ON MAGNETIC INHIBITION OF THERMAL CONVECTION

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Abstract

The effect of an imposed vertical magnetic field on convective transfer in a horizontal Boussinesq layer of fluid heated from below is studied in the mean field approximation. Solutions are found over a wide range of conditions, for free boundaries, by a combination of numerical and analytic techniques. Quantitative estimates are made of the significant modifications to the heat transfer which are brought about by the presence of the magnetic field. It is found that the general properties of nonlinear steady cellular convection seem to persist in the face of magnetic inhibition.

I. Introduction

It has been known for some time that the imposition of a uniform vertical magnetic field on a horizontal convectively unstable layer of fluid can inhibit the onset of steady convection (Thompson 1951; Chandrasekhar 1952, 1961). Of course, even when the steady convection is suppressed, convective instability may arise as a growing oscillation or overstability (Chandrasekhar 1952, 1961; Danielson 1961; Weiss 1964). When convection does arise in the presence of an impressed magnetic field the resultant heat transfer is sensibly less than it would be under the same conditions without the field. Thus, we have Biermann's (1941) explanation of the darkness of sunspots, namely that the strong fields there decrease the normal heat transfer of solar convection.

In view of the solar application of magnetic convection, as well as others in astrophysics and geophysics, it seems worth while to try to estimate the quantitative effects of magnetic fields on convective transfer. J. Wright (personal communication) has recently attempted this by numerically solving the relevant equations for the case of two-dimensional motion. This is a logical extension of works by Parker (1963) and Weiss (1966) which were concerned with distortions of the field by convective motions but which did not include the effect of the field on the motions themselves. However, purely numerical studies do not readily reveal the roles played by the various parameters of the problem, and for this purpose some approximate analytical treatment seems desirable.

The approach taken here is to use the so-called mean field approximation of normal convection theory. In this approximation one defines mean quantities as horizontal averages and decomposes all quantities into mean and fluctuating parts. (In Boussinesq convection, which we shall consider here, the velocity has zero mean.)

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With this decomposition, the nonlinear terms in the equation of motion separate into terms that are bilinear in mean and fluctuating quantities and quadratic in fluctuating quantities. The mean field approximation then consists in neglecting the deviations of the nonlinear terms that are quadratic in fluctuating quantities from their means.

The mean field approximation is suggested indirectly by a theory of Malkus (Spiegel 1962) but it follows also from an appropriate choice of trial function in a variational approach (Roberts 1966) as well as from the Galerkin method (Spiegel 1971; D. O. Gough, personal communication). In normal convection the mean field approximation has been studied quite extensively (Herring 1963, 1964, 1966; Howard 1965; Roberts 1966; Stewartson 1966; Chan 1971; Murphy 1971; Van der Borght 1971) and it seems to give reasonable estimates for convective transfer for high Prandtl numbers (Spiegel 1971).

As in most discussions of the mean field equations, the present work considers only one horizontal mode. It has been possible to include further modes in the nonmagnetic case (Chan 1971; Spiegel 1971) but in the case of free boundaries, with which we shall be mainly concerned, the addition of further modes does not seem to qualitatively alter the convective heat transfer, at least for the nonmagnetic case. In the next section we set forth the equations to be studied. These are restricted to the case of steady convection. In Section III we present the asymptotic solution for highly unstable convection as well as for some intermediate cases, and in Section IV we give some results obtained by numerical solution of the mean field equations.

II. Basic Equations

We consider a horizontal slab of fluid confined between the planes $z = 0$ and d, where z is the vertical coordinate. The boundary temperatures are assumed to be fixed, the lower boundary being warmer, and the impressed temperature difference ΔT across the layer is also fixed. The magnetic field can be written $\mathscr{H}_0 = \langle\mathscr{H}_0\rangle + \boldsymbol{h}$, where the angle brackets denote a horizontal average. Since $\langle\mathscr{H}_0\rangle$ can depend only on z, and $\nabla\cdot\mathscr{H}_0 = 0$, $\langle\mathscr{H}_0\rangle_z$ must be constant and takes on the value of the impressed field. If this value is taken as the unit of field strength, $\mathscr{H}_0 = \hat{\boldsymbol{k}} + \boldsymbol{h}$, where $\hat{\boldsymbol{k}}$ is a unit vector in the vertical direction. Likewise, we take d as the unit of length; d^2/κ as the unit of time, where κ is the thermal diffusivity; κ/d as the unit of velocity; ΔT as the unit of temperature; and $\rho d^2/\nu\kappa$ as the unit of pressure, where ρ is the (effectively constant) density and ν is the kinematic viscosity. The equations of hydromagnetic convection (Chandrasekhar 1961) then take the form

$$\frac{1}{\sigma}\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)-\tau Q\frac{\partial h_i}{\partial z}-\tau Q h_j\frac{\partial h_i}{\partial x_j} = -\frac{\partial\tilde{\omega}}{\partial x_i}+R\theta\hat{k}_i+\nabla^2 u_i+\frac{1}{\sigma}\left\langle u_j\frac{\partial u_i}{\partial x_j}\right\rangle, \tag{1}$$

$$\tilde{\omega} = (p-\langle p\rangle)/\rho+\tfrac{1}{2}\tau Q(\boldsymbol{h}^2-\langle\boldsymbol{h}\rangle^2)+\tau Q h_z+\tau Q\langle h_z\rangle^2, \tag{2}$$

$$\frac{\partial h_i}{\partial t}+u_j\frac{\partial h_i}{\partial x_j} = \frac{\partial u_i}{\partial z}+h_j\frac{\partial u_i}{\partial x_j}+\tau\nabla^2 h_j, \tag{3}$$

$$\partial\langle T\rangle/\partial t+\partial\langle w\theta\rangle/\partial z = \partial^2\langle T\rangle/\partial z^2, \tag{4}$$

$$\partial\theta/\partial t + u_i\,\partial\theta/\partial x_i - \beta w = \nabla^2\theta + \langle u_i\,\partial\theta/\partial x_i\rangle, \tag{5}$$

$$\beta = -\partial\langle T\rangle/\partial z, \qquad T = \langle T\rangle + \theta, \tag{6}$$

$$\partial h_i/\partial x_i = 0, \qquad \partial u_i/\partial x_i = 0, \tag{7}$$

$$R = g\alpha\,\Delta T\,d^3/\kappa\nu, \quad \sigma = \nu/\kappa, \quad Q = \mu\langle\mathscr{H}_0\rangle^2 d^2/4\pi\rho\nu\eta, \quad \tau = \eta/\kappa. \tag{8}$$

