ON THE THEORY OF THE DIFFUSING ELECTRON STREAM IN A GAS

By L. G. H. HUXLEY*

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Abstract

A theoretical discussion is presented of the structure of a stream of electrons moving through a gas under the action of a uniform electric field. The treatment incorporates the phenomenon of longitudinal diffusion in which the coefficient D_L can be different from the isotropic coefficient of diffusion D. The treatment also is not restricted to the case in which the aperture through which the stream enters the diffusion chamber is small. The solution satisfies the boundary condition $n = 0$ at the surface of the anode and everywhere on the cathode except over the plane of the source aperture. The theory therefore provides criteria for the validity of simplified solutions employed hitherto.

I. INTRODUCTION

In the well-known method introduced by Townsend to measure the ratio of the drift velocity W to the coefficient of diffusion D of electrons in a steady state of motion in a gas, it is necessary to determine the distribution $n(x,y,z)$ of the number density of the electrons in a steady stream moving through the gas in a uniform electric field $\boldsymbol{E}$. The source of the stream is an aperture in a plane cathode through which pass electrons from a broad stream (assumed uniform) which moves in a uniform electric field also equal to $\boldsymbol{E}$ and is intercepted by the cathode of the diffusion chamber. The restricted but spreading stream that emerges from the aperture then flows through the gas and is received by a plane anode. The experimental procedure consists in measuring the proportion R of the whole current received by a central region of the anode. It is necessary to determine R as a function of W/D and the known dimensions of the diffusion chamber so that W/D may be deduced from the measured value of R (see e.g. Crompton 1972 for a summary of the development of the theory). To derive the theoretical formula for R it is first necessary to determine the distribution of the number density $n(x,y,z)$ in the stream from the equation of continuity for n and the boundary conditions. In Townsend's analysis the equation of continuity to be satisfied by $n(x,y,z)$ was taken to be

$$\nabla^2 n = (W/D)\partial n/\partial z = 2\lambda\, \partial n/\partial z\,, \tag{1}$$

and the boundary condition to be $n = 0$ over the surface of the cathode except over the aperture. Let the origin of coordinates ($z = 0, \rho = 0$) lie at the centre of the circular aperture (of radius $\rho = a$) in the cathode which lies in the plane $z = 0$,

* Ion Diffusion Unit, Research School of Physical Sciences, Australian National University, P.O. Box 4, Canberra, A.C.T. 2600.

the axis of the stream being $+0z$ and the plane of the anode $z = h$. The diffusion chamber, which was cylindrical in form, was limited laterally by annular rings that served as potential dividers. Townsend also assumed that $n = 0$ at the effective radius of the cylinder $\rho = c$, while he arbitrarily assumed that n was constant over the source aperture, $0 \leqslant \rho \leqslant a$. He supposed the distribution in the stream to be that in a stream proceeding to infinity and then assumed that the current density nW over the anode in the plane $z = h$ was that over the geometrical plane $z = h$ in the uninterrupted stream. Under these conditions he found the appropriate solution of equation (1) for $n(\rho, z)$ in a stream proceeding to infinity in the form of an infinite series whose terms involved Bessel functions of orders zero and unity (e.g. Townsend 1915). Later, Pidduck (1925) derived a solution for the case in which the aperture was a parallel sided slot. He used the correct boundary condition $n = 0$ at the anode $z = h$ and also assumed that n was constant across the aperture.

Much later, Huxley (1940) gave a more convenient form of solution that is appropriate to apparatus in which the aperture is made small, that is, $a/h \ll 1$. Distant solutions of the form

$$n = \exp(\lambda z)(\lambda r)^{-\frac{1}{2}} \sum_{k=0}^{\infty} A_k K_{\frac{1}{2}}(\lambda r)\, P_k(\cos\theta) \tag{2}$$

were assumed for the stream proceeding to infinity, where K is a modified Bessel function of the second kind, $r = (\rho^2 + z^2)^{\frac{1}{2}}$, and $\cos\theta = z/r$. It is found that for $a/h \ll 1$ the aperture behaves as a point source and is a simple dipole source giving

$$n = -A_1 \exp(\lambda z)\, \partial\{r^{-1}\exp(-\lambda r)\}/\partial z\,. \tag{3}$$

With this solution n vanishes over the cathode except at the origin. The condition $n = 0$ over the anode $z = h$ for a stream not proceeding to infinity is satisfied by the addition of supplementary solutions in the form of dipole "image" sources at the positions $\rho = 0$, $z = \pm 2nh$ $(n = 1, 2, \ldots)$, all directed parallel to the original source. In practice, unless z is small, the distribution $n(\rho, z)$ is well represented by the system comprising the original source and the single image source at $z = 2h$.

In what follows, the theory is generalized to remove the restriction in Huxley's formulation that a is small and the special assumption in the work of Townsend (1915) and Pidduck (1925) that n is constant across the aperture. Moreover, it is now known (e.g. Huxley 1972) that the equation of continuity (1), must be modified to take account of the difference between the longitudinal coefficient of diffusion D_{L} and the transverse coefficient D, when electrons diffuse in the presence of an electric field.

II. Integral Solutions

The modified equation of continuity which takes account of the difference between the longitudinal and transverse diffusion coefficients is

$$D\left(\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2}\right) + D_{\mathrm{L}}\frac{\partial^2 n}{\partial z^2} = W\frac{\partial n}{\partial z}\,. \tag{4}$$

Division of all terms in equation (4) by D_L, with the substitutions $x'^2 = (D_L/D)x^2$, $y'^2 = (D_L/D)y^2$, and $2\lambda_L = W/D_L$, then gives

$$\frac{\partial^2 n}{\partial x'^2} + \frac{\partial^2 n}{\partial y'^2} + \frac{\partial^2 n}{\partial z^2} = 2\lambda_L \frac{\partial n}{\partial z}.$$

If we write $n = \exp(\lambda_L z)\, V(x', y', z)$, it is seen that V is a solution of

$$\nabla'^2 V = \lambda_L^2 V, \tag{5}$$

where $\nabla'^2 \equiv (\partial^2/\partial x'^2 + \partial^2/\partial y'^2 + \partial^2/\partial z^2)$.

A solution is required such that $n = 0$ ($V = 0$) on the cathode $z = 0$ except across the aperture $0 \leqslant \rho \leqslant a$, that is to say, $V_{z=0}$ exhibits a discontinuity in its functional form at $\rho = a$. We therefore seek a solution in the form of an integral since there exist classes of integrals regarded as functions of their parameters that show such discontinuous behaviour. It is also evident that cylindrical coordinates should be adopted. If $\rho^2 = x^2 + y^2$ and $\rho'^2 = x'^2 + y'^2 = (D_L/D)\rho^2$ then, since the stream is symmetrical about its axis, equation (5) is equivalent to

$$\frac{1}{\rho'}\frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial V}{\partial \rho'}\right) + \frac{\partial^2 V}{\partial z^2} = \lambda_L^2 V. \tag{6}$$

Put

$$V(\rho', z) = \exp(\pm tz)\, U(\rho'). \tag{7}$$

It then follows from equation (6) that

$$\frac{1}{\rho'}\frac{\mathrm{d}}{\mathrm{d}\rho'}\left(\rho' \frac{\mathrm{d}U}{\mathrm{d}\rho'}\right) + (t^2 - \lambda_L^2)U = 0, \tag{8}$$

which is Bessel's equation of zero order. Consequently a solution finite at $\rho' = 0$ is

$$U \propto J_0\{(t^2 - \lambda_L^2)^{\frac{1}{2}} \rho'\}$$

and the solution of equation (6) is

$$V(\rho', z) \propto \exp(\pm tz) J_0\{(t^2 - \lambda_L^2)^{\frac{1}{2}} \rho'\}. \tag{9}$$

Since t is a parameter, which is arbitrary except for the requirement $t \geqslant \lambda_L$, we may multiply the right-hand side of equation (9) by a function $\phi(t)$ to obtain a more general solution of equation (6). Furthermore, provided the integral is convergent another solution is

$$V \propto \int_{\lambda_L}^{\infty} \exp(\pm tz)\, J_0\{(t^2 - \lambda_L^2)^{\frac{1}{2}} \rho'\}\, \phi(t)\, \mathrm{d}t. \tag{10}$$

Consider first a stream that proceeds in a uniform field to $z = +\infty$, that is to say, the anode is at so great a distance that it has no effect on the structure of the stream at finite distances z from the cathode. We therefore adopt the negative sign in the exponential term in (10) and let the origin of coordinates be the centre of the aperture and the cathode the plane $z = 0$. We also replace the variable of integration by v where $v^2 = t^2 - \lambda_L^2$ with $v\,dv = t\,dt$. Consequently the expression (10) for V becomes

$$V = A\int_0^\infty \frac{\exp\{-z(v^2+\lambda_L^2)^{\frac{1}{2}}\}\,J_0(\rho' v)\,\phi((v^2+\lambda_L^2)^{\frac{1}{2}})\,v\,dv}{(v^2+\lambda_L^2)^{\frac{1}{2}}}, \tag{11}$$

where A is an arbitrary constant. The factor $\phi((v^2+\lambda_L^2)^{\frac{1}{2}})$ may include the case in which it is free from λ_L and of the form $\phi(v)$. We shall, in due course, seek the form of $\phi(v)$ that gives V the appropriate behaviour on the cathode when $z = 0$ but for the moment we note the form of $V(\rho', z)$ when $\phi(v)$ is given particular forms. First let $\phi(v)$ be a constant equal to unity, in which case

$$V = A\int_0^\infty \frac{\exp\{-z(v^2+\lambda_L^2)^{\frac{1}{2}}\}\,J_0(\rho' v)\,v\,dv}{(v^2+\lambda_L^2)^{\frac{1}{2}}}. \tag{12}$$

