SINGULARITIES OF THE TRIANGLE DIAGRAM VERTEX FUNCTION

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Abstract

The definition of a singularity of an integral of Cauchy type is made precise under certain conditions. These conditions are shown to hold for the triangle diagram vertex function, and the positions of its singularities are studied.

I. INTRODUCTION

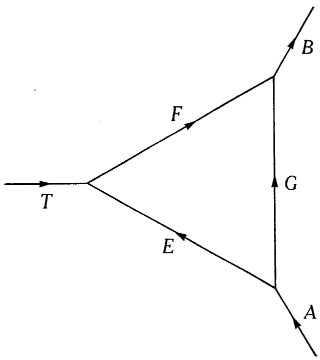

Fig. 1.—Triangle diagram for the vertex function.

In an earlier paper (Frederiksen and Woolcock 1971; hereinafter referred to as VF) we studied in detail the analytic properties of the vertex function arising from the triangle diagram illustrated in Figure 1, with externally stable particles A and B. In particular, we obtained a spectral representation for all allowed mass configurations, including cases for which there is an anomalous threshold. Given the results of VF, one question still remains, namely the precise identification of the singularities of the triangle diagram vertex function for both normal and anomalous threshold cases. We take up this question in the present paper.

Eden *et al.* (1966; with references to earlier work therein) have discussed a heuristic method of obtaining the possible singular points of Feynman amplitudes, but, as pointed out by them and by Hwa and Teplitz (1966), the method unfortunately lacks mathematical precision. However, from the spectral representation obtained in VF we are able to define precisely the physical sheet singularities of the triangle diagram vertex function. To this end, in Section II of the present paper we prove a rather simple mathematical result about a particular kind of integral of Cauchy type. In Section III we show that the triangle diagram vertex function falls within the scope of the theorem, for both normal and anomalous threshold cases. Its singularities can then be identified and their positions studied in relation to the physical regions of the usual kinematical invariant t. In particular we confirm a conclusion obtained by combining results from Norton (1964) and Coleman and Norton (1965), namely that, for externally and internally stable vertices, a singularity of the triangle diagram vertex function occurring in a physical region of t is a normal

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threshold singularity. We also point out that a conjecture of Rechenberg and Sudarshan (1972), that an anomalous threshold singularity is always outside the physical regions of t, has already been proved in VF.

II. Theorem on Integrals of Cauchy Type

Consider an analytic function $F(z)$, regular in the upper half-plane, with the representation

$$F(z) = \pi^{-1} \int_{\bar{I}} \frac{d\xi\, \rho(\xi)}{\xi - z} \qquad \text{for} \qquad \operatorname{Im} z > 0,$$

where $\rho(\xi)$ is a real-valued function on I, the union of a finite number of disjoint open intervals of $\boldsymbol{R}$ (the real line). The closure $\bar{I}$ of I must not be the whole of $\boldsymbol{R}$. The integral is indicated over $\bar{I}$ to show that (independent) limits may need to be taken at each of the end points of the intervals of I (including perhaps $\pm\infty$). The function $F(z)$ is thus represented as a generalized Stieltjes transform.

Now suppose that there exists an analytic function $f(z)$, regular in the upper half-plane, satisfying the conditions:

(i) $f(x+iy)$ approaches a boundary value $f(x)$ as $y \downarrow 0$, for all $x \in I$ and for all except a finite number of values of x in $\boldsymbol{R}-\bar{I}$;

(ii) $f(x) = \rho(x)$ for all $x \in I$;

(iii) $f(z)$ admits the usual representations

$$f(z) = \pi^{-1} \int_{\boldsymbol{R}-I} \frac{d\xi \operatorname{Im} f(\xi)}{\xi - z}$$

$$= (\pi i)^{-1} \int_{\boldsymbol{R}-I} \frac{d\xi \operatorname{Re} f(\xi)}{\xi - z} + (\pi i)^{-1} \int_{\bar{I}} \frac{d\xi\, \rho(\xi)}{\xi - z}$$

for $\operatorname{Im} z > 0$. The last condition implies that (independent) limits may need to be taken at the points of $\bar{I} \cap (\boldsymbol{R}-I)$, at $\pm\infty$, and at the points of $\boldsymbol{R}-\bar{I}$ for which $f(x)$ does not exist. Using the condition (iii) it follows immediately that, for $\operatorname{Im} z > 0$, $F(z)$ also admits the representation

$$F(z) = \pi^{-1} \int_{\boldsymbol{R}-I} \frac{d\xi\, \{-\operatorname{Re} f(\xi) + i \operatorname{Im} f(\xi)\}}{\xi - z}.$$