Here the Latin suffixes denote cartesian components (e.g. $h_3 = h_z$, $w = u_3 = u_z$) and summation over repeated suffixes is implied. The notation is fairly standard: T is the temperature, g the acceleration of gravity, η the magnetic diffusivity, μ the (constant) permeability, and u_i the velocity. The mean field equations can be immediately read off from these by simply omitting the terms that are nonlinear in fluctuating quantities. Thus the mean field equations are

$$\sigma^{-1}\,\partial\boldsymbol{u}/\partial t - \tau Q\,\partial\boldsymbol{h}/\partial z = -\nabla\tilde{\omega} + R\theta\hat{\boldsymbol{k}} + \nabla^2\boldsymbol{u}, \tag{9}$$

$$\partial\boldsymbol{h}/\partial t = \partial\boldsymbol{u}/\partial z + \tau\,\nabla^2\boldsymbol{h}, \tag{10}$$

$$\partial\langle T\rangle/\partial t + \partial\langle w\theta\rangle/\partial z = \partial^2\langle T\rangle/\partial z^2, \tag{11}$$

$$\partial\theta/\partial t - \beta w = \nabla^2\theta, \tag{12}$$

$$\nabla\cdot\boldsymbol{h} = 0, \qquad \nabla\cdot\boldsymbol{u} = 0, \tag{13}$$

$$\tilde{\omega} = (p - \langle p\rangle)/\rho + \tau Q h_z. \tag{14}$$

It is convenient to take the curl of equation (9) twice to eliminate $\tilde{\omega}$ and the horizontal components of velocity. We find

$$\sigma^{-1}\,\nabla^2(\partial w/\partial t) - \tau Q\,\nabla^2(\partial h_z/\partial z) = \nabla^4 w + R\,\nabla_1^2\theta, \tag{15}$$

where $\nabla_1^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$. Also we shall need only the z component of equation (10), that is,

$$\partial h_z/\partial t = \partial w/\partial z + \tau\,\nabla^2 h_z. \tag{16}$$

Thus equations (11), (12), (15), and (16) are four relations for $\langle T\rangle$, θ, w, and h_z which define the basic problem considered here.

The basic equations are separable and admit solutions of the form

$$w = f(x,y)\,W(z,t), \qquad \theta = f(x,y)\,\Theta(z,t), \qquad h_z = f(x,y)\,H(z,t), \tag{17}$$

where

$$\nabla_1^2 f = -a^2 f. \tag{18}$$

In equation (18) a is a separation constant and f is the planform of the cellular pattern with horizontal scale $\sim a^{-1}$. The basic equations then reduce to

$$\sigma^{-1}(\mathrm{D}^2 - a^2)\partial W/\partial t - \tau Q(\mathrm{D}^2 - a^2)\mathrm{D}H = (\mathrm{D}^2 - a^2)^2 W - Ra^2\Theta, \tag{19}$$

$$\partial H/\partial t = \mathrm{D}W + \tau(\mathrm{D}^2 - a^2)H\,, \tag{20}$$

$$\partial\langle T\rangle/\partial t + \mathrm{D}(W\Theta) = \mathrm{D}^2\langle T\rangle\,, \tag{21}$$

$$\partial\Theta/\partial t - \beta W = (\mathrm{D}^2 - a^2)\Theta\,, \tag{22}$$

where $\mathrm{D} \equiv \partial/\partial z$. It should be noted here that the forms of solution (17) are special since in general a sum of many such forms for various a is also a solution. It is for this reason that we refer to the result as a one-mode solution.

One of the main quantities of interest in this problem is the nondimensional heat transfer or Nusselt number

$$N = W\Theta + \beta\,, \tag{23}$$

which is clearly the sum of the convective and conductive heat fluxes. In the case of steady convection, which is the one considered here, equation (21) simply shows that N is a constant. We note also that, since the temperature difference is unity in non-dimensional units, on integrating equation (23) over z we find

$$N = 1 + \int_0^1 W\Theta\,\mathrm{d}z\,. \tag{24}$$

In the steady-state case, equations (19), (20), and (22) become

$$(\mathrm{D}^2 - a^2)^2 W = Ra^2\Theta - \tau Q(\mathrm{D}^2 - a^2)\mathrm{D}H\,, \tag{25}$$

$$\tau(\mathrm{D}^2 - a^2)H = -\mathrm{D}W\,, \tag{26}$$

$$(\mathrm{D}^2 - a^2)\Theta = -\beta W\,. \tag{27}$$

If we eliminate H between equations (25) and (26) we have

$$(\mathrm{D}^2 - a^2)^2 W = Ra^2\Theta + Q\,\mathrm{D}^2 W\,. \tag{28}$$

Similarly we can combine equations (27) and (28) to give

$$(\mathrm{D}^2 - a^2)^3 W = -Ra^2\beta W + Q(\mathrm{D}^2 - a^2)\mathrm{D}^2 W\,. \tag{29}$$