However (Watson 1944, Section 13.47, equation (2)),

$$\int_0^\infty J_\mu(bt)\frac{K_\nu\{a(t^2+z^2)^{\frac{1}{2}}\}}{(t^2+z^2)^{\frac{1}{2}\nu}}\,t^{\mu+1}\,dt = \frac{b^\mu}{a^\nu}\left(\frac{(a^2+b^2)^{\frac{1}{2}}}{z}\right)^{\nu-\mu-1} K_{\nu-\mu-1}\{z(a^2+b^2)^{\frac{1}{2}}\}. \tag{13}$$

It follows from equation (12) that

$$\begin{aligned} V &= A(2z/\pi)^{\frac{1}{2}}\int_0^\infty J_0(\rho' v)\frac{K_{\frac{1}{2}}\{z(v^2+\lambda_L^2)^{\frac{1}{2}}\}}{(v^2+\lambda_L^2)^{\frac{1}{4}}}\,v\,dv \\ &= A(2\lambda_L/\pi)^{\frac{1}{2}}\,K_{\frac{1}{2}}\{\lambda_L(\rho'^2+z^2)^{\frac{1}{2}}\}/(\rho'^2+z^2)^{\frac{1}{4}} \\ &= A\exp(-\lambda_L r')/r', \end{aligned} \tag{14}$$

where $r' = (\rho'^2+z^2)^{\frac{1}{2}}$ and $K_\nu \equiv K_{-\nu}$.

Thus the expression (14) gives the solution of equation (6) that represents the distribution of V from a "pole" source of V at the origin. It also follows from differentiation of (14) with respect to z that additional solutions are

$$\begin{aligned} V &= A\int_0^\infty \frac{\exp\{-z(v^2+\lambda_L^2)^{\frac{1}{2}}\}\,J_0(\rho' v)\,(v^2+\lambda_L^2)^{\frac{1}{2}m}v\,dv}{(v^2+\lambda_L^2)^{\frac{1}{2}}} \\ &= (-1)^m A\frac{\partial^m}{\partial z^m}\left(\frac{\exp(-\lambda_L r')}{r'}\right) = (-1)^m A(2/\pi)^{\frac{1}{2}}\lambda_L\frac{\partial^m}{\partial z^m}\left(\frac{K_{\frac{1}{2}}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}}\right). \end{aligned} \tag{15}$$

In particular, the distribution of V from a "dipole" source of V at the origin is

$$\begin{aligned} V &= -A\frac{\partial}{\partial z}\left(\frac{\exp(-\lambda_L r')}{r'}\right) = -A\left(\frac{2\lambda_L}{\pi}\right)^{\frac{1}{2}}\frac{\partial}{\partial z}\left(\frac{K_{\frac{1}{2}}(\lambda_L r')}{r'^{\frac{1}{2}}}\right) \\ &= A\int_0^\infty \exp\{-z(v^2+\lambda_L^2)^{\frac{1}{2}}\}\,J_0(\rho' v)\,v\,dv. \end{aligned} \tag{16}$$

III. Plane Sources

(a) *Ring Source*

Consider the pole source of equation (14) to be uniformly distributed over a narrow circular annulus in the plane $z = 0$ and with its centre at the origin. At a point $(0, z)$ on the $+0z$ axis the value of V, when the radius of the annulus is q, is from equation (14)

$$V = A\exp\{-\lambda_{\mathrm{L}}(z^2+q'^2)^{\frac{1}{2}}\}/(z^2+q'^2)^{\frac{1}{2}}, \tag{17}$$

where $q' = (D_{\mathrm{L}}/D)^{\frac{1}{2}}q$. The value of V at a point (ρ, z) not on the axis is given by

$$V = A\int_0^\infty \frac{\exp\{-z(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}\}\,J_0(\rho'v)\,\phi(v)\,v\,\mathrm{d}v}{(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}},$$

in which $\phi(v)$ is to be determined. Since V is given by equation (17) when $\rho = 0$ it follows that

$$\int_0^\infty \frac{\exp\{-z(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}\}\,\phi(v)\,v\,\mathrm{d}v}{(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}} = \exp\{-\lambda_{\mathrm{L}}(z^2+q'^2)^{\frac{1}{2}}\}/(z^2+q'^2)^{\frac{1}{2}}.$$

Comparison with equation (14) shows that $\phi(v) = J_0(q'v)$ and therefore that the general expression for V from an annular source of poles is

$$V = A\int_0^\infty \frac{\exp\{-z(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}\}\,J_0(\rho'v)\,J_0(q'v)\,v\,\mathrm{d}v}{(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}}. \tag{18}$$

Similarly the expression for V from a ring of dipole sources is

$$V = A\int_0^\infty \exp\{-z(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}\}\,J_0(\rho'v)\,J_0(q'v)\,v\,\mathrm{d}v. \tag{19}$$

(b) *Infinite Plane Source*

Consider a uniform distribution of pole sources over the whole plane $z = 0$. Let s be the strength of sources per unit area, that is to say, the rate of emission of electrons from an element $\mathrm{d}S$ of the surface is $s\,\mathrm{d}S$. The value of V at a point (ρ, z) is a function of z but not of ρ. It follows from equation (17) that

$$\begin{aligned} V &= A\int_{q=0}^\infty [\exp\{-\lambda_{\mathrm{L}}(z^2+q'^2)^{\frac{1}{2}}\}/(z^2+q'^2)^{\frac{1}{2}}]2\pi q\,\mathrm{d}q \\ &= A(2\pi D/D_{\mathrm{L}})\int_{q'=0}^\infty \exp\{-\lambda_{\mathrm{L}}(z^2+q'^2)^{\frac{1}{2}}\}\,\mathrm{d}\{(z^2+q'^2)^{\frac{1}{2}}\} \\ &= A(2\pi D/D_{\mathrm{L}})[-\exp\{-\lambda_{\mathrm{L}}(z^2+q'^2)^{\frac{1}{2}}\}/\lambda_{\mathrm{L}}]_{q'=0}^\infty \\ &= A(4\pi D/W)\exp(-\lambda_{\mathrm{L}}z), \end{aligned} \tag{20}$$

since $2\lambda_L = W/D_L$. Also,

$$n = \exp(\lambda_L z)\, V = A(4\pi D/W),$$

and at the surface $z = 0$

$$n = V = A(4\pi D/W).$$

However, the rate of emission from an element of surface dS is given both by $s\,dS$ and $nW\,dS = V_{z=0}\,W\,dS$, and consequently

$$s = 4\pi DA \qquad \text{or} \qquad A = s/4\pi D. \tag{21}$$

Thus if V in equation (14) is due to a pole source with strength s then

$$V = (s/4\pi D)\exp(-\lambda_L r')/r'. \tag{22}$$

Similarly, when the annular source of equation (17) carries a total strength $(2\pi q\,dq)s = (2\pi D/D_L)q'\,dq'\,s$ then (17) becomes

$$\begin{aligned} V &= (2\pi D/D_L)(q'\,dq'\,s/4\pi D)\exp\{-\lambda_L(z^2+q'^2)^{\frac{1}{2}}\}/(z^2+q'^2)^{\frac{1}{2}} \\ &= [(s/2D_L)\exp\{-\lambda_L(z^2+q'^2)^{\frac{1}{2}}\}/(z^2+q'^2)^{\frac{1}{2}}]q'\,dq'. \end{aligned} \tag{23}$$

Moreover, equation (18) can then be written

$$V = (s/2D_L)(q'\,dq')\int_0^\infty \frac{\exp\{-z(v^2+\lambda_L^2)^{\frac{1}{2}}\}\,J_0(\rho'v)\,J_0(q'v)\,v\,dv}{(v^2+\lambda_L^2)^{\frac{1}{2}}}. \tag{24}$$

Equation (20) becomes

$$V = (s/W)\exp(-\lambda_L z) \tag{25}$$

and

$$n = s/W.$$

The value of dV/dz is given by

$$dV/dz = -(s\lambda_L/W)\exp(-\lambda_L z) = -(s/2D_L)\exp(-\lambda_L z) \tag{26}$$

when z is positive. For negative values of z the sign of dV/dz is reversed and consequently at the surface there is a discontinuity in dV/dz of an amount

$$\Delta(dV/dz) = s/D_L. \tag{27}$$

Consider the field of V of two infinite parallel plane sources, one on the plane $z = 0$ with source density $+s$ per unit area and the other on the plane $z = -h$ with source density $-s$. The expression for V on the plane $z =$ const. is

$$V = (s/W)\{1-\exp(-\lambda_L h)\}\exp(-\lambda_L z).$$

If $h \to 0$ then

$$V \to (sh\lambda_L/W)\exp(-\lambda_L z) = (m/2D_L)\exp(-\lambda_L z), \tag{28}$$

where m is the "dipole" strength per unit area on the plane $z = 0$. Since V changes sign in crossing the plane $z = 0$ the discontinuity in V at the plane $z = 0$ is

$$\Delta V = m/D_L.$$

Also, on the positive face of the plane $z = 0$

$$n = V = m/2D_{\mathrm{L}}. \tag{29}$$

Thus on the plane $z = 0$ the behaviour of V is that of an electrostatic potential.

