses of the average momentum flux density tensor of the radiation field. The integrations are most conveniently performed over the surface of a sphere of large radius using the asymptotic form of the scattered wave, and the results are obtained in terms of the scattering amplitude $f(\theta,\phi)$. The methods used rely on the fact that, apart from at the boundary of the scatterer, the wave is free, so that (for a uniform non-dissipative medium) the stresses are transmitted to the scatterer alone and to no other boundary or body. Consequently, the methods are normally limited to single scattering in a free field.

An alternative approach to the general problem of calculating average radiation pressure forces and torques is available (Smith 1964, 1965). This formalism leads to or, equivalently, can be incorporated in a generalization of the adiabatic theorem (Smith 1971). If F_x is the generalized radiation pressure force corresponding to a generalized coordinate x, which specifies the configuration of an acoustic system,

$$\omega F_x\,\delta x = -\tfrac{1}{2}\mathrm{i}\iint\limits_S \{p\,\delta(\boldsymbol{v}^*.\boldsymbol{n})+(\boldsymbol{v}.\boldsymbol{n})\,\delta p^*\}\,\mathrm{d}A\,, \tag{1}$$

where p and $\boldsymbol{v}$ are the complex r.m.s. pressure and velocity of the sound field of time dependence $\exp(\mathrm{i}\omega t)$, S is a mathematical surface of unit outward normal $\boldsymbol{n}$ enclosing

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the system, and δ denotes changes resulting from a small adiabatic change δx in the coordinate x. Equation (1) is not restricted to single scattering or affected by the presence of boundary surfaces etc. or the nature of the external excitation but is a result which holds for acoustic systems of arbitrary complexity, provided only that dissipation is negligible.

Equation (1) is used here to derive expressions for the radiation pressure force and torque exerted on an elastic scatterer in a plane wave incident field. It is shown that, when the force and torque are expressed in terms of the scattering amplitude, Westervelt's (1951, 1957) and Maidanik's (1958) formulae are obtained. In addition, alternative formulae for the radiation torque are obtained from the generalized optical theorem (Schiff 1968).

II. Radiation Pressure Force on a Scatterer

For computations using equation (1) it is useful to introduce a velocity potential Ψ, where

$$\boldsymbol{v} = -\nabla\Psi, \qquad p = \mathrm{i}\omega\rho\Psi, \tag{2}$$

ρ being the density of the medium. Equation (1) then becomes

$$F_x\,\delta x = \tfrac{1}{2}\rho\iint_S \{(\partial\Psi/\partial n)\delta\Psi^* - \Psi\,\partial(\delta\Psi^*)/\partial n\}\,\mathrm{d}A\,. \tag{3}$$

To discuss scattering we use polar coordinates (r,θ,ϕ) and take S to be a large sphere centred on the origin. The differentiation $\partial/\partial n$ along the normal in equation (3) then becomes $\partial/\partial r$.

For the scattering of a plane wave of velocity c and wave vector $\boldsymbol{k}$ (with $k=\omega/c$), the incident wave is represented by the velocity potential

$$\Psi_{\mathrm{i}} = \exp(-\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{r}), \tag{4}$$

while the asymptotic form for large r of the scattered wave velocity potential is

$$\Psi_{\mathrm{s}} = r^{-1}\exp(-\mathrm{i}kr)\,f(\theta,\phi), \tag{5}$$

where $f(\theta,\phi)$ is the scattering amplitude for the incident wave (4). The asymptotic form of the total velocity potential is then

$$\Psi = \Psi_{\mathrm{i}}+\Psi_{\mathrm{s}} = \exp(-\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{r}) + r^{-1}\exp(-\mathrm{i}kr)\,f(\theta,\phi)\,. \tag{6}$$

In order that equation (3) may be used to find the force on the scatterer we need an expression for the change $\delta\Psi$ in Ψ on the surface S due to a vector translation $\delta\boldsymbol{x}$ of the scatterer without change of orientation. In this case the incident wave is not affected but the scattered wave potential is changed asymptotically to

$$\Psi_{\mathrm{s}}+\delta\Psi_{\mathrm{s}} = \exp(-\mathrm{i}\boldsymbol{k}\,.\,\delta\boldsymbol{x})\{|\boldsymbol{r}-\delta\boldsymbol{x}|^{-1}\exp(-\mathrm{i}k|\boldsymbol{r}-\delta\boldsymbol{x}|)\}\,f(\theta,\phi)\,. \tag{7}$$

The first factor arises from the phase change in the arrival of the incident wave at the scatterer, the second incorporates changes in the radial distance (only the phase change part proves to be significant), while the angular scattering amplitude remains unchanged.