To complete the formulation of the problem we need to prescribe the boundary conditions. We have already required that the boundary temperatures are fixed, and hence

$$\Theta(0) = 0\,, \qquad \Theta(1) = 0\,. \tag{30}$$

We have also assumed that the fluid is confined between the planes $z = 0$ and 1 (where d is the unit of length), which implies

$$w(0) = 0\,, \qquad w(1) = 0\,. \tag{31}$$

The boundary conditions to be applied to the magnetic field depend on the nature of the medium adjoining the fluid. If the adjoining medium is current-free the external field $\boldsymbol{h}^{\text{ext}}$ is irrotational and derivable from a potential, that is $\boldsymbol{h}^{\text{ext}} = \nabla\chi$. Moreover, $\nabla^2\chi = 0$. The potential χ must be smooth on the boundary of a current-free region so that h_z and $\mathrm{D}h_z$ are continuous (which is equivalent to the statement that $\boldsymbol{h}$ is continuous). Then the external potential must be of the form $\chi = f(x,y)\langle X(z)\rangle$, so that $(\mathrm{D}^2-a^2)\langle X\rangle = 0$. For $z > 0$ this has the solution $\langle X\rangle = \langle X_0\rangle\exp(-az)$ where $\langle X_0\rangle$ is a constant. Smoothness of χ (or continuity of $\boldsymbol{h}$) implies

$$H(1) = -a\langle X_0\rangle\exp(-a)\,, \qquad \mathrm{D}H(1) = a^2\langle X_0\rangle\exp(-a)\,.$$

Hence

$$\mathrm{D}H+aH = 0 \qquad \text{on} \qquad z = 1 \tag{32}$$

and likewise

$$\mathrm{D}H-aH = 0 \qquad \text{on} \qquad z = 0\,. \tag{33}$$

Now a typical current-free adjoining medium is a vacuum and the corresponding boundary is a free boundary. Across such a boundary the tangential stresses are continuous. Since the above conditions show that the magnetic part of the tangential stress is continuous across a free boundary, and moreover there are no viscous stresses outside the fluid, the tangential viscous stresses must vanish on free surfaces and it follows that (Chandrasekhar 1961)

$$\mathrm{D}^2W = 0 \qquad \text{on} \qquad z = 0,1\,. \tag{34}$$

A final point to be made is that we have found steady solutions for equations (23), (27), and (28) with the boundary conditions (30), (31), and (34) only for $R \geqslant R_0$, where R_0 is the Rayleigh number for steady marginally stable convection for given a and Q. The value of R_0 is obtained by solving the linearized equation of motion and is (Chandrasekhar 1961)

$$R_0 = \{(a^2+\pi^2)^3+\pi^2Q(\pi^2+a^2)\}/a^2\,. \tag{35}$$

III. Asymptotic Solutions for Large Rayleigh Number

(a) $Q = O(1)$

The solution of the mean field equations for large R with $Q = 0$ has been considered in various analytic approximations for free boundaries (Howard 1965; Herring 1966; Van der Borght 1971). When $Q = O(1)$ these calculations can be readily extended to the magnetic case, and here we use the method of matched asymptotic expansions following Howard's (1965) treatment of the nonmagnetic case. In doing this we shall consider $a = O(1)$.

Let us introduce the quantity

$$F = N^{-1}\Theta\,. \tag{36}$$

Equations (23), (27), and (28) yield

$$(\mathrm{D}^2-a^2)^2W = Ra^2NF+Q\,\mathrm{D}^2W \tag{37}$$

and

$$(\mathrm{D}^2-a^2)F = -(1-FW)W. \tag{38}$$

Moreover, from equations (23), (28), and (29) we find that

$$(\mathrm{D}^2-a^2)^3W = -Ra^2NW+W^2(\mathrm{D}^2-a^2)^2W-QW^2\mathrm{D}^2W+Q(\mathrm{D}^2-a^2)\mathrm{D}^2W. \tag{39}$$

Now we have $N \geqslant 1$ and $a = O(1)$ so that $Ra^2N \to \infty$ as $R \to \infty$. Away from boundaries we expect the derivatives D to be $O(1)$ so that for equation (39) to have nontrivial solutions we require W to be large; in particular we need $W = O(P^{\frac{1}{4}})$ where $P = Ra^2N$. This tells us that in the main body of the fluid, to leading order, we have

$$\Psi(\mathrm{D}^2-a^2)^2\Psi-Q\Psi\mathrm{D}^2\Psi = 1, \tag{40}$$

where

$$\Psi = (Ra^2N)^{-\frac{1}{4}}W = P^{-\frac{1}{4}}W. \tag{41}$$

In solving (40) we shall apply the free-boundary conditions (31) and (34), which now are

$$\Psi = \mathrm{D}^2\Psi = 0 \qquad \text{on} \qquad z = 0, 1. \tag{42}$$

The main difficulty in this analysis is the solution of equation (40). For $Q = 0$ Howard (1965) used a truncated Fourier sine series for Ψ and obtained an approximate solution. We shall repeat this calculation for $Q = O(1)$.

Let

$$\Psi = A_1\sin\pi z+A_3\sin 3\pi z+\dots. \tag{43}$$

This represents a solution that is symmetric about $z = \frac{1}{2}$ and satisfies the conditions (42). Substitution of this series in equation (40) yields

$$\sum_{n=0}^{\infty}[\{(2n+1)^2\pi^2+a^2\}^2+(2n+1)^2\pi^2Q]A_{2n+1}\sin\{(2n+1)\pi z\}$$

$$= \frac{1}{A_1\sin\pi z}\left(1-\frac{A_3\sin 3\pi z}{A_1\sin\pi z}+\dots\right)$$

$$= \frac{1}{A_1\sin\pi z}\left(1-\frac{A_3(1+2\cos 2\pi z)}{A_1}+\dots\right). \tag{44}$$

If we now multiply by $\sin\{(2m+1)\pi z\}$, integrate over z, and retain only the first two terms in expansion (43), we find

$$\{(\pi^2+a^2)^2+\pi^2Q\}A_1 \simeq (2/A_1)(1-A_3/A_1) \tag{45}$$

and

$$\{(9\pi^2+a^2)^2+9\pi^2Q\}A_3 \simeq (2/A_1)(1-3A_3/A_1). \tag{46}$$

These equations can be solved approximately when $A_3/A_1 \ll 1$, as is to be expected when the two-term sine series is a good approximation. We find

$$A_1 \simeq \left(\frac{2}{(\pi^2+a^2)^2+\pi^2 Q}\right)^{\frac{1}{2}}\left(1-\frac{(\pi^2+a^2)^2+\pi^2 Q}{2\{(9\pi^2+a^2)^2+9\pi^2 Q\}}\right), \tag{47}$$

$$A_3 \simeq \left(\frac{(\pi^2+a^2)^2+\pi^2 Q}{(9\pi^2+a^2)^3+9\pi^2 Q}\right) A_1 . \tag{48}$$

Thus we are provided with an approximation to Ψ in the interior of the fluid. To obtain F we note that for large W equation (38) is satisfied to leading order only if $F = W^{-1} = P^{-\frac{1}{2}}\Psi^{-1}$ in the interior. This conclusion can also be reached by comparing equations (37) and (40).