It follows from the above that equation (16) when related to an isolated dipole with strength m is

$$V = -\frac{m}{4\pi D}\frac{\partial}{\partial z}\left(\frac{\exp(-\lambda_{\mathrm{L}} r')}{r'}\right). \tag{30}$$

Equation (19) when related to an annulus of dipoles with total strength $2\pi mq\,\mathrm{d}q$ becomes

$$V = (m/2D_{\mathrm{L}})(q'\,\mathrm{d}q')\int_0^\infty \exp\{-z(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}\}\,J_0(\rho' v)\,J_0(q'v)\,v\,\mathrm{d}v. \tag{31}$$

(c) *Plane Distribution in which s and m are Functions of q'*

Let s and m be functions of the radial distance q and therefore of $q' = (D_{\mathrm{L}}/D)^{\frac{1}{2}}q$, that is, $s \equiv s(q')$ and $m \equiv m(q')$. It follows from equations (24) and (31) that the expressions for $V(\rho', z)$ produced by the distributions $s(q')$ and $m(q')$ respectively are

$$V(\rho', z) = \int_0^\infty \frac{\exp\{-z(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}\}}{(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}} J_0(\rho' v)\left(\int_0^\infty J_0(q'v)\,\{s(q')/2D_{\mathrm{L}}\}\,q'\,\mathrm{d}q'\right)v\,\mathrm{d}v \tag{32}$$

and

$$V(\rho', z) = \int_0^\infty \exp\{-z(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}\}\,J_0(\rho' v)\left(\int_0^\infty J_0(q'v)\,\{m(q')/2D_{\mathrm{L}}\}\,q'\,\mathrm{d}q'\right)v\,\mathrm{d}v. \tag{33}$$

Also, it follows from equation (32) that

$$-\frac{\partial V(\rho', z)}{\partial z} = \int_0^\infty \exp\{-z(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}\}\,J_0(\rho' v)\left(\int_0^\infty J_0(q'v)\,\{s(q')/2D_{\mathrm{L}}\}\,q'\,\mathrm{d}q'\right)v\,\mathrm{d}v. \tag{34}$$

We see from equation (32) that in equation (11)

$$\phi((v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}) \equiv \phi(v) = \int_0^\infty J_0(q'v)\,\{s(q')/2D_{\mathrm{L}}\}\,q'\,\mathrm{d}q'. \tag{35}$$

On the positive face of the plane $z = 0$, $m/2D_{\mathrm{L}} = V(q') = n(q')$ and consequently, since $z = 0$, equation (33) becomes

$$V(q') = \int_0^\infty J_0(\rho' v)\,\psi(v)\,v\,\mathrm{d}v, \tag{36a}$$

with

$$\psi(v) = \int_0^\infty J_0(q'v)\,V(q')\,q'\,\mathrm{d}q'. \tag{36b}$$

The pair of equations (36a) and (36b) are a particular example of Hankel's inversion theorem (Watson 1944, Section 14.4; Webster 1955, p. 369; Bowman 1958, p. 114).

IV. Distribution of n over Plane $z = 0$

The boundary conditions specify the behaviour of n over the plane $z = 0$, namely

$$n \neq 0, \quad 0 \leqslant \rho \leqslant a; \qquad n = 0, \quad \rho > a.$$

It follows from equation (36b) that if the function $V(q')$ is known then $\psi(v)$ can be determined in principle and $V(\rho', z)$ is then

$$V(\rho', z) = \int_0^\infty \exp\{-z(v^2+\lambda_L^2)^{\frac{1}{2}}\} J_0(\rho' v)\, \psi(v)\, v\, \mathrm{d}v. \tag{37}$$

For instance, Townsend (1915) assumed that on the plane $z = 0$

$$n = \text{const.} = n_0, \quad 0 \leqslant \rho \leqslant a; \qquad n = 0, \quad \rho > a.$$

From equation (36b)

$$\psi(v) = n_0 \int_0^{a'} J_0(q'v)\, q'\, \mathrm{d}q' = n_0(a'/v) J_1(a'v),$$

where $a' = (D_L/D)^{\frac{1}{2}} a$. Consequently

$$V(\rho', z) = n_0\, a' \int_0^\infty \exp\{-z(v^2+\lambda_L^2)^{\frac{1}{2}}\} J_0(\rho' v)\, J_1(a'v)\, \mathrm{d}v$$

and

$$n(\rho', z) = \exp(\lambda_L z)\, V(\rho', z).$$

The assumption that n is constant across the aperture is unrealistic in that it neglects the loss of electrons by diffusion to the edges. On the contrary it is to be expected that n would vary from a maximum value n_0 at the centre to zero at the edges. We note that when λ_L is zero equation (5) reduces to Laplace's equation and that V behaves therefore as an electrostatic potential. The problem under consideration then reduces to the corresponding electrostatic problem of the penetration of a uniform electrostatic field through an aperture in a plane conductor. It is known that in the latter problem the distribution of potential across the aperture is

$$V(q) = V_0(1-q^2/a^2)^{\frac{1}{2}}.$$

We therefore take the distribution of electron number density across the aperture to be

$$V(q') = n(q') = n_0(1-q^2/a^2)^\nu = n_0(1-q'^2/a'^2)^\nu, \tag{38}$$

where ν is, for the moment, unspecified. Equation (36b) then shows that

$$\psi(v) = n_0 \int_0^{a'} J_0(q'v)(1-q'^2/a'^2)^\nu\, q'\, \mathrm{d}q', \tag{39}$$

since $n(q')$ is zero when q' exceeds a'.

In order to evaluate the integral in equation (39) consider Sonine's first finite integral (Watson 1944, Section 12.11) which is

$$\int_0^{\frac{1}{2}\pi} J_\mu(z\sin\theta)\sin^{\mu+1}\theta\cos^{2\nu+1}\theta\,\mathrm{d}\theta = \{2^\nu\Gamma(\nu+1)/z^{\nu+1}\}J_{\mu+\nu+1}(z), \tag{40}$$

for $\mathrm{Re}(\mu)$ and $\mathrm{Re}(\nu) > -1$. Write $\sin\theta = t$ in order to effect the transformation

$$\int_0^1 J_\mu(zt)(1-t^2)^\nu t^{\mu+1}\,\mathrm{d}t = \{2^\nu\Gamma(\nu+1)/z^{\nu+1}\}J_{\mu+\nu+1}(z). \tag{41}$$

It follows from equations (39) and (41) that

$$\begin{aligned}\psi(v) &= n_0 a'^2\int_0^1 J_0\{(q'/a')(a'v)\}(1-q'^2/a'^2)^\nu(q'/a')\,\mathrm{d}(q'/a')\\ &= n_0 a'^2\{2^\nu\Gamma(\nu+1)\}J_{\nu+1}(a'v)/(a'v)^{\nu+1}.\end{aligned} \tag{42}$$

The distribution of the number density n in the stream is from equations (37) and (42)

$$\begin{aligned}n(\rho',z) = {}&\tfrac{1}{2}n_0 a'^2 2^{\nu+1}\Gamma(\nu+1)\exp(\lambda_{\mathrm{L}}z)\\ &\times\int_0^\infty \exp\{-z(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}\}J_0(\rho'v)\{J_{\nu+1}(a'v)/(a'v)^{\nu+1}\}v\,\mathrm{d}v.\end{aligned} \tag{43}$$

When $z = 0$ the integral in this expression becomes an example of a Weber–Shafheitlin discontinuous integral (Watson 1944, Section 13.4). The series expansion of $2^{\nu+1}\Gamma(\nu+1)J_{\nu+1}(a'v)/(a'v)^{\nu+1} = y$ is

$$y = \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(\nu+2)}{\Gamma(m+\nu+2)}\frac{(\frac{1}{2}a'v)^{2m}}{m!}.$$

When $\frac{1}{2}\lambda_{\mathrm{L}}a'$ is small in comparison with unity (see Section VIII) the expression for $n(\rho',z)$ is determined by the first term in the expansion for y, and thus

$$n(\rho',z) \to \tfrac{1}{2}n_0 a'^2\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\int_0^\infty \exp\{-z(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}\}J_0(\rho'v)v\,\mathrm{d}v$$

as $\frac{1}{2}\lambda_{\mathrm{L}}a' \to 0$. From equations (6) and (16), this gives

$$n(\rho',z) \to -\tfrac{1}{2}n_0 a'^2\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\exp(\lambda_{\mathrm{L}}z)\frac{\partial}{\partial z}\left(\frac{\exp(-\lambda_{\mathrm{L}}r')}{r'}\right). \tag{44}$$

Thus a small aperture behaves as a simple "dipole" source, as is usually assumed. It is, however, convenient to postpone consideration of the general case in which $\frac{1}{2}\lambda_{\mathrm{L}}a'$ is no longer very small.