We have thus established that $F(z)$ has two alternative representations for $\operatorname{Im} z > 0$: it may be continued into the lower half-plane either through I or through $\boldsymbol{R}-\bar{I}$. The resulting functions, both regular in the lower half-plane, are of course different. Alternatively, there are two analytic functions, one regular in $\boldsymbol{C}-\bar{I}$ and the other regular in $\boldsymbol{C}-(\boldsymbol{R}-I)$, which coincide for $\operatorname{Im} z > 0$. This result may be expressed by saying that the singularities of $F(z)$ are the points of the set $\{x \mid x \in \bar{I} \cap (\boldsymbol{R}-I)\}$. Further, the point at infinity is (resp. is not) a singularity of $F(z)$ according as the number of finite singularities is odd (resp. even).

If a function $f(z)$, regular in the upper half-plane, has been found whose boundary value on I is $\rho(x)$, the property (iii) above can be most readily established by using the strong form of Cauchy's integral theorem (see e.g. Behnke and Sommer 1955), the contour being a large semicircle of radius R in the upper half-plane, together with the interval $[-R, R]$ of the real axis, indented with small semicircles in the upper half-plane around the exceptional points indicated in (iii). One will then have to check that all the necessary limits can be taken. The equivalence of the two representations in (iii) will follow in the usual way by keeping z in the upper half-plane and taking a second contour that is the mirror image of the upper half-plane contour in the real axis, the analytic function being the conjugate of $f(\bar{z})$, which is regular in $\operatorname{Im} z < 0$.

The result of this section gives a precise characterization of the singularities of the function $F(z)$. It remains in Section III to apply it to the vertex function arising from the triangle diagram.

III. Application to Triangle Diagram Vertex Function

We begin by recalling some results from VF. It is convenient to alter the notation, and to make certain changes of variable, following on the recent study of the box diagram amplitude (Frederiksen and Woolcock 1973*a*, 1973*b*). Firstly the particles are relabelled as in Figure 1 above (cf. Fig. 1 of VF). Further, the complex variable τ which generalizes the kinematical invariant t is replaced by z $(= x+\mathrm{i}y)$, where

$$2EFz = \tau - E^2 - F^2,$$

while the integration variable w which occurs repeatedly in VF is replaced by ξ, where

$$2EF\xi = w - E^2 - F^2.$$

Instead of the external masses A and B, we use quantities a and b defined by

$$2EGa = E^2 + G^2 - A^2, \qquad 2FGb = F^2 + G^2 - B^2.$$

The function $I(\tau)$ of VF is replaced by $F(z)$, where

$$F(z) = \pi^{-1} EF\, I(\tau).$$

The range of t for external particles on the mass shell, namely $t \leqslant (A-B)^2$, becomes $x \leqslant g_-$, while the range of t for an annihilation reaction with initial state $A\bar{B}$, namely $t \geqslant (A+B)^2$, becomes $x \geqslant g_+$, where

$$g_\pm = (2EF)^{-1}\{(A\pm B)^2 - E^2 - F^2\}.$$

The quantities w_0 and w_1 defined in equation (32) of VF are replaced by $\xi_{1\pm}$, where

$$\xi_{1+} = (2EF)^{-1}(w_0 - E^2 - F^2), \qquad \xi_{1-} = (2EF)^{-1}(w_1 - E^2 - F^2).$$

In terms of the quantities a and b, $\xi_{1\pm}$ may be written as

$$\xi_{1\pm} = -ab \pm \{(1-a^2)(1-b^2)\}^{\frac{1}{2}}. \tag{1}$$

We now put the spectral representations given in Section 5 of VF in terms of the quantities defined above. For the normal threshold case, $F(z)$ has the representation

$$F(z) = \pi^{-1}\int_1^\infty \frac{\mathrm{d}\xi\,\rho(\xi)}{\xi - z}.$$

For $\xi > \max\{1, g_+\}$ and for $1 < \xi < g_-$, $\rho(\xi)$ is given by

$$\rho(\xi) = \tfrac{1}{2}\{\Phi(\xi)\}^{-\frac{1}{2}}\ln\left(\frac{\Psi(\xi)+\{\xi^2-1\}^{\frac{1}{2}}\{\Phi(\xi)\}^{\frac{1}{2}}}{\Psi(\xi)-\{\xi^2-1\}^{\frac{1}{2}}\{\Phi(\xi)\}^{\frac{1}{2}}}\right), \tag{2}$$