Noting that

$$|\boldsymbol{r}-\delta\boldsymbol{x}| = r-\delta\boldsymbol{x}\,.\,\hat{\boldsymbol{r}}\,,$$

where the circumflex denotes a unit vector, from equation (7) we can write

$$\delta\Psi_{\mathrm{s}} = \mathrm{i}kr^{-1}\exp(-\mathrm{i}kr)\{(\delta\boldsymbol{x}\,.\,\hat{\boldsymbol{r}}-\delta\boldsymbol{x}\,.\,\hat{\boldsymbol{k}})+r^{-1}\,\delta\boldsymbol{x}\,.\,\hat{\boldsymbol{r}}\}\,f(\theta,\phi)\,. \tag{8}$$

Since only terms in r^{-1} are significant asymptotically, equation (8) becomes

$$\delta\Psi_{\mathrm{s}} = \mathrm{i}kr^{-1}\exp(-\mathrm{i}kr)\,(\delta\boldsymbol{x}\,.\,\hat{\boldsymbol{r}}-\delta\boldsymbol{x}\,.\,\hat{\boldsymbol{k}})\,f(\theta,\phi) \tag{9}$$

$$= \delta x\,\mathrm{i}kr^{-1}\exp(-\mathrm{i}kr)\,(\cos\beta-\cos\alpha)\,f(\theta,\phi)\,, \tag{10}$$

where α and β are the angles between $\delta\boldsymbol{x}$ and $\boldsymbol{k}$ and $\delta\boldsymbol{x}$ and $\boldsymbol{r}$ respectively.

In order to integrate the incident wave potential Ψ_{i}, the asymptotic δ-function resolution of a plane wave into incoming and outgoing spherical waves (Morse and Feshbach 1953) is used to give

$$\Psi_{\mathrm{i}} = \exp(-\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{r})$$

$$= 2\pi\left(\frac{\exp(\mathrm{i}kr)}{\mathrm{i}kr}\,\delta(\pi+u-\theta)\,\delta(v-\phi)-\frac{\exp(-\mathrm{i}kr)}{\mathrm{i}kr}\delta(u-\theta)\,\delta(v-\phi)\right), \tag{11}$$

where the polar angles u and v correspond to the direction of $\boldsymbol{k}$ and the δ-function distribution is defined for continuous functions g by

$$\iint g(\theta,\phi)\,\delta(u-\theta)\,\delta(v-\phi)\,\mathrm{d}\Omega = g(u,v)\,,$$

$\mathrm{d}\Omega$ being an element of solid angle which is integrated over all directions.

Finally, for substitution in equation (3), the total velocity potential may now be written

$$\Psi = 2\pi\left(\frac{\exp(\mathrm{i}kr)}{\mathrm{i}kr}\,\delta(\pi+u-\theta)\,\delta(v-\phi)-\frac{\exp(-\mathrm{i}kr)}{\mathrm{i}kr}\,\delta(u-\theta)\,\delta(v-\phi)\right)$$

$$+\frac{\exp(-\mathrm{i}kr)}{r}f(\theta,\phi) \tag{12}$$

and we have

$$\delta\Psi = \delta\Psi_{\mathrm{s}}\,. \tag{13}$$

Substitution of (12) and (13) into (3) gives, on retaining only terms of order r^{-2},

$$F_x\,\delta x = \tfrac{1}{2}\rho\,\delta x \iint_S \Bigg[\Bigg\{2\pi\bigg(\frac{\exp(\mathrm{i}kr)}{r}\,\delta(\pi+u-\theta)\,\delta(v-\phi)+\frac{\exp(-\mathrm{i}kr)}{r}\,\delta(u-\theta)\,\delta(v-\phi)\bigg)$$

$$-\frac{\mathrm{i}k\exp(-\mathrm{i}kr)}{r}f(\theta,\phi)\Bigg\}\bigg(\frac{-\mathrm{i}k\exp(\mathrm{i}kr)}{r}f^*(\theta,\phi)\,(\cos\beta-\cos\alpha)\bigg)$$