V. Representation of $n(\rho', z)$ from Source displaced along $0z$ Axis

If $\psi(v)$ is replaced by $\exp\{l(v^2+\lambda_L^2)^{\frac{1}{2}}\}\psi(v)$ in equation (37), the new expression for V then is

$$V(\rho', z, l) = \int_0^\infty \exp\{-(z-l)(v^2+\lambda_L^2)^{\frac{1}{2}}\} J_0(\rho' v)\,\psi(v)\,v\,\mathrm{d}v, \tag{45}$$

in which $\psi(v)$ retains its original form and l is less than z. Evidently equations (45) and (37) are the same but for the replacement of z by $z-l$ and consequently

$$V(\rho', z, l) \equiv V(\rho', z-l).$$

The solution is therefore that appropriate to the displacement of the source plane from $z = 0$ to l. The expression for n is

$$\begin{aligned} n &= \exp\{\lambda_L(z-l)\}\, V(\rho', z-l) = \exp(\lambda_L z)\exp(-\lambda_L l)\, V(\rho', z-l) \\ &= \exp(\lambda_L z)\, V_1(\rho', z-l), \end{aligned} \tag{46}$$

in which V_1 is what $V(\rho', z-l)$ becomes when the strength of the dipole distribution over the source plane is everywhere changed by the same factor $\exp(-\lambda_L l)$.

VI. Solution when Stream terminates on Anode in Plane $z = h$

In this case the solution must be such that n and therefore V are zero over the plane $z = h$. We refer to equation (9) and note the alternative possibilities of $\pm tz$ in the exponential. In seeking an expression for a stream proceeding to $z = +\infty$ it was necessary to exclude the positive exponent. However, when the stream is limited in length it is legitimate to include terms in both exponents and indeed necessary to do so in order to meet the requirement $n = V = 0$ when $z = h$. In what follows, the terms are associated to form a hyperbolic function. For a stream originating from a distribution of pole sources over the plane $z = 0$ we replace equation (32) by

$$V(\rho', z) = \int_0^\infty \frac{\sinh\{(h-z)(v^2+\lambda_L^2)^{\frac{1}{2}}\}}{\sinh\{h(v^2+\lambda_L^2)^{\frac{1}{2}}\}} \frac{J_0(\rho' v)}{(v^2+\lambda_L^2)^{\frac{1}{2}}}\,\phi(v)\,v\,\mathrm{d}v, \tag{47}$$

where, as before, $\phi(v)$ is given by equation (35). It will be noted that $V(\rho', z) = 0$ on the plane $z = h$ and that when $z = 0$

$$V(q') = \int_0^\infty \frac{J_0(q' v)}{(v^2+\lambda_L^2)^{\frac{1}{2}}}\,\phi(v)\,v\,\mathrm{d}v,$$

which is what equation (32) gives when $z = 0$. The conditions over the plane $z = 0$ are therefore the same as for the uninterrupted stream that proceeds to infinity.

Similarly, the stream that originates from a distribution of dipoles over the plane $z = 0$ is represented by the following modification of equation (37),

$$V(\rho', z) = \int_0^\infty \frac{\sinh\{(h-z)(v^2+\lambda_L^2)^{\frac{1}{2}}\}}{\sinh\{h(v^2+\lambda_L^2)^{\frac{1}{2}}\}} J_0(\rho' v)\,\psi(v)\,v\,\mathrm{d}v, \tag{48}$$

where

$$\psi(v) = \int_0^\infty J_0(q'v)\{m(q')/2D_{\mathrm{L}}\}q'\,\mathrm{d}q'.$$

Again, $V(\rho',h) = 0$ and the original distribution of sources in the plane $z = 0$ remains unmodified. The distribution of number density in each stream (pole or dipole) is found to be

$$n(\rho',z) = \exp(\lambda_{\mathrm{L}} z)\,V(\rho',z).$$

In particular when the stream enters through a circular aperture in the cathode it follows from what has immediately preceded and from equation (43) that

$$n(\rho',z) = \tfrac{1}{2}n_0\,a'^2\,2^{\nu+1}\,\Gamma(\nu+1)\exp(\lambda_{\mathrm{L}} z)$$
$$\times \int_0^\infty \frac{\sinh\{(h-z)(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}\}}{\sinh\{h(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}\}} J_0(\rho'v)\frac{J_{\nu+1}(a'v)}{(a'v)^{\nu+1}}\,v\,\mathrm{d}v. \qquad (49)$$

When $\frac{1}{2}\lambda_{\mathrm{L}}\,a' \to 0$ (see Section VIII) this expression approaches the form

$$n(\rho',z) \to \tfrac{1}{2}n_0\,a'^2\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\int_0^\infty \frac{\sinh\{(h-z)(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}\}}{\sinh\{h(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}\}} J_0(\rho'v)\,v\,\mathrm{d}v, \qquad (50)$$

which represents the distribution of electrons in the stream from a small aperture such that $\frac{1}{2}\lambda_{\mathrm{L}}\,a' \ll 1$.

VII. Representation in Terms of Supplementary Image Solutions

We return to equation (48) which is an expression for $V(\rho',z)$ appropriate to a stream that falls on an anode in the plane $z = h$, the number density being $n(\rho',z) = \exp(\lambda_{\mathrm{L}} z)\,V(\rho',z)$. The factor in the integrand that is the ratio of two hyperbolic functions can be expanded into a series as

$$\frac{\sinh\{(h-z)(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}\}}{\sinh\{h(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}\}}$$
$$= \sum_{k=0}^{\infty}\left(\exp\{-(z+2kh)(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}\} - \exp[-\{2(k+1)h-z\}(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}]\right). \qquad (51)$$

An equivalent form for the number density from equation (48) is therefore

$$n(\rho',z) = \exp(\lambda_{\mathrm{L}} z)\,V(\rho',z)$$
$$= \exp(\lambda_{\mathrm{L}} z)\left(\sum_{k=0}^{\infty}\int_0^\infty \exp\{-(z+2kh)(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}\}\,J_0(\rho',v)\,\psi(v)\,v\,\mathrm{d}v\right.$$
$$\left. - \sum_{k=0}^{\infty}\int_0^\infty \exp[-\{2(k+1)h-z\}(v^2+\lambda_{\mathrm{L}}^2)^{\frac{1}{2}}]\,J_0(\rho'v)\,\psi(v)\,v\,\mathrm{d}v\right). \qquad (52)$$

It follows from Section IV that the first sum of integrals on the right-hand side of equation (52) is the contribution to $n(\rho', z)$ between the electrodes of the original distribution of dipole sources over the plane $z = 0$, with supplementary contributions from image distributions of dipoles over the planes $z = -2kh$, $k = 1,2,3,\ldots$, whereas the second sum of integrals is the contribution of plane image distributions of dipoles over the planes $z = 2(k+1)h$, $k = 0,1,2,\ldots$. The sense of polarization is the same on every plane and is that of the source plane $z = 0$, that is, in the direction $+0z$.

In the same way equation (47) is seen to be equivalent to the contribution from the original distribution of pole sources over the plane $z = 0$, together with image distributions of poles over the planes $z = \pm 2kh$. The sign of the poles on the planes $z = +2kh$ is negative and that on the planes $z = -2kh$ is positive.

When $(\lambda_L h)$ is large, as is usually the case in practice, the number density $n(\rho', z)$ in the stream except near the source, is given with adequate accuracy by the sum of the contributions of the source distribution over the plane $z = 0$ and of that over the plane $z = +2h$. In the case of the stream emerging from a circular aperture in the cathode, the source and its images behave as simple dipoles when the radius a is such that $(\frac{1}{2}\lambda_L a') \ll 1$ in accordance with equation (44). The use of image solutions has been frequently made in practice (Huxley 1940; Huxley and Crompton 1955; Crompton and Jory 1962; Crompton *et al.* 1965).

VIII. Aperture with Finite Radius

We seek a representation in series of the right-hand side of equation (43) in which the restriction $a/z \ll 1$ is removed. The factor $2^{\nu+1} J_{\nu+1}(a'v)/(a'v)^{\nu+1}$ in the integrand can be replaced by its series expansion

$$2^{\nu+1}\frac{J_{\nu+1}(a'v)}{(a'v)^{\nu+1}} = \frac{1}{\Gamma(\nu+2)}\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(\nu+2)}{\Gamma(\nu+2+m)}\frac{(\frac{1}{2}a'v)^{2m}}{m!}$$

$$= {}_0F_1\{\nu+2; -\tfrac{1}{4}(a'v)^2\}/\Gamma(\nu+2), \tag{53}$$

where ${}_0F_1\{\nu+2; -\frac{1}{4}(a'v)^2\}$ is a generalized hypergeometric function expressed in Pochhammer's notation (Watson 1944, Section 4.4). In this notation

$${}_pF_q(\alpha_1,\alpha_2,\ldots,\alpha_p;\beta_1,\beta_2,\ldots,\beta_q;z) = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\frac{z^n}{n!},$$

where each $(\alpha)_n$ is given by

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$$

and $(\alpha)_0 = 1$, with a corresponding interpretation of $(\beta)_n$.

