THE VLASOV–MAXWELL OPERATORS IN SELF-ADJOINT FORM

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Abstract

A systematic method for obtaining the scalar product with respect to which a given Vlasov–Maxwell operator is self-adjoint is illustrated by considering the operator introduced by Shure (1964) to describe one-dimensional longitudinal oscillations in a bounded plasma. By separation of variables the problem is reduced to the solution of a singular integral equation for the weight function of the scalar product. The solution of this equation is not explicit but is in the form of a simple linear Fredholm integral equation of the second kind which is easily solved. The method should be applicable to similar operators in this field.

I. Introduction

Lanczos (1961), for example, showed that any second-order linear differential operator can be regarded as self-adjoint provided the weight function in the scalar product is chosen appropriately. In the present paper consideration is given to the possibility of finding scalar products with respect to which the linear integro-differential Vlasov–Maxwell operators which occur in plasma physics can be regarded as self-adjoint, and it is shown that such a scalar product can be found for a particular case. The method can be generalized to deal with other operators in this field.

The problem can be stated as follows. Instead of starting with an arbitrary scalar product $\langle\, , \rangle$, usually with weight function unity, and finding the adjoint operator $L^\dagger$ with the same spectrum as the original operator L such that

$$\langle \psi^\dagger_{\nu'}, L\psi_\nu \rangle = \langle L^\dagger \psi^\dagger_{\nu'}, \psi_\nu \rangle\,, \tag{1}$$

where $\psi^\dagger_{\nu'}$ and ψ_ν are the eigenfunctions of $L^\dagger$ and L respectively, we would like instead to find, for the given operator, the scalar product $(\, ,)$ for which

$$(\psi_{\nu'}, L\psi_\nu) = (L\psi_{\nu'}, \psi_\nu)\,. \tag{2}$$

Using such a scalar product with respect to which the operator is self-adjoint avoids the introduction of the adjoint operator and the derivation of the adjoint eigenfunctions. Also the derivation of the scalar product is constructive as it leads immediately to a weighted orthogonality property of the eigenfunctions of the given operator, which allows the expansion coefficients in an eigenfunction expansion of an arbitrary vector to be obtained.

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Work along the above lines has been done in neutron transport theory by Zweifel (1967) and Kaper (1969). Cercignani (1969) has studied operators in the kinetic theory of gases while Case (1965) has discussed the analogue of the spectral theorem for transport operators in plasma physics when considering constants of the motion. In plasma physics the situation is slightly more complex than in neutron transport theory since the operator usually includes, besides a transport equation, Maxwell's equations for the electric and magnetic fields, and thus the scalar product must also contain additional terms due to the presence of these fields.

II. Physical Model and Mathematical Formulation

The general procedure to be followed in plasma physics can be conveniently described by considering a particular problem. Perhaps the simplest boundary value problem in this topic is the one-dimensional problem of penetration of an electric field into a plasma. This has been considered by many authors (e.g. Aamodt and Case 1963; Shure 1964; Case 1967) and the same physical model as described in those papers will be used here. The perturbation $f(x,u,t)$ in the electron distribution function $n_0 f_0$ and the electric field $E(x,t)$ are given by the linearized equations

$$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+\frac{n_0 e E(x,t)}{m}\frac{\mathrm{d}F}{\mathrm{d}u}=0 \tag{3}$$

and

$$\partial E/\partial x = 4\pi e\int_{-\infty}^{\infty} f(x,u,t)\,\mathrm{d}u\,, \tag{4}$$

where e and m are the electronic charge and mass respectively and

$$F(u^2)=\int_{-\infty}^{\infty}\mathrm{d}v_y\int_{-\infty}^{\infty} f_0(u^2+v_y^2+v_z^2)\,\mathrm{d}v_z\,. \tag{5}$$

The electron velocity v_x in the x direction has been denoted by u to simplify the notation and the equilibrium distribution function $n_0 f_0$ is assumed to be isotropic in velocity space, i.e. to be a function of the square of the velocity. Also, for simplicity, f_0 is assumed to be nonzero for finite velocities and to tend to zero "sufficiently rapidly" as the velocity tends to infinity.

Shure (1964) has shown, using a slightly different notation, that with time dependence $\exp(-\mathrm{i}\omega t)$ and spatial dependence $\exp(\mathrm{i}\omega z/v)$ equations (3) and (4) can be rewritten in the operator form

$$L\psi_v = (\mathrm{i}\omega/v)\rho\,\psi_v\,, \tag{6}$$

where

$$L=\begin{bmatrix}\mathrm{i}\omega & -(n_0 e/m)\,\mathrm{d}F/\mathrm{d}u\\ 4\pi e\int_{-\infty}^{\infty}\mathrm{d}u & 0\end{bmatrix},\quad \rho=\begin{bmatrix}u & 0\\ 0 & 1\end{bmatrix},\quad \psi_v=\begin{bmatrix}f(v,u,\omega)\\ E(v,\omega)\end{bmatrix}. \tag{7}$$

Although (6) is not exactly in the form of an eigenvalue equation, ψ_v will be referred to as the eigenfunction of L corresponding to the eigenvalue v. The spectrum and eigenfunctions of L have been derived by Shure (1964) and Case (1967) and are listed in the Appendix, together with a definition of the function $\Lambda(z,\omega)$ and its properties.