$$-\Bigg\{2\pi\bigg(\frac{\exp(\mathrm{i}kr)}{\mathrm{i}kr}\,\delta(\pi+u-\theta)\,\delta(v-\phi)-\frac{\exp(-\mathrm{i}kr)}{\mathrm{i}kr}\,\delta(u-\theta)\,\delta(v-\phi)\bigg)$$

$$+\frac{\exp(-\mathrm{i}kr)}{r}f(\theta,\phi)\Bigg\}\bigg(\frac{k^2\exp(\mathrm{i}kr)}{r}f^*(\theta,\phi)\,(\cos\beta-\cos\alpha)\bigg)\Bigg]\,\mathrm{d}A$$

$$= -\tfrac{1}{2}\rho\,\delta x\iint_S(\cos\beta-\cos\alpha)\bigg(\frac{4\pi\mathrm{i}k}{r^2}f^*(\theta,\phi)\,\delta(u-\theta)\,\delta(v-\phi)+\frac{2k^2}{r^2}f(\theta,\phi)\,f^*(\theta,\phi)\bigg)\,\mathrm{d}A$$

$$= \delta x\,k^2\rho\iint(\cos\alpha-\cos\beta)\{\,|f(\theta,\phi)|^2-(2\pi/\mathrm{i}k)\,f^*(\theta,\phi)\,\delta(u-\theta)\,\delta(v-\phi)\}\,\mathrm{d}\Omega$$

$$= \delta x\,k^2\rho\iint(\cos\alpha-\cos\beta)\,|f(\theta,\phi)|^2\,\mathrm{d}\Omega \tag{14}$$

since $\beta=\alpha$ for $\theta=u$ and $\phi=v$. Hence the force F_x in the direction of $\delta\boldsymbol{x}$ is

$$F_x = k^2\rho\iint(\cos\alpha-\cos\beta)\,|f(\theta,\phi)|^2\,\mathrm{d}\Omega \tag{15}$$

for unit r.m.s. amplitude of the incident potential. From equations (2) this corresponds to an energy flux

$$p\boldsymbol{v}^*\cdot\hat{\boldsymbol{k}} = \rho\omega k,$$

so that for unit energy flux, equation (15) becomes

$$F_x = c^{-1}\iint(\cos\alpha-\cos\beta)\,|f(\theta,\phi)|^2\,\mathrm{d}\Omega. \tag{16}$$

In the usual description of scattering, $\boldsymbol{k}$ is taken in the $\theta=0$ direction and the force in the direction of $\boldsymbol{k}$ then becomes

$$F_{\parallel} = c^{-1}\iint(1-\cos\theta)\,|f(\theta,\phi)|^2\,\mathrm{d}\Omega, \tag{17}$$

since $\alpha=0$ and $\beta=\theta$, while the force in a perpendicular direction is

$$F_{\perp} = -c^{-1}\iint\cos\beta\,|f(\theta,\phi)|^2\,\mathrm{d}\Omega, \tag{18}$$

where β is now the angle between the direction in which $F_{\perp}$ is being evaluated and the current radius vector in the solid angle integration. Equations (17) and (18), which have been obtained here from the generalized adiabatic theorem, are Westervelt's (1951, 1957) formulae for zero absorption. They have the obvious physical interpretation (Olsen, Wergeland, and Westervelt 1958) that the force results from the momentum reaction exerted on the scatterer in compensation for the change in momentum of the scattered radiation. Although the force equation (1) is in some respects of very general applicability, it cannot accommodate dissipation. The addition to equation (17) of a term corresponding to the momentum absorbed by the scatterer gives Westervelt's results for inelastic scattering.

III. Radiation Pressure Torque

Let $\boldsymbol{N}$ be the torque about the origin exerted on the scatterer and $\delta\boldsymbol{\Theta}$ a vector representing a rotation of the scatterer through a small angle $\delta\Theta$ about an axis through the origin in the direction of $\delta\boldsymbol{\Theta}$. Then, instead of equation (3), we obtain from equation (1)

$$\boldsymbol{N}\,.\,\delta\boldsymbol{\Theta} = \tfrac{1}{2}\rho \iint_S \{(\partial\Psi/\partial n)\,\delta\Psi^* - \Psi\,\partial(\delta\Psi^*)/\partial n\}\,\mathrm{d}A\,, \tag{19}$$

where, as before, S is a large sphere centred on the origin.