where

$$\Phi(\xi) = (\xi - g_+)(\xi - g_-),$$

$$\Psi(\xi) = \xi^2 - 1 + (EF)^{-1}G\{(Ea+Fb)\xi + Eb + Fa\}.$$

For $\max\{1, g_-\} < \xi < g_+$, the form of $\rho(\xi)$ becomes

$$\rho(\xi) = \{-\Phi(\xi)\}^{-\frac{1}{2}}\left\{\tfrac{1}{2}\pi - \arctan\left(\frac{\Psi(\xi)}{\{\xi^2-1\}^{\frac{1}{2}}\{-\Phi(\xi)\}^{\frac{1}{2}}}\right)\right\}. \tag{3}$$

For an anomalous threshold, with $g_- \leqslant \xi_{1-} < \xi_{1+} < 1 < g_+$, $F(z)$ has the representation

$$F(z) = \pi^{-1}\int_{\xi_{1+}}^1 \frac{\mathrm{d}\xi\,\sigma(\xi)}{\xi - z} + \pi^{-1}\int_1^\infty \frac{\mathrm{d}\xi\,\rho(\xi)}{\xi - z}.$$

For $\xi > g_+$, $\rho(\xi)$ is given by equation (2) and, for $1 < \xi < g_+$, $\rho(\xi)$ is given by equation (3). For $\xi_{1+} \leqslant \xi \leqslant 1$, $\sigma(\xi)$ is given by

$$\sigma(\xi) = \pi/\{-\Phi(\xi)\}^{\frac{1}{2}}. \tag{4}$$

We consider firstly the normal threshold case and look for an analytic function $f(z)$, regular in $\mathrm{Im}\, z > 0$, whose boundary value $f(x)$ from above is just $\rho(x)$ for $x > 1$. The appropriate function lies ready at hand in equation (2). Let us formally write

$$f(z) = \tfrac{1}{2}\{\Phi(z)\}^{-\frac{1}{2}}\log\left(\frac{\Psi(z)+\{z^2-1\}^{\frac{1}{2}}\ \Phi(z)\}^{\frac{1}{2}}}{\Psi(z)-\{z^2-1\}^{\frac{1}{2}}\{\Phi(z)\}^{\frac{1}{2}}}\right). \tag{5}$$

For z real and greater than $\max\{1, g_+\}$, the square roots are the positive ones and the logarithm is the natural logarithm of a positive real number. To see that the argument of the logarithm does not vanish in this range, we note that (cf. equation (64) of VF)

$$\{\Psi(z)\}^2 - (z^2-1)\Phi(z) = E^{-2}F^{-2}G^2(2EFz + E^2 + F^2)(z - \xi_{1+})(z - \xi_{1-}), \tag{6}$$

and that ξ_{1+} (equation (1)) is either complex (with ξ_{1-} its complex conjugate) or, if real, is less than or equal to unity.

To continue $f(z)$ away from the semi-infinite interval of the real axis on which it has been defined precisely, we specify that the function $\{z^2-1\}^{\frac{1}{2}}$ has a cut along the real axis from -1 to $+1$ while $\{\Phi(z)\}^{\frac{1}{2}}$ has a cut along this axis from g_- to g_+. The other points where difficulty arises are $z = \xi_{1\pm}$ and $-(2EF)^{-1}(E^2+F^2)$, when either the numerator or denominator of the logarithm vanishes. If the quantities $\xi_{1\pm}$ are complex, $f(z)$ cannot be regular in the upper half-plane and the theorem of Section II will not apply. We therefore make the further restriction that, at the vertices involving the particles A and B, each particle be stable against decay into the other two. This amounts to imposing the inequalities