We turn next to the solutions in the boundary layer in which we use the scaled independent variable

$$\zeta = P^{\frac{1}{4}} z . \tag{49}$$

To obtain the appropriate matching condition we note that near $z = 0$ equations (43), (47), and (48) imply $\Psi \sim Az$, where

$$A \simeq \left(\frac{2\pi^2}{(\pi^2+a^2)^2+\pi^2 Q}\right)^{\frac{1}{2}}\left(1+\frac{5\{(\pi^2+a^2)^2+\pi^2 Q\}}{2\{(9\pi^2+a^2)^2+9\pi^2 Q\}}\right). \tag{50}$$

Hence in the boundary layer we let

$$W = P^{\frac{1}{4}}\psi, \qquad F = P^{-\frac{1}{4}} f, \tag{51}$$

so that equations (37) and (38) become

$$P^{\frac{1}{2}}\psi'''' = f + Q\psi'' \tag{52}$$

and

$$f'' = -(1-f\psi)\psi , \tag{53}$$

where the primes denote differentiation with respect to ζ.

The boundary conditions are

$$\psi = \psi'' = f = 0 \qquad \text{at} \qquad \zeta = 0, \tag{54}$$

while the matching conditions are

$$\psi \to A\zeta, \quad f \to (A\zeta)^{-1} \qquad \text{as} \qquad \zeta \to \infty . \tag{55}$$

Similar considerations apply near $z = 1$.

To leading order the appropriate solution of equation (52) is

$$\psi = A\zeta \tag{56}$$

and equation (53) becomes

$$f'' = -A(1-A\zeta f)\zeta, \tag{57}$$

which is the same as the corresponding equation for $Q = 0$. Howard's (1965) particular integral of this equation is

$$f = \tfrac{1}{2}\zeta \int_0^1 \exp(-\tfrac{1}{2}A\zeta^2 t)\,(1-t^2)^{-\frac{1}{4}}\,\mathrm{d}t. \tag{58}$$

Since this integral satisfies the boundary and matching conditions it represents the desired solution.

Now, on introducing equation (36) into equation (24) we find

$$N^{-1} = \int_0^1 (1-FW)\,\mathrm{d}z. \tag{59}$$

Since $FW = 1$ in the interior, we need only consider the contributions from the boundary layers. The two boundary layers contribute equally and hence on introducing the definitions (49) and (51) into (59) we see that

$$N^{-1} = 2P^{-\frac{1}{4}} \int_0^\infty (1-f\psi)\,\mathrm{d}\zeta, \tag{60}$$

where ψ and f are given by equations (56) and (58) respectively. If these latter results are also introduced, we find after some straightforward integrations that N is given by

$$N^{-1} = 2^{\frac{1}{2}} P^{-\frac{1}{4}} A^{-\frac{1}{2}}\{\Gamma(\tfrac{3}{2})\}^2. \tag{61}$$

This leads to

$$N \simeq \left(\frac{1}{2\cdot 124}\right)^{4/3}\left(1+\frac{5\{(\pi^2+a^2)^2+\pi^2Q\}}{(9\pi^2+a^2)^2+9\pi^2Q}\right)^{\frac{1}{3}}\left(\frac{2\pi^2a^2}{(\pi^2+a^2)^2+\pi^2Q}\right)^{\frac{1}{3}} R^{\frac{1}{3}}, \tag{62}$$

which reduces to Howard's (1965) result for $Q = 0$. For $Q = O(1)$ this expression for N has its maximum value for $a_{\max} \simeq \pi(1+Q/\pi^2)^{\frac{1}{4}}$.

To complete this phase of the analysis we consider the form of H as implied by equation (26). In the interior of the fluid this becomes

$$\tau(\mathrm{D}^2-a^2)H \simeq -P^{\frac{1}{2}}(\pi A_1 \cos\pi z + 3\pi A_3\cos 3\pi z), \tag{63}$$

which admits as the solution satisfying condition (32)

$$H = \frac{\pi P^{\frac{1}{2}}}{\tau}\left\{\frac{A_1\cos\pi z}{\pi^2+a^2} + \frac{3A_3\cos 3\pi z}{9\pi^2+a^2} + \tfrac{1}{2}\left(\frac{A_1}{\pi^2+a^2}+\frac{3A_3}{9\pi^2+a^2}\right)\left(\exp\{a(1-z)\}-\exp(-az)\right)\right\}. \tag{64}$$

We could also solve equation (26) in the boundary layer, but as equation (64) already satisfies the boundary conditions no boundary layer contribution is needed. We see that H, which represents the deformation of the magnetic field imposed by the motion, is quite large ($O(P^{\frac{1}{2}}/\tau)$) and tends to be constant near the boundaries.

(b) $Q = O(R)$

The analysis in subsection (a) considered the case of strong convection in which convective transport is the dominant mode of heat transfer in the interior of the fluid. However, we know from linear theory that, for $a = O(1)$, as Q approaches $a^2R/\{\pi^2(\pi^2+a^2)\}$ the steady convection is suppressed. We now consider some aspects of the transition between the kind of solution discussed in (a) and the situation of marginal stability.

Let

$$\lambda = R/Q, \qquad \gamma = \lambda a^2 N. \tag{65}$$

Clearly, as the stabilizing effects of the field increase N approaches unity and, in at least part of the transition regime, $N = O(1)$. We now consider only that part of the regime and hence assume $\lambda = O(1)$, but we continue to suppose that Q and R are large.