To calculate the torque on the scatterer we take Ψ to be the potential when the scatterer is in its initial orientation (i.e. as described by equation (12)) and then $\Psi+\delta\Psi$ must correspond to a solution when the scatterer is rotated through $\delta\boldsymbol{\Theta}$. The easiest way to find such a neighbouring solution is to rotate both the scatterer and the fields together. This is a valid procedure because there is no angular momentum associated with the incident plane wave which itself undergoes rotation too. Thus the potential $\Psi+\delta\Psi$ at some point $\boldsymbol{r}$ on the sphere S arises from the potential Ψ at the point obtained by rotating $\boldsymbol{r}$ through $-\delta\boldsymbol{\Theta}$, i.e.

$$\Psi+\delta\Psi = R(-\delta\boldsymbol{\Theta})\,\Psi, \tag{20}$$

where $R(\boldsymbol{\Theta})\,\Psi$ indicates the value of Ψ at a point obtained by rotating through a vector angle $\boldsymbol{\Theta}$ from the current direction. For functions that can be expanded in Taylor series, the rotation operator R can be written as

$$R(\boldsymbol{\Theta}) = \exp(\boldsymbol{\Theta}\,.\,\boldsymbol{L}), \qquad \boldsymbol{L} = \boldsymbol{r}\times\nabla\,. \tag{21}$$

The component of the operator $\boldsymbol{L}$ in any direction is the derivative with respect to angle about that direction as axis. A familiar explicit form is

$$L_x = -\left(\sin\phi\,\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\,\frac{\partial}{\partial\phi}\right), \quad L_y = \left(\cos\phi\,\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\,\frac{\partial}{\partial\phi}\right), \quad L_z = \frac{\partial}{\partial\phi}. \tag{22}$$

For infinitesimal rotations,

$$R(\delta\boldsymbol{\Theta}) = 1+\delta\boldsymbol{\Theta}\,.\,\boldsymbol{L} \tag{23}$$

so that from equations (20) and (23)

$$\delta\Psi = \{R(-\delta\boldsymbol{\Theta})-1\}\Psi = (-\delta\boldsymbol{\Theta}\,.\,\boldsymbol{L})\Psi\,. \tag{24}$$

Substitution of equation (24) in (19) gives

$$\boldsymbol{N}.\delta\boldsymbol{\Theta} = \delta\boldsymbol{\Theta}.\tfrac{1}{2}\rho \iint\limits_S \{\Psi \boldsymbol{L}(\partial\Psi^*/\partial r) - (\partial\Psi/\partial r)\boldsymbol{L}\Psi^*\}\,\mathrm{d}A. \tag{25}$$

Because of the relationship of $\boldsymbol{L}$ to infinitesimal rotations,

$$\iint\limits_S \boldsymbol{L}(f_1 f_2)\,\mathrm{d}A = 0, \qquad \text{or} \qquad \iint\limits_S f_1\boldsymbol{L}f_2\,\mathrm{d}A = -\iint\limits_S f_2\boldsymbol{L}f_1\,\mathrm{d}A. \tag{26}$$

Using (26), equation (25) can be written in a variety of ways, e.g.

$$\boldsymbol{N}.\delta\boldsymbol{\Theta} = -\delta\boldsymbol{\Theta}.\tfrac{1}{2}\rho \iint\limits_S \{(\partial\Psi^*/\partial r)\boldsymbol{L}\Psi + (\partial\Psi/\partial r)\boldsymbol{L}\Psi^*\}\,\mathrm{d}A, \tag{27}$$

which is equation (15) of Maidanik (1958).

If $\boldsymbol{N}$ is expressed in terms of the scattering amplitude by substitution from equation (12) into (25) then, on retaining only terms of order r^{-2},