Two separate problems are considered. When the boundary condition on the distribution function is other than the specular reflection boundary condition, a superposition of eigenfunctions with eigenvalues in the set $\mathscr{S}^+ = \{[0,\infty], -v_0\}$ is usually considered. To deal with such superpositions of eigenfunctions, half-range orthogonality and completeness properties have to be established. We therefore try to find a scalar product $(\ ,\ )_\mathrm{h}$ of the form

$$(\psi_{v'},\psi_v)_\mathrm{h} = \int_0^\infty W(u)\,f(v',u,\omega)\,f(v,u,\omega)\,\mathrm{d}u + w_0\,E(v')\,E(v) \tag{8}$$

such that

$$(\psi_{v'},L\psi_v)_\mathrm{h} = (L\psi_{v'},\psi_v)_\mathrm{h} \qquad \text{for all} \qquad v',v\in\mathscr{S}^+. \tag{9}$$

Alternatively when the boundary condition on the distribution function is the specular reflection boundary condition, the problem can usually be solved by considering a superposition of eigenfunctions with eigenvalues in the set $\mathscr{S} = \{[-\infty,\infty], -v_0, +v_0\}$ and, in order to deal with such superpositions, full-range orthogonality and completeness properties have to be established. Therefore the second problem considered is to find a scalar product $(\ ,\ )_\mathrm{f}$ of the form

$$(\psi_{v'},\psi_v)_\mathrm{f} = \int_{-\infty}^\infty W(u)\,f(v',u,\omega)\,f(v,u,\omega)\,\mathrm{d}u + w_0\,E(v')\,E(v) \tag{10}$$

such that

$$(\psi_{v'},L\psi_v)_\mathrm{f} = (L\psi_{v'},\psi_v)_\mathrm{f} \qquad \text{for all} \qquad v',\,v\in\mathscr{S}. \tag{11}$$

Now the expansion coefficients in an eigenfunction expansion depend only on the ratio of scalar products (see equation (65) in Section IV below). Thus in each case the scalar product need only be determined to within an arbitrary constant. The coefficient w_0 of $E(v')\,E(v)$ can therefore be taken as unity, and this will be done here.

Once the scalar products have been obtained, orthogonality properties immediately follow. Omitting the subscripts on the scalar products, we have in each case from equation (6)

$$(\psi_{v'},L\psi_v) = (\mathrm{i}\omega/v)(\psi_{v'},\rho\psi_v) \tag{12}$$

and

$$(L\psi_{v'},\psi_v) = (\mathrm{i}\omega/v')(\rho\psi_{v'},\psi_v) = (\mathrm{i}\omega/v')(\psi_{v'},\rho\psi_v), \tag{13}$$

since ρ is diagonal. Subtraction of (13) from (12) then gives

$$0 = (1/v - 1/v')(\psi_{v'},\rho\psi_v). \tag{14}$$

If the above-mentioned eigenfunction expansions are valid then the expansion coefficients can be obtained immediately from these orthogonality properties.

III. Half-range Orthogonality

In this section, only the first problem formulated in Section II will be considered and the weight function of the scalar product with respect to which L can be regarded as self-adjoint for eigenfunctions with eigenvalues in the set $\mathscr{S}^+$ will now be obtained. Consider first $(\psi_{v'}, L\psi_v)_{\mathrm{h}}$ for $v, v' \in [0, \infty[$. We have by (7) and equation (A1) of the Appendix

$$L\psi_v = \begin{bmatrix} \mathrm{i}\omega\, f(v,u,\omega) - (v\omega_{\mathrm{p}}^2/\mathrm{i}\omega)\,\mathrm{d}F/\mathrm{d}u \\ 4\pi e \int_{-\infty}^{\infty} f(v,u,\omega)\,\mathrm{d}u \end{bmatrix} = \begin{bmatrix} \mathrm{i}\omega\, f(v,u,\omega) - (v\omega_{\mathrm{p}}^2/\mathrm{i}\omega)\,\mathrm{d}F/\mathrm{d}u \\ 4\pi e \end{bmatrix}, \tag{15}$$

in which the normalization condition (A2) has been used. Thus, using the definition (8) of the scalar product $(\, , \,)_{\mathrm{h}}$, we get

$$\begin{aligned}(\psi_{v'}, L\psi_v)_{\mathrm{h}} &= \mathrm{i}\omega \int_0^{\infty} W(u)\, f(v',u,\omega)\, f(v,u,\omega)\,\mathrm{d}u \\ &\quad - \frac{\omega_{\mathrm{p}}^2 v}{\mathrm{i}\omega} \int_0^{\infty} W(u)\,\frac{\mathrm{d}F}{\mathrm{d}u}\, f(v',u,\omega)\,\mathrm{d}u + \frac{(4\pi e)^2 v'}{\mathrm{i}\omega}\end{aligned} \tag{16}$$

and

$$\begin{aligned}(L\psi_{v'}, \psi_v)_{\mathrm{h}} &= \mathrm{i}\omega \int_0^{\infty} W(u)\, f(v',u,\omega)\, f(v,u,\omega)\,\mathrm{d}u \\ &\quad - \frac{\omega_{\mathrm{p}}^2 v'}{\mathrm{i}\omega} \int_0^{\infty} W(u)\,\frac{\mathrm{d}F}{\mathrm{d}u}\, f(v,u,\omega)\,\mathrm{d}u + \frac{(4\pi e)^2 v}{\mathrm{i}\omega}.\end{aligned} \tag{17}$$