Introducing the scalings

$$W = (Ra^2N/Q)^{\frac{1}{2}}\Omega, \qquad F = (Q/Ra^2N)^{\frac{1}{2}}\Phi, \tag{66}$$

equations (37) and (38) then become

$$\mathrm{D}^2\Omega+\Phi = 0 \tag{67}$$

and

$$(\mathrm{D}^2-a^2)\Phi = -\gamma(1-\Phi\Omega)\Omega. \tag{68}$$

It is now preferable to eliminate Φ from equations (67) and (68) to yield

$$\mathrm{D}^4\Omega-(a^2+\gamma\Omega^2)\mathrm{D}^2\Omega-\gamma\Omega = 0. \tag{69}$$

Now, in analogy with subsection (a) we approximate Ω as

$$\Omega \simeq A_1\sin\pi z+A_3\sin 3\pi z+\dots. \tag{70}$$

If we substitute the expansion (70) into equation (69), multiply separately by $\sin\pi z$ and $\sin 3\pi z$, and integrate, we obtain the two equations for A_1 and A_3

$$4(\pi^4+\pi^2a^2-\gamma) = -\pi^2A_1^2\gamma(3-11\,A_3/A_1) \tag{71}$$

and

$$4A_3(81\pi^4+9\pi^2a^2-\gamma) = -\pi^2A_1^3\gamma(-1+22\,A_3/A_1), \tag{72}$$

where we have treated A_3/A_1 as small.

Likewise we expand Φ as

$$\Phi = \alpha_1\sin\pi z+\alpha_3\sin 3\pi z+\dots \tag{73}$$

and introduce equations (70) and (73) into (68). After then multiplying the resultant equation by $\sin \pi z$ and $\sin 3\pi z$ and integrating, we derive the formulae for α_1 and α_3

$$\alpha_1 = \frac{\gamma A_1}{(\pi^2+a^2)+\frac{1}{4}\gamma A_1^2\{3-(\alpha_3/\alpha_1+2A_3/A_1)\}} \tag{74}$$

and

$$\frac{\alpha_3}{\alpha_1} = \frac{\gamma A_1^2(1-4A_3/A_1)+(A_3/A_1)\{4(\pi^2+a^2)+\gamma A_1^2(3-2A_3/A_1)\}}{2\gamma A_1^2+4(9\pi^2+a^2)+\gamma A_3 A_1}. \tag{75}$$

This completes the interior solution and we must now consider the boundary layer solution. It is easy to see that these are boundary layers of thickness $Q^{-\frac{1}{2}}$. However, as the interior solution already satisfies all the boundary conditions, it supplies an adequate description of the entire solution and there is no real need to write down the boundary layer part separately. We should remark though that this is a purely viscous boundary layer, not the thermal one that occurs in normal strong convection.

We may next compute N from the relation

$$N^{-1} = \int_0^1 (1-\Phi\Omega)\,\mathrm{d}z, \tag{76}$$

which gives

$$N^{-1} = 1-\tfrac{1}{2}(\alpha_1 A_1+\alpha_3 A_3). \tag{77}$$

This equation together with the formulae for A_1, A_3, α_1, and α_3 then specify N as a function of Ra^2/Q.

IV. Numerical Solutions

The analytic approximations of Section III give the solutions in various limiting cases. To obtain more precise solutions over a wide range of the parameters and to exhibit graphically the nature of the solutions, we turn to numerical methods. For the free-boundary case such methods are available which have been used in the analogous problem without magnetic fields (Herring 1963; Murphy 1971). Following these procedures, we introduce the expansions

$$\left.\begin{aligned} W = \sum_{n=1}^{M} W_n \sin\{(2n-1)\pi z\}, \qquad \Theta = \sum_{n=1}^{M} f_n \sin\{(2n-1)\pi z\}, \\ \langle T\rangle = \sum_{n=1}^{M} t_n \sin(2\pi n z) - z. \end{aligned}\right\} \tag{78}$$

These representations satisfy the free-boundary conditions and correspond to solutions that are symmetric about the plane $z = \frac{1}{2}$.

In principle we could in this fashion deal with time-dependent solutions, but here we shall consider only the steady case and thus use the corresponding form of (21)