The right-hand side of equation (43) can be represented in series form in several ways, but because the limiting case where $a/z \ll 1$ as expressed in equation (44) depends upon the effective polar distance $r' = (\rho'^2+z^2)^{\frac{1}{2}}$ we seek an expansion that is a function of r'. To this end we transform the right-hand side of equation (53) to become a function of $a'^2(v^2+\lambda_L^2)$ in order to employ equation (15). If we write

$u = \frac{1}{4}a'^2(v^2+\lambda_L^2)$, or $\frac{1}{4}a'^2v^2 \equiv u-(\frac{1}{2}\lambda_L a')^2$, it then follows from Taylor's theorem that

$$\begin{aligned}\frac{1}{\Gamma(\nu+2)}\,{}_0F_1\{\nu+2;\,-(\tfrac{1}{2}a'v)^2\} &= \frac{1}{\Gamma(\nu+2)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\tfrac{1}{2}\lambda_L a')^{2k}\frac{d^k}{du^k}\Big({}_0F_1\{\nu+2;\,-u\}\Big)\\ &= \frac{1}{\Gamma(\nu+2)}\sum_{k=0}^{\infty}(\tfrac{1}{2}\lambda_L a')^{2k}\frac{{}_0F_1\{\nu+2+k;\,-u\}}{k!(\nu+2)_k}\\ &= \sum_{k=0}^{\infty}\frac{(\frac{1}{2}\lambda_L a')^{2k}}{k!\,\Gamma(\nu+2+k)}\,{}_0F_1\{\nu+2+k;\,-\tfrac{1}{4}a'^2(v^2+\lambda_L^2)\}.\end{aligned} \tag{54}$$

We now assemble the right-hand side of equation (54) to form a series in ascending powers of $\frac{1}{4}a'^2(v^2+\lambda_L^2)$. The coefficient of $(-1)^m\{\frac{1}{4}a'^2(v^2+\lambda_L^2)\}^m/m!$ is

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(\frac{1}{2}\lambda_L a')^{2k}}{k!}\frac{1}{\Gamma(\nu+2+k)}\frac{\Gamma(\nu+2+k)}{\Gamma(\nu+2+k+m)} &= \sum_{k=0}^{\infty}\frac{(\frac{1}{2}\lambda_L a')^{2k}}{k!\,\Gamma(\nu+1+m+1+k)}\\ &= I_{\nu+1+m}(\tfrac{1}{2}\lambda_L a')/(\tfrac{1}{2}\lambda_L a')^{\nu+1+m},\end{aligned}$$

where $I_{\nu+1+m}$ is a modified Bessel function of the first kind. Consequently

$$\frac{1}{\Gamma(\nu+2)}\,{}_0F_1\{\nu+2;\,-(\tfrac{1}{2}a'v)^2\} = \sum_{m=0}^{\infty}(-1)^m I_{\nu+1+m}(\tfrac{1}{2}\lambda_L a')\frac{\{\frac{1}{4}a'^2(v^2+\lambda_L^2)\}^m}{(\frac{1}{2}\lambda_L a')^{\nu+1+m}\,m!}. \tag{55}$$

It follows that equation (43) can be replaced by

$$\begin{aligned}n(\rho',z) = \tfrac{1}{2}n_0 a'^2\exp(\lambda_L z)\int_0^{\infty}&\exp\{-z(v^2+\lambda_L^2)^{\frac{1}{2}}\}J_0(\rho'v)\\ &\times\left(\Gamma(\nu+1)\sum_{m=0}^{\infty}(-1)^m I_{\nu+1+m}(\tfrac{1}{2}\lambda_L a')\frac{\{\frac{1}{4}a'^2(v^2+\lambda_L^2)\}^m}{(\frac{1}{2}\lambda_L a')^{\nu+1+m}\,m!}\right)v\,dv.\end{aligned} \tag{56}$$

We conclude therefore from equations (56) and (15) that

$$\begin{aligned}n(\rho',z) &= \exp(\lambda_L z)\,V(\rho',z)\\ &= n_0\exp(\lambda_L z)(2/\pi)^{\frac{1}{2}}\Gamma(\nu+1)\sum_{m=0}^{\infty}(-1)^m(\tfrac{1}{2}\lambda_L a')^{2(m+1)}\frac{I_{\nu+1+m}(\frac{1}{2}\lambda_L a')}{(\frac{1}{2}\lambda_L a')^{\nu+1+m}\,m!}\\ &\qquad\times\frac{1}{\lambda_L^{2m+1}}\frac{\partial^{2m+1}}{\partial z^{2m+1}}\left(\frac{K_{\frac{1}{2}}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}}\right).\end{aligned} \tag{57}$$

As shown in the Appendix, the derivatives

$$\frac{\partial^{2m}}{\partial z^{2m}}\left(\frac{K_{\frac{1}{2}}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}}\right)\quad\text{and}\quad\frac{\partial^{2m+1}}{\partial z^{2m+1}}\left(\frac{K_{\frac{1}{2}}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}}\right)$$

can be represented respectively as the sum of terms of the forms

$$\{K_{2\mu+1/2}(\lambda_L r')/(\lambda_L r')^{\frac{1}{2}}\}P_{2\mu}(\cos\theta)\quad\text{and}\quad\{K_{2\mu+3/2}(\lambda_L r')/(\lambda_L r')^{\frac{1}{2}}\}P_{2\mu+1}(\cos\theta).$$

The coefficients $A_{2\mu}$ and $A_{2\mu+1}$ in their expansions are given by equations (A8) and (A9) of the Appendix. It follows from equation (A9) that (57) can be transformed by rearrangement to an ascending series in $\{K_{2\mu+3/2}(\lambda_L r')/(\lambda_L r')^{\frac{1}{2}}\}P_{2\mu+1}(\cos\theta)$ as

$$\frac{n(\rho', z)}{n_0\,\Gamma(\nu+1)(2/\pi)^{\frac{1}{2}}\exp(\lambda_L z)}$$
$$= \Gamma(\tfrac{3}{2})\sum_{\mu=0}^{\infty}\left(\sum_{m=\mu}^{\infty}(-1)^m\frac{4(m+1)\Gamma(m+\frac{3}{2})}{\Gamma(\frac{1}{2})}\frac{(\frac{1}{2}\lambda_L a')^{2(m+1)}}{(\frac{1}{2}\lambda_L a')^{\nu+1+m}}\right.$$
$$\left.\times I_{\nu+1+m}(\tfrac{1}{2}\lambda_L a')\frac{4\mu+3}{(m-\mu)!\,\Gamma(m+\mu+\frac{5}{2})}\right)\frac{K_{2\mu+3/2}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}}P_{2\mu+1}(\cos\theta). \tag{58}$$

When $\frac{1}{2}\lambda_L a' \to 0$ the series on the right-hand side reduces to the leading term for which $\mu = m = 0$. Equation (58) is then seen to reduce to equation (44) as required.

We note that for $r > a$ and $\theta = \frac{1}{2}\pi$ the corresponding points lie on the metal surface of the cathode. Since the Legendre polynomials in equation (58) are all of odd order they vanish when $\theta = \frac{1}{2}\pi$. Thus n vanishes on the surface of the cathode as required. When $\lambda_L a'$ is small but finite the factor $I_{\nu+1+m}(\frac{1}{2}\lambda_L a')/(\frac{1}{2}\lambda_L a')^{\nu+1+m}$ in equation (58) may be replaced by $1/\Gamma(\nu+2+m)$.

We shall require in the following section a formula for $-\partial V(\rho, z)/\partial z$. From equation (57) we have

$$-\frac{\partial V}{\partial z} = \frac{n_0}{a'}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\Gamma(\nu+1)\sum_{m=0}^{\infty}(-1)^m\frac{(\frac{1}{2}\lambda_L a')^{2m+3}}{m!}\frac{I_{\nu+1+m}(\frac{1}{2}\lambda_L a')}{(\frac{1}{2}\lambda_L a')^{\nu+1+m}}$$
$$\times\frac{1}{\lambda_L^{2(m+1)}}\frac{\partial^{2(m+1)}}{\partial z^{2(m+1)}}\left(\frac{K_{\frac{1}{2}}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}}\right). \tag{59}$$

After application of equations (A6) and (A8) of the Appendix and rearrangement it can be shown that

$$-\frac{\partial V}{\partial z} = \frac{n_0}{a'}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\Gamma(\nu+1)\Gamma(\tfrac{3}{2})\sum_{\mu=1}^{\infty}\sum_{m=\mu}^{\infty}(-1)^m\frac{\{2(m+1)\}!\,(\frac{1}{2}\lambda_L a')^{2m+3}}{2^m\,m!}\frac{I_{\nu+1+m}(\frac{1}{2}\lambda_L a')}{(\frac{1}{2}\lambda_L a')^{\nu+1+m}}$$
$$\times\frac{4\mu+1}{(m+1-\mu)!\,\Gamma(m+\mu+\frac{5}{2})}\frac{K_{2\mu+\frac{1}{2}}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}}P_{2\mu}(\cos\theta). \tag{60}$$

Consequently when $\theta = \frac{1}{2}\pi$ the derivative $-\partial V/\partial z$ does not vanish on the cathode for $r > a$.