Suppose now that both v and v' are nonzero. Then subtracting (17) from (16), rearranging terms, and dividing through by $\mathrm{i}\omega v v'$ gives

$$\frac{\sigma^2}{v'} \int_0^{\infty} W(u)\,\frac{\mathrm{d}F}{\mathrm{d}u}\, f(v',u,\omega)\,\mathrm{d}u - \frac{\beta}{v'} = \frac{\sigma^2}{v} \int_0^{\infty} W(u)\,\frac{\mathrm{d}F}{\mathrm{d}u}\, f(v,u,\omega)\,\mathrm{d}u - \frac{\beta}{v}, \tag{18}$$

where $\beta = (4\pi e/\mathrm{i}\omega)^2$. The left-hand side of equation (18) is a function of v' only while the right-hand side is a function of v only. If (18) is to hold for all values of v and v' belonging to $]0, \infty[$ then each side must equal a constant, α say. Thus $W(u)$ must satisfy, for $v \in]0, \infty[$,

$$\sigma^2 \int_0^{\infty} W(u)(\mathrm{d}F/\mathrm{d}u)\, f(v,u,\omega)\,\mathrm{d}u - \beta = \alpha v. \tag{19}$$

Also, $W(u)$ must satisfy equation (19) for $v = 0$. To see this we put $v = 0$ in equations (16) and (17), equate, cancel like terms, and finally obtain

$$v'\left(\sigma^2 \int_0^{\infty} W(u)(\mathrm{d}F/\mathrm{d}u)\, f(0,u,\omega)\,\mathrm{d}u - \beta\right) = 0. \tag{20}$$

For this equation to hold for all admissible values of v', equation (19) with $v = 0$ must be satisfied as claimed. Thus using equation (A1) for $f(v,u,\omega)$, we obtain the singular integral equation for $W(u)$

$$\sigma^2(\mathrm{d}F/\mathrm{d}u)\,\lambda(v,\omega)\,W(v) + \sigma^4 v^2 \int_0^\infty \frac{W(u)(\mathrm{d}F/\mathrm{d}u)^2\,\mathrm{d}u}{u-v} = \alpha v + \beta, \qquad 0 \leqslant v < \infty. \tag{21}$$

Now equation (9) must hold for $v' = \infty$, $0 \leqslant v < \infty$ and $v = \infty$, $0 \leqslant v' < \infty$, while using equations (7), and (A8) for ψ_∞, we have

$$L\psi_\infty = \begin{bmatrix} 0 \\ 4\pi e \int_{-\infty}^{\infty} (\mathrm{d}F/\mathrm{d}u)\,\mathrm{d}u \end{bmatrix} = 0, \tag{22}$$

as F vanishes at infinity. Thus the condition which $W(u)$ must satisfy in this case is

$$(L\psi_{v'}, \psi_\infty)_{\mathrm{h}} = 0. \tag{23}$$

As $(L\psi_v, \psi_\infty)_{\mathrm{h}} \equiv (\psi_\infty, L\psi_v)_{\mathrm{h}}$ this is the only condition.

Using (A8) for ψ_∞, equation (23) becomes

$$\sigma^2 \int_0^\infty W(u)(\mathrm{d}F/\mathrm{d}u)\,f(v,u,\omega)\,\mathrm{d}u - \beta = -v\sigma^4 \int_0^\infty W(u)(\mathrm{d}F/\mathrm{d}u)^2\,\mathrm{d}u. \tag{24}$$

Comparison of equations (24) and (19) shows that we must take

$$\alpha = -\sigma^4 \int_0^\infty W(u)(\mathrm{d}F/\mathrm{d}u)^2\,\mathrm{d}u. \tag{25}$$

Now, as the discrete eigenvalues $\pm v_0$ are absent from the spectrum for $\omega^2 > \omega_{\mathrm{p}}^2$, in this case $\mathscr{S}^+ = [0, \infty]$. Therefore there are no more conditions to be satisfied when $\omega^2 > \omega_{\mathrm{p}}^2$ so that the required weight function then is obtained by solving the singular integral equation (21) with α given by (25).

When $\omega^2 < \omega_{\mathrm{p}}^2$, however, the discrete eigenvalues $\pm v_0$ exist. As $-v_0$ is contained in $\mathscr{S}^+$, the following two further conditions must be satisfied

$$(\psi_v, L\psi_{-v_0})_{\mathrm{h}} = (L\psi_v, \psi_{-v_0})_{\mathrm{h}}, \qquad v \in \mathscr{S}^+, \tag{26a}$$

$$(L\psi_{-v_0}, \psi_\infty)_{\mathrm{h}} = 0. \tag{26b}$$

Condition (26a) gives

$$\sigma^2 \int_0^\infty W(u)\frac{\mathrm{d}F}{\mathrm{d}u} f(v,u,\omega)\,\mathrm{d}u - \beta = -\frac{v}{v_0}\left(\sigma^2 \int_0^\infty W(u)\frac{\mathrm{d}F}{\mathrm{d}u} f(-v_0,u,\omega)\,\mathrm{d}u - \beta\right). \tag{27}$$