$$\mathrm{D}^2\langle T\rangle = \mathrm{D}(W\Theta), \tag{79}$$

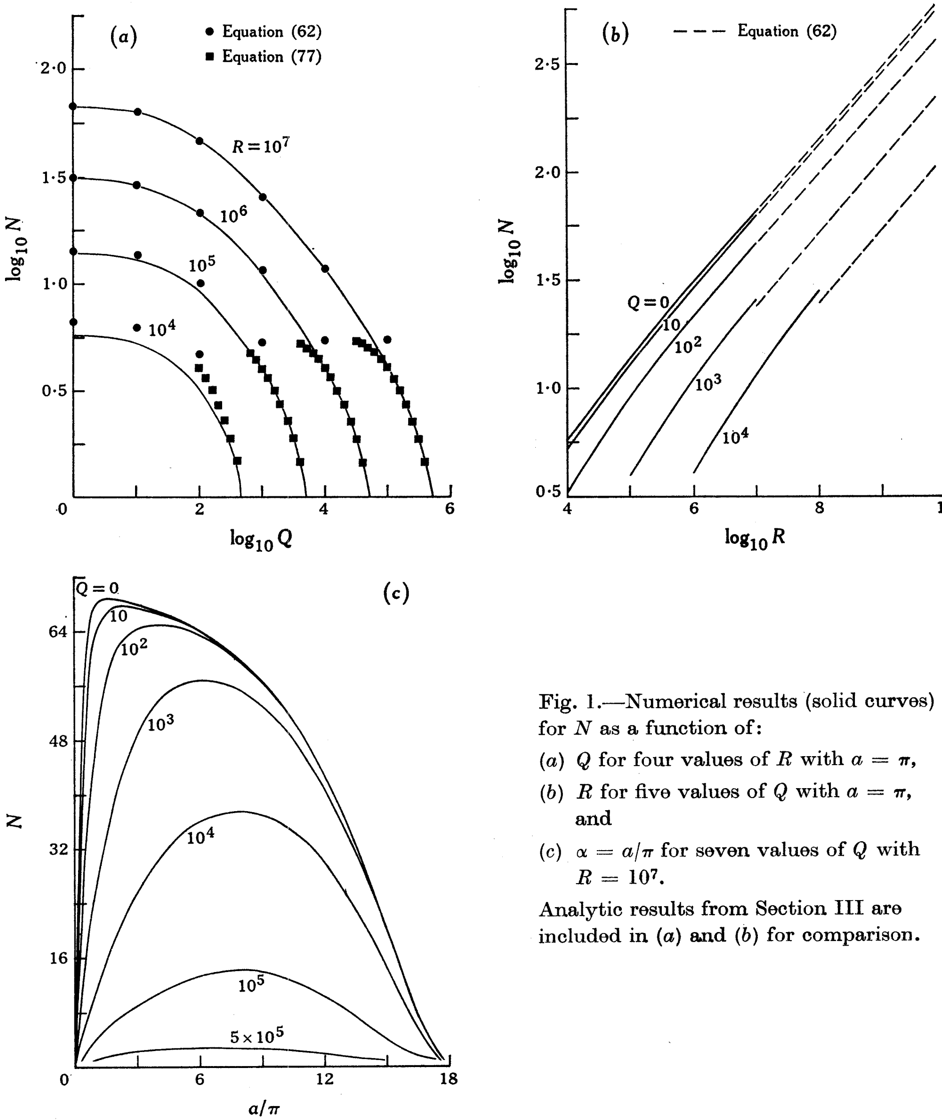


Fig. 1.—Numerical results (solid curves) for N as a function of:

(a) Q for four values of R with $a = \pi$,

(b) R for five values of Q with $a = \pi$, and

(c) $\alpha = a/\pi$ for seven values of Q with $R = 10^7$.

Analytic results from Section III are included in (a) and (b) for comparison.

as well as equations (27) and (28) in which $\beta = -\mathrm{D}\langle T\rangle$. Substitution of the Fourier representations and projection onto the various components leads to the system of nonlinear algebraic equations:

$$[\{(2m-1)^2+\alpha^2\}^2+(Q/\pi^2)(2m-1)^2]w_m = R_1\,\alpha^2 f_m\,, \tag{80}$$

$$\{(2m-1)^2+\alpha^2\}f_m = w_m - \pi \sum_{l=1}^{M} l t_l\{w_{m+l} + Y(2m-2l-1)\, w_{\frac{1}{2}|2m-2l-1|+\frac{1}{2}}\}\,, \tag{81}$$

$$m t_m = \tfrac{1}{4}\pi \sum_{l=1}^{M} w_l\{f_{m+l} + Y(2l-2m-1) f_{\frac{1}{2}|2m-2l+1|+\frac{1}{2}}\}\,, \tag{82}$$

where

$$R_1 = R/\pi^4, \qquad \alpha = a/\pi, \qquad w_n = W_n/\pi^2,$$

and

$$\left.\begin{aligned} Y(n) &= 1 \quad \text{for} \quad n > 0, \\ &= 0 \qquad\qquad n = 0, \\ &= -1 \qquad\quad n < 0. \end{aligned}\right\} \tag{83}$$

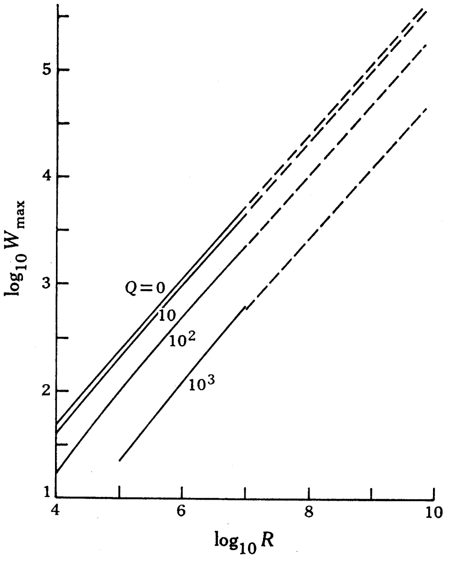


Fig. 2.—Numerical results (solid curves) for the variation of the amplitude of the vertical velocity. The dependence of the maximum value of W (over z) on R is illustrated for four values of Q with $a = \pi$. The dashed curves represent the analytic results from Section III(a).

For a given M these equations were solved and N was computed from equation (23). The value of M was then increased until N was constant to an accuracy of $0 \cdot 1\%$. In practice this required values of M ranging from 30 to 90 as R varied from 10^4 to 10^7. In addition, H was computed from the expression

$$H = \tfrac{1}{2}\tau^{-1}\Phi(0)\{\exp(az-a)-\exp(-az)\}+\tau^{-1}\phi(z), \tag{84}$$

where

$$\phi(z) = \sum_{n=1}^{M} \frac{(2n-1)W_n}{\pi\{(2n-1)^2+\alpha^2\}}\cos\{(2n-1)\pi z\}. \tag{85}$$

These expressions are readily derived from equation (26) with the help of the conditions (32) and (33). The solutions were found for R up to 10^7 and for a large range of Q.

Figure 1(a) shows the variation of N with Q for four values of R with $a = \pi$. The inhibition of convective flux by a magnetic field is well illustrated. For comparison, the analytic results from Sections III(a) and III(b) are also shown, and these agree quite well for $R = 10^7$. Another aspect of the results is indicated by Figure 1(b), which shows N as a function of R for five values of Q with $a = \pi$. The change in heat transport as a function of $\alpha = a/\pi$ is also shown in Figure 1(c) for $R = 10^7$ and a range of values of Q. The main feature to be noted is the increase with Q of the value of a that maximizes N.