$$\begin{aligned}\boldsymbol{N}.\delta\boldsymbol{\Theta} = \tfrac{1}{2}\rho \iint\limits_S \Bigg[&\Bigg\{2\pi\Bigg(\frac{\exp(\mathrm{i}kr)}{\mathrm{i}kr}\,\delta(\pi+u-\theta)\,\delta(v-\phi) - \frac{\exp(-\mathrm{i}kr)}{\mathrm{i}kr}\,\delta(u-\theta)\,\delta(v-\phi)\Bigg)\\
&\quad + \frac{\exp(-\mathrm{i}kr)}{r}f(\theta,\phi)\Bigg\}\\
&\times \delta\boldsymbol{\Theta}.\boldsymbol{L}\Bigg\{2\pi\Bigg(\frac{\exp(-\mathrm{i}kr)}{r}\,\delta(\pi+u-\theta)\,\delta(v-\phi) + \frac{\exp(\mathrm{i}kr)}{r}\,\delta(u-\theta)\,\delta(v-\phi)\Bigg)\\
&\quad + \frac{\mathrm{i}k\exp(\mathrm{i}kr)}{r}f^*(\theta,\phi)\Bigg\}\\
&-\Bigg\{2\pi\Bigg(\frac{\exp(\mathrm{i}kr)}{r}\,\delta(\pi+u-\theta)\,\delta(v-\phi) + \frac{\exp(-\mathrm{i}kr)}{r}\,\delta(u-\theta)\,\delta(v-\phi)\Bigg)\\
&\quad - \frac{\mathrm{i}k\exp(-\mathrm{i}kr)}{r}f(\theta,\phi)\Bigg\}\\
&\times \delta\boldsymbol{\Theta}.\boldsymbol{L}\Bigg\{2\pi\Bigg(\frac{\exp(\mathrm{i}kr)}{\mathrm{i}kr}\,\delta(u-\theta)\,\delta(v-\phi) - \frac{\exp(-\mathrm{i}kr)}{\mathrm{i}kr}\,\delta(\pi+u-\theta)\,\delta(v-\phi)\Bigg)\\
&\quad + \frac{\exp(\mathrm{i}kr)}{r}f^*(\theta,\phi)\Bigg\}\Bigg]\,\mathrm{d}A.\end{aligned} \tag{28}$$

In equation (28) only terms in $f(\theta,\phi)$ contribute since, physically, $\boldsymbol{N}$ is zero when f is. To mathematically interpret the operation of $\delta\boldsymbol{\Theta}.\boldsymbol{L}$ on δ-functions, it is necessary to use the more basic definition $1-R(-\delta\boldsymbol{\Theta})$ implied by equations (24). Thus

$$\begin{aligned}\boldsymbol{N}.\delta\boldsymbol{\Theta} = \rho \iint [&2\pi\{f(\theta,\phi)\,\delta\boldsymbol{\Theta}.\boldsymbol{L}\delta(u-\theta)\,\delta(v-\phi) - \delta(u-\theta)\,\delta(v-\phi)\,\delta\boldsymbol{\Theta}.\boldsymbol{L}f^*(\theta,\phi)\}\\
&+\mathrm{i}k\,f(\theta,\phi)\,\delta\boldsymbol{\Theta}.\boldsymbol{L}f^*(\theta,\phi)]\,\mathrm{d}\Omega,\end{aligned} \tag{29}$$

or, by virtue of equations (26),

$$\boldsymbol{N}\cdot\delta\boldsymbol{\Theta} = -2\pi\rho\iint \delta(u-\theta)\,\delta(v-\phi)\,\delta\boldsymbol{\Theta}\cdot\boldsymbol{L}\{f(\theta,\phi)+f^*(\theta,\phi)\}\,\mathrm{d}\Omega$$
$$+\mathrm{i}k\rho\iint f(\theta,\phi)\,\delta\boldsymbol{\Theta}\cdot\boldsymbol{L}f^*(\theta,\phi)\,\mathrm{d}\Omega\,.$$

Since this is true for all $\delta\boldsymbol{\Theta}$,

$$\boldsymbol{N} = -4\pi\rho\,\mathrm{Re}\Big(\boldsymbol{L}f(\theta,\phi)\Big)_{\substack{\theta=u\\ \phi=v}} + \mathrm{i}k\rho\iint f\,\boldsymbol{L}f^*\,\mathrm{d}\Omega\,, \tag{30}$$

which is Maidanik's (1958) result. For unit incident flux,

$$\boldsymbol{N} = -(4\pi c/\omega^2)\mathrm{Re}\Big(\boldsymbol{L}f(\theta,\phi)\Big)_{\substack{\theta=u\\ \phi=v}} + (\mathrm{i}/\omega)\iint f\,\boldsymbol{L}f^*\,\mathrm{d}\Omega\,. \tag{31}$$

The second term in equation (31) is the reaction to the angular momentum carried off by the scattered field, while the first term represents the interference between the scattered wave and the outgoing component of the incident plane wave.