Using equation (A9) for $f(-v_0,u,\omega)$ and comparing (27) with (19) gives

$$\alpha = -v_0^{-1}\left(\sigma^4 v_0^2 \int_0^\infty \frac{W(u)(\mathrm{d}F/\mathrm{d}u)^2\,\mathrm{d}u}{u+v_0} - \beta\right). \tag{28}$$

Consider now the condition (26b). Using equations (7) for L, (A9) for ψ_{-v_0}, and (A8) for ψ_∞, we obtain for this condition

$$-v_0^{-1}\left(\sigma^4 v_0^2 \int_0^\infty \frac{W(u)(\mathrm{d}F/\mathrm{d}u)\,\mathrm{d}u}{u+v_0} - \beta\sigma^2 v_0^2 \int_{-\infty}^\infty \frac{(\mathrm{d}F/\mathrm{d}u)\,\mathrm{d}u}{u+v_0}\right)$$
$$= -\sigma^4 \int_0^\infty W(u)(\mathrm{d}F/\mathrm{d}u)^2\,\mathrm{d}u\,. \tag{29}$$

Since $\Lambda(-v_0) = 0$, we have (see equation (A4))

$$\sigma^2 v_0^2 \int_{-\infty}^\infty \frac{(\mathrm{d}F/\mathrm{d}u)\,\mathrm{d}u}{u+v_0} = 1\,. \tag{30}$$

Putting equation (30) into (29) merely gives us the condition we would obtain by equating the right-hand sides of (25) and (28), and we can conclude that (26b) therefore gives us no new condition on $W(u)$. Thus for $\omega^2 < \omega_p^2$ the required weight function is obtained by solving the singular integral equation (21) together with the relations (25) and (28). That both (25) and (28) are necessary is seen in subsection (ii) below.

The two cases $\omega^2 > \omega_p^2$ and $\omega^2 < \omega_p^2$ have to be treated separately.

(i) $\omega^2 > \omega_p^2$

Let

$$\Phi_1(z) = \frac{1}{2\pi\mathrm{i}} \int_0^\infty \frac{W(u)(\mathrm{d}F/\mathrm{d}u)^2\,\mathrm{d}u}{u-z}. \tag{31}$$

The Plemelj formulae (Muskhelishvili 1953) give for $0 \leqslant v < \infty$

$$W(v)(\mathrm{d}F/\mathrm{d}u)^2 = \Phi_1^+(v) - \Phi_1^-(v)\,, \tag{32a}$$

$$\frac{1}{\pi\mathrm{i}} \int_0^\infty \frac{W(u)(\mathrm{d}F/\mathrm{d}u)^2\,\mathrm{d}u}{u-v} = \Phi_1^+(v) + \Phi_1^-(v)\,. \tag{32b}$$

Thus by equation (21), $\Phi_1(z)$ must be a solution which vanishes at infinity of the nonhomogeneous Hilbert boundary problem

$$\Phi_1^+(v) = \left(\frac{\Lambda^+(v)}{\Lambda^-(v)}\right)\Phi_1^-(v) + \frac{(\alpha v+\beta)\,\mathrm{d}F/\mathrm{d}v}{\sigma^2\Lambda^-(v)}, \qquad 0 \leqslant v < \infty\,. \tag{33}$$

Division by $\Lambda^-(v)$ is possible as it is shown to be nonzero in the Appendix. Conversely, Muskhelishvili (1953) shows that any sectionally holomorphic solution of (33) which vanishes at infinity leads to a solution of the original singular integral equation (21). To obtain such a solution of (33), we introduce the fundamental solution $X(z)$ of the corresponding homogeneous Hilbert boundary problem

$$X^+(v) = (\Lambda^+(v)/\Lambda^-(v))X^-(v)\,, \qquad 0 \leqslant v < \infty\,. \tag{34}$$

This is the solution of (34) which is holomorphic in the whole plane except for a cut along the positive real axis and which does not vanish anywhere in the finite plane. If we take logarithms of both sides of (34) and apply the Plemelj formulae we see that (Muskhelishvili 1953)

$$X_0(z) = \exp\left[\frac{1}{2\pi \mathrm{i}}\int_0^\infty \log\left(\frac{\Lambda^+(u)}{\Lambda^-(u)}\right)\frac{\mathrm{d}u}{u-z}\right] \tag{35}$$

is a solution of (34), that branch of the logarithm being chosen which vanishes at infinity. It remains to examine the behaviour of $X_0(z)$ at the origin and this can be done using an argument similar to that of Mason (1972). We have

$$\begin{aligned} X_0(z) &= \exp\left[\frac{1}{2\pi \mathrm{i}}\int_0^\infty \left\{\log\left(\frac{\Lambda^+(u)}{\Lambda^-(u)}\right) - \log\left(\frac{\Lambda^+(0)}{\Lambda^-(0)}\right) + \log\left(\frac{\Lambda^+(0)}{\Lambda^-(0)}\right)\right\}\frac{\mathrm{d}u}{u-z}\right] \\ &= Y(z)\, z^{-(1/2\pi \mathrm{i})\log(\Lambda^+(0)/\Lambda^-(0))}, \end{aligned} \tag{36}$$