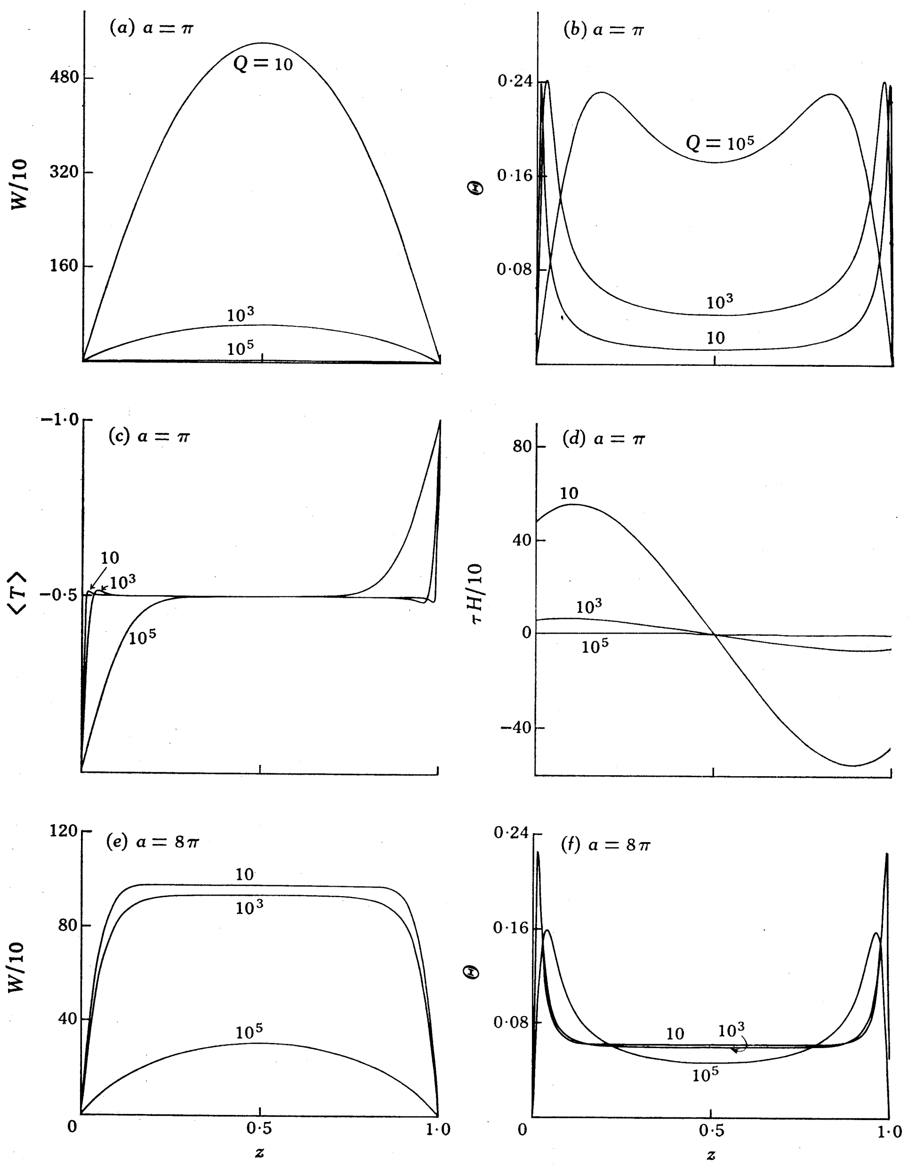
The results for the variation of the amplitude of the vertical velocity as a function of R are given in Figure 2, which plots the maximum value $W_{\max}$ of W (over z) for four values of Q.

The graphs in Figures 3(a)–3(l) show a variety of solutions for various parameter values and are self-explanatory. The solutions for given R and Q are qualitatively like those for the nonmagnetic case but with R reduced by an amount depending on the value of Q. Of course, we have now in addition the disturbance magnetic field H which gives a total field $1 + H(z)\sqrt{2}\cos ax$. What is interesting about this field is that, for the most part, in strong convection H is quite small in the mid regions of the fluid and the magnetic field has its original imposed value. However, near the boundaries (both inside and outside the layer) the perturbation field can be comparable with or even much larger than the original field. Clearly then, measurements of a magnetic field just at the edge of a convective layer may give a completely misleading expression of the internal field if this feature of magnetoconvection is neglected.