IX. Flux Density at Anode

The flux of electrons across an element of surface $\mathrm{d}S$ of a geometrical plane $z = \text{const.}$ is

$$F\,\mathrm{d}S = \left(nW - D_L\frac{\partial n}{\partial z}\right)\mathrm{d}S = \exp(\lambda_L z)\left((W - D_L\lambda_L)V - D_L\frac{\partial V}{\partial z}\right)\mathrm{d}S$$
$$= \exp(\lambda_L z)\left(\tfrac{1}{2}WV - D_L\frac{\partial V}{\partial z}\right)\mathrm{d}S = D_L\exp(\lambda_L z)\left(\lambda_L V - \frac{\partial V}{\partial z}\right)\mathrm{d}S, \tag{61}$$

since $2\lambda_L = W/D_L$. At the anode where $z = h$ and $V = 0$ the expression for the flux density becomes

$$F\,dS = -D_L \exp(\lambda_L h)(dV/dz)_{z=h}\,dS.$$

When the general expression for $V(\rho', z)$ from equation (48) is introduced into equation (61) the flux is seen to be

$$F\,dS = \left(D_L \exp(\lambda_L z) \int_0^\infty \frac{\lambda_L \sinh\{(h-z)(v^2+\lambda_L^2)^{\frac{1}{2}}\} + (v^2+\lambda_L^2)^{\frac{1}{2}} \cosh\{(h-z)(v^2+\lambda_L^2)^{\frac{1}{2}}\}}{\sinh\{h(v^2+\lambda_L^2)^{\frac{1}{2}}\}} \times J_0(\rho' v)\,\psi(v)\,v\,dv\right) dS,$$

which on the anode, where $z = h$, reduces to

$$F\,dS = \left(D_L \exp(\lambda_L h) \int_0^\infty \frac{(v^2+\lambda_L^2)^{\frac{1}{2}} J_0(\rho' v)\,\psi(v)\,v\,dv}{\sinh\{h(v^2+\lambda_L^2)^{\frac{1}{2}}\}}\right) dS.$$

The factor $\sinh\{h(v^2+\lambda_L^2)^{\frac{1}{2}}\}$ is equivalent to

$$\tfrac{1}{2}\exp\{h(v^2+\lambda_L^2)^{\frac{1}{2}}\}[1-\exp\{-2h(v^2+\lambda_L^2)^{\frac{1}{2}}\}].$$

Since $\exp\{-2h(v^2+\lambda_L^2)^{\frac{1}{2}}\}$ is less than $\exp(-2\lambda_L h)$ and in practice λ_L is of the order of magnitude 10 cm^{-1} and h is several centimetres (commonly 10 cm), it is evident that $\exp\{-2h(v^2+\lambda_L^2)^{\frac{1}{2}}\}$ is extremely small in comparison with unity. Consequently $F\,dS$ at the cathode is accurately represented by

$$F\,dS = \left(2D_L \exp(\lambda_L h) \int_0^\infty (v^2+\lambda_L^2)^{\frac{1}{2}} \exp\{-h(v^2+\lambda_L^2)^{\frac{1}{2}}\}\, J_0(\rho' v)\,\psi(v)\,v\,dv\right) dS. \qquad (62)$$

It is evident from equation (37) that this value of the flux is twice the value of the quantity $-D_L \exp(\lambda_L h)(\partial V/\partial z)_{z=h}\,dS$ in an uninterrupted stream across a surface element of the geometrical plane $z = h$. We can therefore calculate the actual flux to a surface element dS of the metal anode from the value of this quantity for an uninterrupted stream as if the anode were absent.

The flux to a central disc of the anode with radius b is

$$-2D_L \exp(\lambda_L h)\, 2\pi \int_0^b (\partial V/\partial z)_{z=h}\,\rho\,d\rho,$$

where $V(\rho, z)$ is appropriate to the unimpeded stream. It can be seen from equation (59) that over the fixed plane $z = h$ of the anode $-\partial V/\partial z$ becomes a function of r' and z/r' since $\cos\theta = h/r'$. Also on this plane $r'^2 = h^2+\rho'^2$ and $\rho'\,d\rho' = r'\,dr'$. Thus

$$\frac{dV(r', z)}{dz} = \frac{z}{r'}\frac{\partial V(r', z/r')}{\partial r'} + \frac{\partial V(r', z/r')}{\partial z}, \qquad z = h.$$

The flux to the central disc is therefore, with $d' = (h^2+b'^2)^{\frac{1}{2}}$,

$$-4\pi D_{\mathrm{L}} \exp(\lambda_{\mathrm{L}} h)(D/D_{\mathrm{L}}) \int_h^{d'} h \, \mathrm{d}V(r', h/r') + \frac{\mathrm{d}V(r', h/r')}{\mathrm{d}h} r' \, \mathrm{d}r'$$

$$= -4\pi D \exp(\lambda_{\mathrm{L}} h)\left(h V(r', h/r')|_h^{d'} + \int_h^{d'} \frac{\mathrm{d}V(r', h/r')}{\mathrm{d}h} r' \, \mathrm{d}r'\right)$$

$$= 4\pi D \exp(\lambda_{\mathrm{L}} h)\left(h\{V(h,1) - V(d', h/d')\} + \int_{d'}^{h} \frac{\mathrm{d}V(r', h/r')}{\mathrm{d}h} r' \, \mathrm{d}r'\right).$$

The total flux is obtained by setting $d' = \infty$ and noting that $V(d', h/d')_{d'=\infty} = 0$. The proportion R of the total flux that is received by the central disc is

$$R = \frac{h\{V(h,1) - V(d', h/d')\} + \int_{d'}^{h} \frac{\mathrm{d}V(r', h/r')}{\mathrm{d}h} r' \, \mathrm{d}r'}{h V(h,1) + \int_{\infty}^{h} \frac{\mathrm{d}V(r', h/r')}{\mathrm{d}h} r' \, \mathrm{d}r'}. \tag{63}$$

When $\frac{1}{2}\lambda_{\mathrm{L}} a'$ in equations (57) and (59) is so small that the first term on the right-hand side of each equation, for which $m = 0$, is the dominating term then

$$V(r', z/r') \propto \partial\{K_{\frac{1}{2}}(\lambda_{\mathrm{L}} r')/(\lambda_{\mathrm{L}} r')^{\frac{1}{2}}\}/\partial z.$$

Thus from equation (A10a) of the Appendix

$$V(r', z/r') \propto (z/r') K_{3/2}(\lambda_{\mathrm{L}} r')/(\lambda_{\mathrm{L}} r')^{\frac{1}{2}}.$$

The ratio R is then

$$R = \frac{h\{K_{3/2}(\lambda_{\mathrm{L}} h)/(\lambda_{\mathrm{L}} h)^{\frac{1}{2}} - (h/d') K_{3/2}(\lambda_{\mathrm{L}} d')/(\lambda_{\mathrm{L}} d')^{\frac{1}{2}}\} + \int_{d'}^{h} \{K_{3/2}(\lambda_{\mathrm{L}} r')/(\lambda_{\mathrm{L}} r')^{\frac{1}{2}}\} \, \mathrm{d}r'}{h K_{3/2}(\lambda_{\mathrm{L}} h)/(\lambda_{\mathrm{L}} h)^{\frac{1}{2}} + \int_{\infty}^{h} \{K_{3/2}(\lambda_{\mathrm{L}} r')/(\lambda_{\mathrm{L}} r')^{\frac{1}{2}}\} \, \mathrm{d}r'}.$$

But

$$K_{3/2}(\lambda_{\mathrm{L}} r')/(\lambda_{\mathrm{L}} r')^{\frac{1}{2}} = -\mathrm{d}\{K_{1/2}(\lambda_{\mathrm{L}} r')/(\lambda_{\mathrm{L}} r')^{\frac{1}{2}}\}/\mathrm{d}(\lambda_{\mathrm{L}} r')$$

and consequently

$$\int_{d'}^{h} \{K_{3/2}(\lambda_{\mathrm{L}} r')/(\lambda_{\mathrm{L}} r')^{\frac{1}{2}}\} \, \mathrm{d}r' = -\lambda_{\mathrm{L}}^{-1}[K_{1/2}(\lambda_{\mathrm{L}} r')/(\lambda_{\mathrm{L}} r')^{\frac{1}{2}}]_{d'}^{h}.$$

On replacement of the Bessel functions $K_{1/2}$ and $K_{3/2}$ by their equivalent forms in terms of exponentials and after reduction it is found that

$$R = 1 - \left(\frac{h}{d'} - \frac{1 - h^2/d'^2}{\lambda_{\mathrm{L}} h}\right) \frac{h}{d'} \exp\{-\lambda_{\mathrm{L}}(d' - h)\}. \tag{64}$$

The formula (64) for R is the standard expression for a dipole point source and has often been discussed in the literature (see e.g. Crompton 1972; Huxley 1972, Section IX, and references therein). This particular example illustrates the applicability of equation (63) in general. In order to apply it to a stream for which $\frac{1}{2}\lambda a'$ is not negligible it would be necessary to evaluate the dominant terms of equations (57) or (58).