where $Y(z)$ takes a definite nonzero value at $z = 0$. We can express this in terms of N, the number of zeros of $\Lambda(z)$ in the upper (or lower) half-plane, in the usual way by considering the continuous change in the argument of $\Lambda(z)$ around a contour consisting of the whole real line from $-\infty$ to $+\infty$ and a semicircle at infinity in the upper half-plane. We have, as there is no change in the argument of $\Lambda(z)$ on this large semicircle,

$$N = (1/2\pi)\Delta_{-\infty,\infty}\arg(\Lambda^+(u)), \tag{37}$$

where $\Delta_{-\infty,\infty}$ denotes the value at $+\infty$ minus the value at $-\infty$, the difference being due to a continuous change along the real axis from $-\infty$ to $+\infty$. Thus

$$\begin{aligned} N &= (1/4\pi)\Delta_{-\infty,\infty}\arg\left(\frac{\Lambda^+(u)}{\Lambda^-(u)}\right) && \text{as} \quad \Lambda^-(u) = \Lambda^+(-u) \\ &= (1/4\pi \mathrm{i})\Delta_{-\infty,\infty}\log\left(\frac{\Lambda^+(u)}{\Lambda^-(u)}\right) && \text{as} \quad \log\left|\frac{\Lambda^+(\infty)}{\Lambda^-(\infty)}\right| = \log\left|\frac{\Lambda^+(-\infty)}{\Lambda^-(-\infty)}\right| \\ &= (1/2\pi \mathrm{i})\Delta_{0,\infty}\log\left(\frac{\Lambda^+(u)}{\Lambda^-(u)}\right) && \text{as} \quad \Lambda^\pm(u) = \Lambda^\mp(-u) \\ &= -(1/2\pi \mathrm{i})\log\left(\frac{\Lambda^+(0)}{\Lambda^-(0)}\right) \end{aligned} \tag{38}$$

as that branch of the logarithm for which $\log(\Lambda^+(\infty)/\Lambda^-(\infty)) = 0$ was chosen. Thus $X_0(z)$ behaves like z^N at the origin. We therefore take as the fundamental solution $z^{-N}X_0(z)$. This function does not vanish anywhere in the finite plane. As $N = 0$ for $\omega^2 > \omega_p^2$, the fundamental solution in this case, which we denote by $X_1(z)$, is just

$$X_1(z) = X_0(z). \tag{39}$$

Now using (34), equation (33) becomes

$$\frac{\Phi_1^+(v)}{X_1^+(v)} = \frac{\Phi_1^-(v)}{X_1^-(v)} + \frac{(\alpha v+\beta)\, dF/dv}{\sigma^2 \Lambda^-(v)\, X_1^+(v)}, \qquad 0 \leqslant v < \infty. \tag{40}$$

Let

$$R_1(z) = \frac{1}{2\pi i \sigma^2}\int_0^\infty \frac{(\alpha u+\beta)(dF/du)\, du}{\Lambda^-(u)\, X_1^+(u)(u-z)}. \tag{41}$$

Applying the Plemelj formula (32a) to $R_1(z)$ allows us to write equation (40) as

$$\Phi_1^+(v)/X_1^+(v) - R_1^+(v) = \Phi_1^-(v)/X_1^-(v) - R_1^-(v), \qquad 0 \leqslant v < \infty. \tag{42}$$

Thus the function

$$\Phi_1(z)/X_1(z) - R_1(z) \tag{43}$$

is holomorphic in the whole plane and vanishes at infinity. By Liouville's theorem, it must be identically zero so that

$$\Phi_1(z) = X_1(z)\, R_1(z). \tag{44}$$

As $X_1(z)$ tends to unity as z tends to infinity, the right-hand side of (44) vanishes at infinity as required. Using equations (A5) and (A6) to express dF/du in terms of $\Lambda^+(u)$ and $\Lambda^-(u)$, and also equation (34), $R_1(z)$ can be rewritten as

$$R_1(z) = \frac{\alpha}{\sigma^4(2\pi i)^2}\int_0^\infty \left(\frac{1}{X_1^+(u)} - \frac{1}{X_1^-(u)}\right)\frac{du}{u(u-z)}$$
$$+ \frac{\beta}{\sigma^4(2\pi i)^2}\int_0^\infty \left(\frac{1}{X_1^+(u)} - \frac{1}{X_1^-(u)}\right)\frac{du}{u^2(u-z)}. \tag{45}$$

The integrals in (45) can be evaluated using contour integration in the complex plane, the contour consisting of a large circle at infinity, a small circle round the origin, and two lines from zero to $+\infty$ joining the two circles, with one line just below and the other just above the positive real axis. The result is

$$R_1(z) = \frac{\alpha}{2\pi i \sigma^4}\left(\frac{1}{z X_1(z)} - \frac{1}{z X_1(0)}\right)$$
$$+ \frac{\beta}{2\pi i \sigma^4}\left(\frac{1}{z^2 X_1(z)} + \frac{X_1'(0)}{z X_1^2(0)} - \frac{1}{z^2 X_1(0)}\right), \tag{46}$$

where the prime denotes differentiation with respect to z. Thus

$$\Phi_1(z) = \frac{\alpha}{2\pi i \sigma^4}\left(\frac{1}{z} - \frac{X_1(z)}{z X_1(0)}\right)$$
$$+ \frac{\beta}{2\pi i \sigma^4}\left(\frac{1}{z^2} + \frac{X_1'(0)\, X_1(z)}{z X_1^2(0)} - \frac{X_1(z)}{z^2 X_1(0)}\right). \tag{47}$$