An alternative derivation, which initially yields a different expression for the torque $\boldsymbol{N}$, is obtained by regarding the direction of the incident plane wave as unchanged in the variation. The scattering amplitude is then best regarded as $f(\boldsymbol{k},\theta,\phi)$, that is, as a function of both (θ,ϕ) and $\boldsymbol{k}$ in the body fixed axes. As well as the rotation operator used above, a corresponding operator $\boldsymbol{L}_k$ for changes in the $\boldsymbol{k}$ direction is required. Since only the scattered wave undergoes change, we have, for substitution in equation (19) $\delta\Psi = \delta\Psi_{\mathrm{s}}$, where $\delta\Psi_{\mathrm{s}}$ is composed of two contributions:

(1) the incident wave arriving from a different direction relative to the body, and

(2) the radiation scattered relative to the body being rotated relative to the fixed surface of integration.

The combined effect of these when the scatterer is rotated through $\delta\boldsymbol{\Theta}$ is

$$\delta\Psi_{\mathrm{s}} = -r^{-1}\exp(-\mathrm{i}kr)\,\delta\boldsymbol{\Theta}\cdot(\boldsymbol{L}+\boldsymbol{L}_k)f(\boldsymbol{k},\theta,\phi) \tag{32}$$

with

$$\boldsymbol{L}_k = \boldsymbol{k}\times\nabla_k\,. \tag{33}$$

Substitution of equation (32) into (19) leads to

$$\boldsymbol{N} = \mathrm{i}k\rho\iint f(\boldsymbol{L}+\boldsymbol{L}_k)f^*\,\mathrm{d}\Omega - 2\pi\rho\iint \delta(u-\theta)\,\delta(v-\phi)\,(\boldsymbol{L}+\boldsymbol{L}_k)f^*\,\mathrm{d}\Omega\,. \tag{34}$$

However, the generalized optical theorem for elastic scattering (Schiff 1968) may be used to show that

$$\iint f\,\boldsymbol{L}_k f^*\,\mathrm{d}\Omega = (2\pi/\mathrm{i}k)\iint \delta(u-\theta)\,\delta(v-\phi)\,(\boldsymbol{L}_k f^* - \boldsymbol{L}f)\,\mathrm{d}\Omega\,, \tag{35}$$

which may then be used to eliminate $\boldsymbol{L}_k$ from equation (34), thus reducing it to the previously obtained equation (30).

Alternatively, although it cannot be entirely eliminated, $\boldsymbol{L}$ can be removed from the interference term to give

$$\boldsymbol{N} = 2k\rho \iint \mathrm{Im}\Big(f^*(\tfrac{1}{2}\boldsymbol{L}+\boldsymbol{L}_k)f\Big)\,\mathrm{d}\Omega - 4\pi\rho\,\mathrm{Re}\Big(\boldsymbol{L}_k f\Big)_{\substack{\theta=u\\ \phi=v}} \tag{36}$$

as another form for the radiation pressure torque. In this case, the interference term depends on the variation of the forward scattering amplitude as the incident wave is rotated.

IV. Conclusions

A generalized radiation pressure force theory has been applied to the calculation of forces and torques exerted on a scattering body and the expressions of Westervelt (1951, 1957) and Maidanik (1958) for the average force and torque on a non-absorbing scatterer have been obtained. For the radiation torque, some alternative formulae, which depend on variations in the scattering amplitude for changes in the direction of the incident plane wave, have also been obtained by means of the generalized optical theorem.

Although the generalized radiation pressure theory may be applied to an acoustic system of arbitrary complexity, it is strictly valid only when dissipation is absent. In the present application the scattering is required to be elastic, i.e. the net average energy flux to the scatterer is zero. On the other hand, Westervelt's and Maidanik's methods, although effectively limited to simple systems (e.g. single scattering), can accommodate absorption by the scatterer. In all cases, however, the assumption of zero absorption in the medium is made. Radiation pressure is usually discussed using some such idealized model and if there is any significant departure from ideal behaviour it may sometimes be estimated as a correction. However, within its domain of applicability the present generalized theory has been shown to include the Westervelt and Maidanik formulae as special cases.

V. References

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