The exact representation of the function $K_{k+\frac{1}{2}}(x)/(\pi/2x)^{\frac{1}{2}}$ is given by

$$\frac{K_{k+\frac{1}{2}}(x)}{(\pi/2x)^{\frac{1}{2}}} = \exp(-x)\sum_{l=0}^{k}\frac{(k+l)!}{l!\,(k-l)!}\frac{1}{(2x)^{l}},$$

and consequently with large values of the argument x the functions $K_{k+\frac{1}{2}}(x)$ all approach the form $K_{\frac{1}{2}}(x)$, especially when k is not large. Thus when $\lambda_L r'$ is large the Bessel functions in the leading terms of equations (57) and (58) all approach the form $K_{\frac{1}{2}}(\lambda_L r')$. Moreover, when $b/h \ll 1$, $\cos\theta \approx 1$ over the central disc and we conclude that when $\lambda_L h$ is large and b/h is small the structure of the stream approaches that from a pole point source even when $\frac{1}{2}\lambda_L a'$, although small, is not negligible. In these circumstances the formula for the ratio R is

$$R = 1-(h/d')\exp\{-\lambda_L(d'-h)\}. \tag{65}$$

Typical magnitudes of the parameters are $h = 10$ cm, $\lambda_L \approx 10\ \text{cm}^{-1}$, $b = 0\cdot 5$ cm, and consequently $\lambda_L h$ $(< \lambda_L r') \approx 100$; also over the disc $\cos\theta \approx 1-\frac{1}{2}(\frac{1}{20})^2 = 1-\frac{1}{800} \approx 1$. The usual size of the radius of the aperture is $a = 0\cdot 05$ cm and consequently $\frac{1}{2}\lambda_L a' \leqslant \frac{1}{4}$, which is small enough to suppress the higher order terms in equations (57) and (58). It is found in practice that equation (65) accurately describes the measured values of R when the experimental parameters have magnitudes of the above order. The ratio R is then independent of the size of the source, as is found to be the case. At smaller values of h, the variation of $\cos\theta$ across the disc becomes significant but, provided $\frac{1}{2}\lambda_L a'$ is sufficiently small to suppress all terms in equation (57) following the first, the structure of the stream is that from a dipole point source and the ratio R is given by equation (64). At smaller distances and with apertures such that $\frac{1}{2}\lambda_L a'$ is not negligible, the ratio R would not be given accurately by formulae (64) or (65). The present analysis permits the values of R to be calculated in these less restricted circumstances.

The ratio R is the important experimental quantity since from it the value of λ_L, or at greater distances $\lambda = W/2D$ (see e.g. Huxley 1972), can be derived. The related quantity D/μ, where μ is the mobility W/E, is an important physical magnitude in the study of the motion of electrons in gases.

X. Dependence of ν upon $(\lambda_L a')$

It was remarked in Section IV that in the limiting case where $\lambda_L \to 0$, so that equation (5) assumes the form $\nabla'^2 V = 0$, the problem reduces to the corresponding electrostatic problem. The distribution of number density across the source aperture is then given by equation (38), that is

$$n(q') = V(q') = n_0(1-q'^2/a'^2)^{\frac{1}{2}},$$

where $z = 0$ and $0 \leqslant q' \leqslant a'$. Nevertheless the investigation was carried out intentionally in a general manner by replacing the specific index $\frac{1}{2}$ by a general symbol ν. It can be seen, however, that the same value $\nu = \frac{1}{2}$ is not correct when λ_L is not small. Small values of $\lambda_L = W/2D_L$ imply that the distribution near the aperture is dominated by diffusion to the cathode through the coefficient D_L rather than by transport through the aperture from the drift W in the electric field; such are the conditions with small gas pressures and weak electric fields. As also diffusion becomes more effective as the radius a of the aperture is reduced, we replace the criterion $\lambda_L \to 0$ by $\lambda_L a' \to 0$. At the other extreme where $\lambda_L \to \infty$, loss of electrons from the stream through diffusion to the cathode is negligible in comparison with the number transported through the aperture by the drift W. Moreover when λ_L is finite the proportional loss by diffusion becomes less as a is increased. The criterion for small loss is therefore $\lambda_L a' \to \infty$. When $\lambda_L = W/2D_L$ approaches large values it follows that equation (4) approaches the form $\partial n/\partial z \to 0$, or $n \to$ const. In these circumstances the distribution of n across the aperture is constant, a behaviour which is consistent with the distribution $n = n_0(1-q'^2/a'^2)^\nu$ if $\nu \to 0$ as $\lambda_L \to \infty$.

In view of the above considerations it is evident that to adopt a fixed value of ν whatever the value of $\lambda_L a'$ does not give a faithful description of the properties of the stream. Since the change in ν is only from $\frac{1}{2}$ to zero as $\lambda_L a'$ varies from zero to infinity, we adopt the following empirical formula for the distribution of n across the aperture for any value of $\lambda_L a'$:

$$\left.\begin{aligned} n &= n_0(1-q'^2/a'^2)^\nu, \quad 0 \leqslant q \leqslant a; \qquad \nu = (2+\lambda_L a')^{-1}; \\ &= 0, \qquad\qquad\qquad\qquad a < q. \end{aligned}\right\} \tag{66}$$

We now consider the matter in more specific terms. Equation (43) is equivalent to

$$n(\rho', z) = n_0\, a'(2/a')^\nu\, \Gamma(\nu+1) \exp(\lambda_L z)$$

$$\times \int_0^\infty \exp\{-z(v^2+\lambda_L^2)^{\frac{1}{2}}\}\, J_0(\rho' v)\, \{J_{\nu+1}(a'v)/v^\nu\}\, dv. \tag{67}$$

As $\lambda_L \to \infty$ the factor $\exp\{-z(v^2+\lambda_L^2)^{\frac{1}{2}}\}$ in the integrand approaches the form $\exp(-\lambda_L z)$ and is cancelled by the factor $\exp(\lambda_L z)$ that prefixes the integral. Equation (67) therefore approaches the form, since $\nu \to 0$,

$$n(\rho', z) = n_0\, a' \int_0^\infty J_0(\rho' v)\, J_1(a'v)\, dv, \qquad \lambda_L \to \infty,$$

$$\left.\begin{aligned} &= n_0\, a' \times 1/a' = n_0, \qquad 0 \leqslant \rho \leqslant a, \\ &= 0, \qquad\qquad\qquad\qquad a < \rho. \end{aligned}\right\} \tag{68}$$

The stream is a nondiffusing column with constant radius a and number density n_0.

In general when $\lambda_{\mathrm{L}} a'$ is finite so that $0 < \nu \leqslant \frac{1}{2}$, equation (67) gives for the distribution of n on the plane $z = 0$, where $q' \equiv \rho'$,

$$n(q',z) = n_0(a'^2/2)(2/a')^{\nu+1}\,\Gamma(\nu+1)\,I, \tag{69}$$

where

$$I = \int_0^\infty J_0(q'v)\{J_{\nu+1}(a'v)/v^\nu\}\,\mathrm{d}v\,.$$

The integral I is a special case of the class of discontinuous integrals of Weber and Shafheitlin (Watson 1944, Section 13.4; Magnus and Oberhettinger 1949, p. 35; Abramowitz and Stegun 1965, p. 487), which are of the form

$$J = \int_0^\infty J_\mu(q'v)\{J_\theta(a'v)/v^\gamma\}\,\mathrm{d}v\,.$$

In particular, it can be deduced from the general expressions for these integrals J that

$$I = \frac{(\frac{1}{2}a')^\nu}{a'\,\Gamma(\nu+1)}\,{}_2\mathrm{F}_1(-\nu,1;1;q'^2/a'^2) = \frac{(\frac{1}{2}a')^\nu(1-q'^2/a'^2)^\nu}{a'\,\Gamma(\nu+1)}$$

when $0 \leqslant q' < a'$, but that $I = 0$ when $a' \leqslant q'$. Thus from equation (67) when $z = 0$, $n = n_0(1-q'^2/a'^2)^\nu$ across the aperture but is zero on the cathode where $q > a$. This is the required behaviour.

The appropriate value of $\nu = \lambda_{\mathrm{L}} a' = (\lambda\lambda_{\mathrm{L}})^{\frac{1}{2}} a$ is found from the experimental values of λ and λ_{L}. When the momentum transfer cross section $q_{\mathrm{m}}(c)$ is, in effect, independent of the speed c of the electrons then $\lambda_{\mathrm{L}} \approx 2\lambda$ where $\frac{1}{2}m\langle c^2\rangle$ greatly exceeds $3/2\kappa T$, the mean energy of a molecule. In that event $\nu \approx \sqrt{2}\,\lambda a$. From the above discussion it is a good approximation to the truth to suppose that n is constant across the aperture when $\sqrt{2}\,\lambda a \gg 2$, that is, $\lambda a \gg \sqrt{2}$. In practice $a = 0{\cdot}05$ cm and consequently we require $\lambda \gg 28$. To meet this condition requires, in diatomic gases, the use of gas pressures of several tens of torr and, in monatomic gases, hundreds of torr. Thus in general it must be assumed that n is far from constant across the aperture.