Applying the Plemelj formula (32a) to equation (47) gives

$$W(u)\left(\frac{\mathrm{d}F}{\mathrm{d}u}\right)^2 = -\frac{\alpha}{2\pi\mathrm{i}\sigma^4}\,\frac{X_1^+(u)-X_1^-(u)}{u\,X_1(0)} + \frac{\beta}{2\pi\mathrm{i}\sigma^4}\left(\frac{X_1'(0)}{u\,X_1^2(0)}-\frac{1}{u^2X_1(0)}\right)\left(X_1^+(u)-X_1^-(u)\right). \tag{48}$$

Now α depends on $W(u)$ through equation (25) so that the relation (48) is a simple linear Fredholm integral equation of the second kind for $W(u)$, which can be solved immediately. If equation (48) is integrated from zero to infinity with respect to u and the integrals involving $X_1(z)$ are evaluated by the contour used to obtain equation (46) then, taking account of the relation (25), we obtain

$$\alpha = \beta\,X_1'(0)/X_1(0). \tag{49}$$

Substitution of (49) back into equation (48) gives

$$W(u)\left(\frac{\mathrm{d}F}{\mathrm{d}u}\right)^2 = -\frac{\beta\{X_1^+(u)-X_1^-(u)\}}{2\pi\mathrm{i}\sigma^4u^2\,X_1(0)}, \tag{50}$$

which can be simplified using equations (34) and (A5) and (A6) to

$$W(u) = -\frac{4\pi m\,X_1^+(u)}{n_0\,X_1(0)\,\Lambda^+(u)\,\mathrm{d}F/\mathrm{d}u}. \tag{51}$$

(ii) $\omega^2 < \omega_\mathrm{p}^2$

At first sight it appears that this case is overdetermined as there are two expressions, (25) and (28), for α. It turns out, however, that both are needed. The analysis proceeds as for $\omega^2 > \omega_\mathrm{p}^2$ and equation (42) (with suffix 1 replaced by 2) is again obtained. This time, however, since $N = 1$ the fundamental solution $X_2(z)$ is given by

$$X_2(z) = z^{-1}\,X_0(z). \tag{52}$$

This solution behaves like z^{-1} at infinity. Thus

$$\lim_{z\to\infty}\left(\frac{\Phi_2(z)}{X_2(z)}-R_2(z)\right) = -\frac{1}{2\pi\mathrm{i}}\int_0^\infty W(u)(\mathrm{d}F/\mathrm{d}u)^2\,\mathrm{d}u = \alpha/2\pi\mathrm{i}\sigma^4 \tag{53}$$

by equation (25).

Consider now the function

$$\Phi_2(z)/X_2(z)-R_2(z)-\alpha/2\pi\mathrm{i}\sigma^4. \tag{54}$$

By equation (42) it is holomorphic in the whole plane and by (53) it vanishes at infinity. Therefore because of Liouville's theorem it must be identically zero. It then follows that

$$\Phi_2(z) = X_2(z)\{R_2(z)+\alpha/2\pi\mathrm{i}\sigma^4\}. \tag{55}$$

The right-hand side of this expression vanishes at infinity and so leads to a solution of the original singular integral equation (21). Evaluating $R_2(z)$ in the same way as for $R_1(z)$, we get

$$R_2(z) = \frac{\alpha}{2\pi i\sigma^4}\left(\frac{1}{z\,X_2(z)} - \frac{1}{z\,X_2(0)} - 1\right) + \frac{\beta}{2\pi i\sigma^4}\left(\frac{1}{z^2\,X_2(z)} + \frac{X_2'(0)}{z\,X_2^2(0)} - \frac{1}{z^2\,X_2(0)}\right). \tag{56}$$

Substitution of equation (56) into (55) and use of the Plemelj formula (32a) then gives

$$W(u)\left(\frac{dF}{du}\right)^2 = -\frac{\alpha}{2\pi i\sigma^4}\,\frac{X_2^+(u)-X_2^-(u)}{u\,X_2(0)} + \frac{\beta}{2\pi i\sigma^4}\left(\frac{X_2'(0)}{u\,X_2^2(0)} - \frac{1}{u^2\,X_2(0)}\right)\left(X_2^+(u)-X_2^-(u)\right). \tag{57}$$

Equation (57) is again a linear Fredholm integral equation of the second kind for $W(u)$. If we try to solve it as before by integrating from zero to infinity with respect to u, merely an identity is obtained and we must therefore make use of the remaining condition (28). Operating on both sides of equation (57) by

$$\int_0^\infty (u+u_0)^{-1}\,du$$

and using the condition (28) leads to the expression

$$\alpha = \beta\{v_0^{-1} + X_2'(0)/X_2(0)\}. \tag{58}$$

Substitution of (58) into (57) and simplification as before gives the result

$$W(u) = -\frac{4\pi m(u+v_0)X_2^+(u)}{n_0\,v_0\,X_2(0)\,\Lambda^+(u)\,dF/du}. \tag{59}$$

Thus both expressions (25) and (28) have been required, the first to determine the behaviour of the unknown function at infinity and the second to solve the final integral equation.