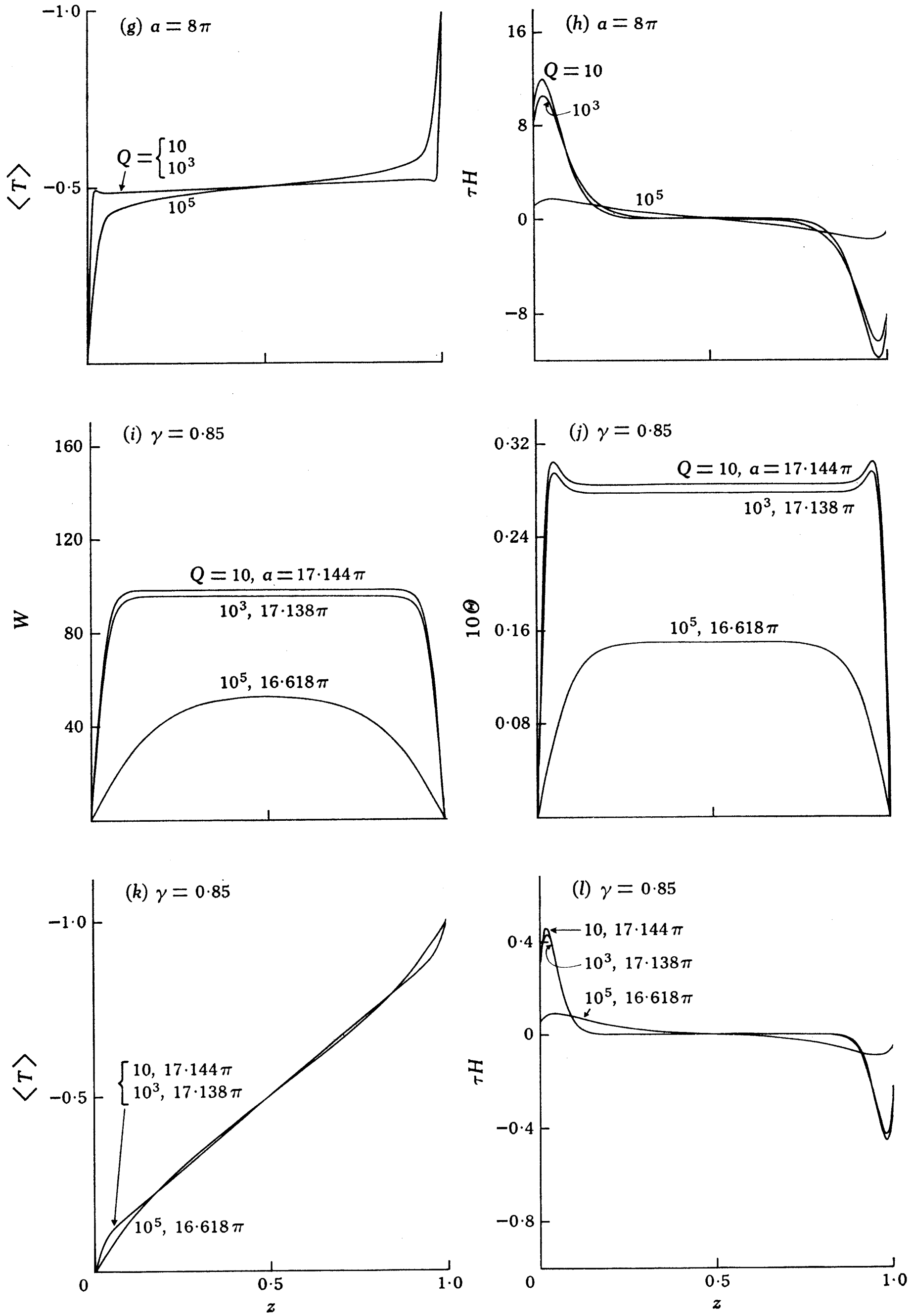
V. Concluding Remarks

From the present study of the behaviour of convection under the influence of externally impressed magnetic fields using the mean field approximation, which is known to give quite reasonable results for laboratory convection, we have gained some insight into the continued influence of magnetic effects even when the convective instability is pronounced. The solutions have shown that, for a given impressed field and if the Rayleigh number is large enough, convection in the fluid interior is practically unhampered by the field. However, even without the presence of the field, convective motions and transport must diminish near the boundaries. It is in these regions that the field can make itself felt. In the regions of transition between the strong interior convection and the vanishing motion at the boundaries, the convection can be further inhibited by the impressed field. The result is that the boundary layers in which conductive transfer dominates are thickened. Since the Nusselt number varies inversely as the thickness of the boundary layers, magnetic fields have an inhibiting effect on the total transport. This is likely to be true no matter how large the Rayleigh number becomes, so long as the flow remains laminar. Of course, at high enough Rayleigh numbers the convection will become turbulent and these conclusions could be modified, particularly if the boundary layers become turbulent. Also, the possibility of corrections from time-dependent effects, even in the mean field approximation, remains to be examined.

The most concrete result of the calculations is that in expressions for N at large R the chief effect of imposed magnetic fields is the replacement of Ra^2 by Ra^2/Q in the expression for N. Apart from dissipation coefficients, the quantity R/Q contains the important factor $(g\alpha\,\Delta T\,d)(4\pi\rho)/\mu\langle\mathscr{H}_0\rangle^2$. The factor $g\alpha\,\Delta T\,d \equiv V_{\mathrm{c}}^2$ gives a measure of the convective velocity V_{c}, while $\mu\langle\mathscr{H}_0\rangle^2/4\pi\rho \equiv V_{\mathrm{A}}^2$ is the square of an Alfvén speed. Thus R/Q is proportional to $(V_{\mathrm{c}}/V_{\mathrm{A}})^2$, which is the sort of parameter one might have expected to find playing an important role here. However, it must also be appreciated that the preferred value of a can be affected by the magnetic field. For example, if we wish to estimate the maximum heat transport then we expect $a = O(Q^{\frac{1}{4}})$ for large Q and in that case $RQ^{-\frac{1}{2}}$ is a key parameter in estimating N. Thus, we expect to find that the maximum heat transport in this kind of problem varies as $R^{\frac{1}{3}}/Q^{\frac{1}{6}}$.



Figs. 3(*a*)–3(*l*).—Solutions of W, Θ, $\langle T\rangle$, and τH as functions of z, each for $R = 10^7$ and $\log_{10} Q = 1$, 3, and 5 with $a = \pi$ and 8π and $\gamma = (\pi^2+a^2)\{(\pi^2+a^2)^2+\pi^2 Q\}/a^2R = 0{\cdot}85$.



(g) $a = 8\pi$
$Q = \begin{cases} 10 \\ 10^3 \end{cases}$
10^5
$\langle T \rangle$
−1·0
−0·5
(h) $a = 8\pi$
$Q = 10$
10^3
10^5
τH
16
8
0
−8
(i) $\gamma = 0{\cdot}85$
$Q = 10,\ a = 17{\cdot}144\pi$
$10^3,\ 17{\cdot}138\pi$
$10^5,\ 16{\cdot}618\pi$
W
160
120
80
40
(j) $\gamma = 0{\cdot}85$
$Q = 10,\ a = 17{\cdot}144\pi$
$10^3,\ 17{\cdot}138\pi$
$10^5,\ 16{\cdot}618\pi$
10Θ
0·32
0·24
0·16
0·08
(k) $\gamma = 0{\cdot}85$
$\begin{cases} 10,\ 17{\cdot}144\pi \\ 10^3,\ 17{\cdot}138\pi \end{cases}$
$10^5,\ 16{\cdot}618\pi$
$\langle T \rangle$
−1·0
−0·5
0
0·5
1·0
z
(l) $\gamma = 0{\cdot}85$
$10,\ 17{\cdot}144\pi$
$10^3,\ 17{\cdot}138\pi$
$10^5,\ 16{\cdot}618\pi$
τH
0·4
0
−0·4
−0·8
0
0·5
1·0
z

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