XI. References

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Appendix

Derivation of Expression for $\partial^n\{K_{\frac{1}{2}}(\lambda_L r')/(\lambda_L r')^{\frac{1}{2}}\}/\partial z^n$

Let

$$I = \int_0^\infty \exp\{-z(v^2+\lambda_L^2)^{\frac{1}{2}}\} J_0(\rho' v)\{\phi((v^2+\lambda_L^2)^{\frac{1}{2}})/(v^2+\lambda_L^2)^{\frac{1}{2}}\} v \, dv$$

and suppose that $\phi((v^2+\lambda_L^2)^{\frac{1}{2}})$ is the convergent series

$$\phi((v^2+\lambda_L^2)^{\frac{1}{2}}) = \sum_{n=0}^{\infty} A_n (v^2+\lambda_L^2)^{\frac{1}{2}n}.$$

It then follows from equation (15) that

$$I = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \lambda_L \phi \frac{\partial}{\partial z}\left(-\frac{K_{\frac{1}{2}}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}}\right) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \lambda_L \left\{\sum_{n=0}^{\infty} (-1)^n A_n \frac{\partial^n}{\partial z^n}\left(\frac{K_{\frac{1}{2}}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}}\right)\right\}. \tag{A1}$$

Let

$$\phi((v^2+\lambda_L^2)^{\frac{1}{2}}) = \exp\{a(v^2+\lambda_L^2)^{\frac{1}{2}}\} = \sum_{n=0}^{\infty} a^n (v^2+\lambda_L^2)^{\frac{1}{2}n}/n!,$$

which gives, for $a < z$,

$$I = \int_0^\infty \frac{\exp\{-(z-a)(v^2+\lambda_L^2)^{\frac{1}{2}}\}}{(v^2+\lambda_L^2)^{\frac{1}{2}}} J_0(\rho' v) v \, dv = \frac{\exp[-\lambda_L\{(z-a)^2+\rho'^2\}^{\frac{1}{2}}]}{\{(z-a)^2+\rho'^2\}^{\frac{1}{2}}}. \tag{A2}$$

Alternatively,

$$I = \int_0^\infty \left(\frac{\exp\{-z(v^2+\lambda_L^2)^{\frac{1}{2}}\}}{(v^2+\lambda_L^2)^{\frac{1}{2}}} J_0(\rho' v) \sum_{n=0}^{\infty} a^n (v^2+\lambda_L^2)^{\frac{1}{2}n}/n!\right) v \, dv$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \lambda_L \sum_{n=0}^{\infty} (-1)^n \frac{a^n}{n!} \frac{\partial^n}{\partial z^n}\left(\frac{K_{\frac{1}{2}}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}}\right). \tag{A3}$$

However, in equation (A2)

$$(z-a)^2+\rho'^2 = (z^2+\rho'^2)-2az+a^2 = r'^2-2ar'\cos\theta+a^2,$$

where $r' = (z^2+\rho'^2)^{\frac{1}{2}}$ and $\cos\theta = z/r'$. Now (Watson 1944, Section 11.41, equation (11))

$$\frac{\exp\{-\lambda_L(r'^2+a^2-2ar'\cos\theta)^{\frac{1}{2}}\}}{(r'^2+a^2-2ar'\cos\theta)^{\frac{1}{2}}} = \sum_{m=0}^{\infty} (2m+1) \frac{K_{m+\frac{1}{2}}(\lambda_L r')}{r'^{\frac{1}{2}}} \frac{I_{m+\frac{1}{2}}(\lambda_L a)}{a^{\frac{1}{2}}} P_m(\cos\theta), \tag{A4}$$

and consequently

$$\sum_{m=0}^{\infty} \lambda_L(2m+1) \frac{K_{m+\frac{1}{2}}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}} \frac{I_{m+\frac{1}{2}}(\lambda_L a)}{(\lambda_L a)^{\frac{1}{2}}} P_m(\cos\theta) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \lambda_L \sum_{n=0}^{\infty} (-1)^n \frac{a^n}{n!} \frac{\partial^n}{\partial z^n}\left(\frac{K_{\frac{1}{2}}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}}\right). \tag{A5}$$

Consider the expansion

$$I_{s+\frac{1}{2}}(\lambda_L a)/(\lambda_L a)^{\frac{1}{2}} = \sum_{p=0}^{\infty} (\tfrac{1}{2}\lambda_L a)^{s+p}/2^{\frac{1}{2}}\Gamma(s+p+\tfrac{3}{2})\,p!\,.$$

Let $s+2p = n$. When $n = 2m$ is an even number it is found that the coefficient of $a^n = a^{2m}$ on the left-hand side of equation (A5) is

$$(\tfrac{1}{2}\lambda_L)^{2m} \sum_{p=0}^{m} \frac{4(m-p)+1}{p!\,\Gamma(2m-p+\frac{3}{2})} \frac{K_{2(m-p)+1/2}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}} P_{2(m-p)}(\cos\theta)$$

and when n is an odd integer $(2m+1)$ the coefficient of a^{2m+1} is

$$(\tfrac{1}{2}\lambda_L)^{2m+1} \sum_{p=0}^{m} \frac{4(m-p)+3}{p!\,\Gamma(2m-p+\frac{5}{2})} \frac{K_{2(m-p)+3/2}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}} P_{2(m-p)+1}(\cos\theta)\,.$$

When the coefficients of a^n on each side of equation (A5) are equated we derive the two formulae

$$\frac{\partial^{2m}}{\partial z^{2m}}\left(\frac{K_{\frac{1}{2}}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}}\right) = (2m)!\,(\tfrac{1}{2}\lambda_L)^{2m}\,\Gamma(\tfrac{3}{2})$$

$$\times \sum_{p=0}^{m} \frac{4(m-p)+1}{p!\,\Gamma(2m-p+\frac{3}{2})} \frac{K_{2(m-p)+1/2}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}} P_{2(m-p)}(\cos\theta) \qquad \text{(A6)}$$

for $n = 2m$ and

$$\frac{\partial^{2m+1}}{\partial z^{2m+1}}\left(\frac{K_{\frac{1}{2}}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}}\right) = -(2m+1)!\,(\tfrac{1}{2}\lambda_L)^{2m+1}\,\Gamma(\tfrac{3}{2})$$

$$\times \sum_{p=0}^{m} \frac{4(m-p)+3}{p!\,\Gamma(2m-p+\frac{5}{2})} \frac{K_{2(m-p)+3/2}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}} P_{2(m-p)+1}(\cos\theta) \qquad \text{(A7)}$$

for $n = 2m+1$. These are the formulae sought.

If $m-p = \mu$ it follows from equation (A6) that when $n = 2m$ the coefficient $A_{2\mu}$ of $\{K_{2\mu+1/2}(\lambda_L r')/(\lambda_L r')^{\frac{1}{2}}\}P_{2\mu}(\cos\theta)$ in the expansion of $\partial^{2m}\{K_{\frac{1}{2}}(\lambda_L r')/(\lambda_L r')^{\frac{1}{2}}\}/\partial z^{2m}$ is

$$A_{2\mu} = (2m)!\,(\tfrac{1}{2}\lambda_L)^{2m}\,\Gamma(\tfrac{3}{2})\,(4\mu+1)/(m-\mu)!\,\Gamma(m+\mu+\tfrac{3}{2}), \qquad \text{(A8)}$$

and similarly from (A7) that the coefficient $A_{2\mu+1}$ of $\{K_{2\mu+3/2}(\lambda_L r')/(\lambda_L r')^{\frac{1}{2}}\}P_{2\mu+1}(\cos\theta)$ in the expansion of $\partial^{2m+1}\{K_{\frac{1}{2}}(\lambda_L r')/(\lambda_L r')^{\frac{1}{2}}\}/\partial z^{2m+1}$ is

$$A_{2\mu+1} = -(2m+1)!\,(\tfrac{1}{2}\lambda_L)^{2m+1}\,\Gamma(\tfrac{3}{2})\,(4\mu+3)/(m-\mu)!\,\Gamma(m+\mu+\tfrac{5}{2}) \qquad \text{(A9)}$$

for $\mu \leqslant m$.

Simple special cases of equations (A6) and (A7) are, with

$$F(\lambda_L r') = K_{\frac{1}{2}}(\lambda_L r')/(\lambda_L r')^{\frac{1}{2}},$$

$$-\frac{\partial F}{\partial z} = \lambda_L \frac{K_{3/2}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}} P_1(\cos\theta), \tag{A10a}$$

$$\frac{\partial^2 F}{\partial z^2} = \tfrac{1}{3}\lambda_L^2\left(\frac{K_{1/2}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}} + \frac{2K_{5/2}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}} P_2(\cos\theta)\right), \tag{A10b}$$

$$-\frac{\partial^3 F}{\partial z^3} = \tfrac{1}{5}\lambda_L^3\left(\frac{3K_{3/2}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}} P_1(\cos\theta) + \frac{2K_{7/2}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}} P_3(\cos\theta)\right), \tag{A10c}$$

$$\frac{\partial^4 F}{\partial z^4} = \lambda_L^4\left(\frac{K_{1/2}(\lambda_L r')}{5(\lambda_L r')^{\frac{1}{2}}} + \frac{4K_{5/2}(\lambda_L r')}{7(\lambda_L r')^{\frac{1}{2}}} P_2(\cos\theta) + \frac{8K_{9/2}(\lambda_L r')}{35(\lambda_L r')^{\frac{1}{2}}} P_4(\cos\theta)\right). \tag{A10d}$$