To summarize, the weight function for the inner product (8) is given by equation (51) for $\omega^2 > \omega_p^2$ and by equation (59) for $\omega^2 < \omega_p^2$, the constant w_0 being taken as unity in each case.

IV. Half-range Completeness

We now consider briefly how the results derived above can be used constructively. The orthogonality relations (14) can be written for $v', v \in \mathscr{S}^+$ as

$$(\psi_{v'}, \rho\psi_v)_h = N(v)\,\delta(v-v'). \tag{60}$$

For the scalar product derived in the previous section it can be shown that for both $\omega^2 > \omega_p^2$ and $\omega^2 < \omega_p^2$

$$N(\nu) = (\psi_\nu, \rho\psi_\nu)_h = \nu\, W(\nu)\, \Lambda^+(\nu)\, \Lambda^-(\nu), \qquad 0 \leqslant \nu < \infty \tag{61}$$

and also

$$N(\infty) = (\psi_\infty, \rho\psi_\infty)_h = \{X_1(0)\}^{-1}, \qquad \omega^2 > \omega_p^2, \tag{62a}$$

$$= \{\nu_0\, X_2(0)\}^{-1}, \qquad \omega^2 < \omega_p^2, \tag{62b}$$

while for $\omega^2 < \omega_p^2$

$$N(-\nu_0) = (\psi_{-\nu_0}, \rho\psi_{-\nu_0})_h = (4\pi e\nu_0/i\omega)^2 X_2(-\nu_0)/X_2(0). \tag{63}$$

The Poincaré–Bertrand formula (Muskhelishvili 1953) has been used to obtain the relation (61). These results lead to a proof of the following completeness property of a particular subset of the eigenfunctions of L.

Theorem 1

Any vector ψ_0 defined by

$$\psi_0 = \begin{bmatrix} g_0(u) \\ E_0 \end{bmatrix},$$

where $g_0(u)$ is "sufficiently smooth", can be expanded for $u \geqslant 0$ in the form

$$\psi_0 = \int_0^\infty c(\nu)\,\psi_\nu\, \mathrm{d}\nu + c_\infty\, \psi_\infty + c_-\, \psi_{-\nu_0} \tag{64}$$

for some $c(\nu)$, c_∞, and c_-, the latter expansion coefficient being zero for $\omega^2 > \omega_p^2$.

Proof. If such an expansion is possible, then by the orthogonality relations expressed in (60) the expansion coefficients must be

$$c(\nu) = \frac{(\psi_\nu, \rho\psi_0)}{(\psi_\nu, \rho\psi_\nu)}, \qquad c_\infty = \frac{(\psi_\infty, \rho\psi_0)}{(\psi_\infty, \rho\psi_\infty)}, \qquad c_- = \frac{(\psi_{-\nu_0}, \rho\psi_0)}{(\psi_{-\nu_0}, \rho\psi_{-\nu_0})}. \tag{65}$$

With this clue the proof merely consists in substituting these values for the expansion coefficients in the right-hand side of (64) and verifying that the expansion does in fact give ψ_0. The Poincaré–Bertrand formula must be used when interchanging the order of integration of integrals defined as Cauchy principal values.

This completeness property gives an alternative method of solution to, for example, the problem discussed by Aamodt and Case (1963) of penetration of an electric field into a plasma confined to a region $x > 0$ by a diffuse reflecting wall. Analogous orthogonality and completeness properties can be established for the eigenfunctions of L with eigenvalues in the set $\mathscr{S}^- = \{[-\infty, 0], \nu_0\}$.

V. Full-range Orthogonality and Completeness

Proceeding as in Sections III and IV above, it can be shown that for condition (11) to be satisfied the weight function of the scalar product must be

$$W(u) = -4m/n_0(\mathrm{d}F/\mathrm{d}u) \tag{66}$$

for both $\omega^2 > \omega_p^2$ and $\omega^2 < \omega_p^2$. The analysis is simpler as the function $X(z)$ need not be introduced. Equation (66) gives basically the same scalar product as that used by Shure (1964) and by Case (1967).

Orthogonality relations of the form (60) (with the suffix h replaced by f) hold for $v', v \in \mathscr{S}$ with, for both $\omega^2 > \omega_p^2$ and $\omega^2 < \omega_p^2$,

$$N(v) = -\frac{4\pi m v \Lambda^+(v)\Lambda^-(v)}{n_0 \, dF/dv}, \qquad -\infty < v < \infty, \tag{67}$$

$$N(\infty) = 1-\sigma^2, \tag{68}$$

and, for $\omega^2 < \omega_p^2$,

$$N(-v_0) = N(v_0) = -(4\pi e/i\omega)^2 v_0^3 \Lambda'(v_0). \tag{69}$$

These results can be used to prove the following completeness theorem.

Theorem 2

Any vector ψ_0 defined by

$$\psi_0 = \begin{bmatrix} g_0(u) \\ E_0 \end{bmatrix},$$

where $g_0(u)$ is "sufficiently smooth", can be expanded for $-\infty \leqslant u \leqslant \infty$ in the form

$$\psi_0 = \int_{-\infty}^{\infty} c(v)\psi_v \, dv + c_\infty \psi_\infty + c_+ \psi_{v_0} + c_- \psi_{-v_0} \tag{70}$$

for some $c(v)$, c_∞, c_+, and c_-, the latter two expansion coefficients being zero for $\omega^2 > \omega_p^2$.

This theorem can be established in the same way as Theorem 1. As $\psi_{-\infty} = \psi_\infty$, only one expansion coefficient c_∞ need be considered. Theorem 2 leads to an alternative solution to the slab problem considered by Shure (1964) and the half-space problem with a specular reflection boundary condition considered by Case (1967).

VI. Conclusions

The method described here can be used to derive the scalar products with respect to which similar operators in plasma physics may be regarded as self-adjoint. This avoids the introduction of adjoint operators and the derivation of their eigenfunctions. When working with either initial value or boundary value problems requiring full-range orthogonality conditions, however, the desired scalar product can usually be obtained most simply by inspection. Otherwise, the present procedure or a similar method will have to be used. Although the derivation of the scalar product can be complicated, it is constructive, for besides giving properties of the operator it also leads immediately to the solution of physical problems.

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Appendix

Spectrum and Eigenfunctions of L

(i) $-\infty < v < \infty$

The corresponding eigenfunction ψ_v is

$$\psi_v = \begin{bmatrix} f(v,u,\omega) \\ E(v,\omega) \end{bmatrix} = \begin{bmatrix} \sigma^2 v^2 (\mathrm{d}F/\mathrm{d}u)/(u-v) + \lambda(v,\omega)\,\delta(u-v) \\ (4\pi e/\mathrm{i}\omega)v \end{bmatrix}, \tag{A1}$$

where $\sigma^2 = \omega_p^2/\omega^2$ and the electron plasma frequency ω_p is equal to $(4\pi n_0 e^2/m)^{\frac{1}{2}}$. The "eigendistributions" $f(v,u,\omega)$ satisfy the normalization condition

$$\int_{-\infty}^{\infty} f(v,u,\omega)\,\mathrm{d}u = 1\,. \tag{A2}$$

Also

$$\lambda(v,\omega) = 1 - \sigma^2 v^2 \int_{-\infty}^{\infty} \frac{(\mathrm{d}F/\mathrm{d}u)\,\mathrm{d}u}{u-v}, \tag{A3}$$

the Cauchy principal value of the integral in (A3) being understood. The quantity $\lambda(v,\omega)$ is related to the function

$$\Lambda(z) = 1 - \sigma^2 z^2 \int_{-\infty}^{\infty} \frac{(\mathrm{d}F/\mathrm{d}u)\,\mathrm{d}u}{u-z} \tag{A4}$$

by the equations

$$\Lambda^+(v) = \lambda(v,\omega) - \mathrm{i}\pi\sigma^2 v^2\,\mathrm{d}F/\mathrm{d}v\,, \qquad -\infty \leqslant v \leqslant \infty\,, \tag{A5}$$

$$\Lambda^-(v) = \lambda(v,\omega) + \mathrm{i}\pi\sigma^2 v^2\,\mathrm{d}F/\mathrm{d}v\,, \qquad -\infty \leqslant v \leqslant \infty\,, \tag{A6}$$

where the plus and minus indices denote the limits of $\Lambda(z)$ as z approaches v on the real axis from above and below that axis respectively. We note in passing that Backus (1960) has shown that if it is assumed that the equilibrium distribution f_0 is isotropic in velocity space, i.e. is a function of the square of the velocity, then

$$\mathrm{d}F/\mathrm{d}v = -2\pi v\, f_0(v^2)\,. \tag{A7}$$

As we have further assumed that f_0 is nonzero for finite velocities, it is easily seen that $\Lambda^+(v)$ and $\Lambda^-(v)$ are both nonzero for $-\infty \leqslant v \leqslant \infty$.

(ii) $v = \pm\infty$

The corresponding eigenfunctions $\psi_{+\infty}$ and $\psi_{-\infty}$ are the same and are

$$\psi_{\pm\infty} = \begin{bmatrix} f(\pm\infty, u, \omega) \\ E(\pm\infty, \omega) \end{bmatrix} = \begin{bmatrix} (n_0\, e/mi\omega)\, \mathrm{d}F/\mathrm{d}u \\ 1 \end{bmatrix}. \tag{A8}$$

(iii) $v = \pm v_0$

For $\omega^2 < \omega_\mathrm{p}^2$, the function $\Lambda(z)$ has two purely imaginary zeros $\pm v_0$ where $v_0 = \mathrm{i}\gamma$, say, γ being real and positive. The corresponding eigenfunctions $\psi_{\pm v_0}$ are

$$\psi_{\pm v_0} = \begin{bmatrix} f(\pm v_0, u, \omega) \\ E(\pm v_0, \omega) \end{bmatrix} = \begin{bmatrix} \sigma^2 v_0^2(\mathrm{d}F/\mathrm{d}u)/(u \mp v_0) \\ \pm 4\pi e v_0/\mathrm{i}\omega \end{bmatrix}. \tag{A9}$$

It should be noted that $\Lambda(z, \omega)$ has no zeros for $\omega^2 > \omega_\mathrm{p}^2$ so that the $\psi_{\pm v_0}$ exist only for $\omega^2 < \omega_\mathrm{p}^2$.