$$|a| < 1, \qquad |b| < 1. \tag{7}$$

If angles α and β in the range $(0, \pi)$ are defined by

$$a = \cos\alpha, \qquad b = \cos\beta,$$

the quantities $\xi_{1\pm}$ of equation (1) are given by

$$\xi_{1\pm} = -\cos(\alpha \pm \beta).$$

Thus we have

$$-1 \leqslant \xi_{1-} < \xi_{1+} \leqslant 1. \tag{8}$$

Other important inequalities which hold if the conditions (7) apply are

$$g_- \leqslant \xi_{1-}, \qquad \xi_{1+} \leqslant g_+. \tag{9}$$

The inequalities (9) are not proved in VF, but it can be readily shown that

$$g_+ - \xi_{1+} = (1-a^2)^{\frac{1}{2}}(1-b^2)^{\frac{1}{2}}\{-1+\cosh(\chi_1+\chi_2)\},$$

$$\xi_{1-} - g_- = (1-a^2)^{\frac{1}{2}}(1-b^2)^{\frac{1}{2}}\{-1+\cosh(\chi_1-\chi_2)\},$$

where

$$\sinh\chi_1 = \frac{G-Ea}{E(1-a^2)^{\frac{1}{2}}}, \qquad \sinh\chi_2 = \frac{G-Fb}{F(1-b^2)^{\frac{1}{2}}}.$$

With the inequalities (7) satisfied by a and b, and with the precise specification of $f(x)$ for $x > \max\{1, g_+\}$, namely $f(x) = \rho(x)$ as given by equation (2), the formal definition (5) of $f(z)$ becomes a precise definition of a function regular in $\boldsymbol{C} - (-\infty, \max\{1, g_+\}]$, which satisfies the Schwarz reflection principle. The function $f(x+\mathrm{i}y)$ has a limit $f(x)$ as $y \downarrow 0$ for all x except perhaps the points ± 1, $g_\pm$, $\xi_{1\pm}$, and $-(2EF)^{-1}(E^2+F^2)$. In what follows we write down the functional form of $f(x)$ for $x < \max\{1, g_+\}$ and study its behaviour at these seven special points (which we take to be all different). The results are obtained by straightforward but tedious complex analysis.

For the interval $\min\{1, g_+\} < x < \max\{1, g_+\}$ we consider the cases $1 < g_+$ and $g_+ < 1$ separately. For $1 < g_+$ we have

$$\operatorname{Re} f(x) = \tfrac{1}{2}\{-\Phi(x)\}^{-\frac{1}{2}} \arg\left(\frac{\Psi(x)+\mathrm{i}\{x^2-1\}^{\frac{1}{2}}\{-\Phi(x)\}^{\frac{1}{2}}}{\Psi(x)-\mathrm{i}\{x^2-1\}^{\frac{1}{2}}\{-\Phi(x)\}^{\frac{1}{2}}}\right), \quad \operatorname{Im} f(x) = 0, \tag{10}$$

when $1 < x < g_+$. The arg function is specified by fixing it at zero for $x = g_+$. Thus

$\mathrm{Re}\, f(x)$ as given by (10) is exactly the $\rho(x)$ of equation (3). Now

$$\mathrm{Re}\, f(x) \to (g_+^2 - 1)^{\frac{1}{2}}/\Psi(g_+) \tag{11}$$

as $x \to g_+$ from either side. Note that equation (6) shows $\Psi(g_+) > 0$. The function $f(x)$ is real for $x > 1$ and the cut in the function $f(z)$ extends along the real axis from 1 to $-\infty$. The functional form of $f(x)$ changes at $x = g_+$ but by defining $f(g_+)$ equal to the right-hand side of (11) we specify $f(x)$ for all $x > 1$ in such a way that it has continuous derivatives of all orders for all $x > 1$.

For the case $g_+ < 1$, we have, for $g_+ < x < 1$,

$$\mathrm{e}\, f(x) = 0, \quad \mathrm{Im}\, f(x) = \tfrac{1}{2}\{\Phi(x)\}^{-\frac{1}{2}} \arg\left(\frac{\Psi(x) + \mathrm{i}\{1-x^2\}^{\frac{1}{2}}\{\Phi(x)\}^{\frac{1}{2}}}{\Psi(x) - \mathrm{i}\{1-x^2\}^{\frac{1}{2}}\{\Phi(x)\}^{\frac{1}{2}}}\right).$$

The arg function is specified by fixing it at zero when $x = 1$. Thus

$$f(x) \sim 2^{\frac{1}{2}}(x-1)^{\frac{1}{2}}/\Psi(1) \qquad \text{for} \qquad x \downarrow 1,$$

$$\mathrm{Im}\, f(x) \sim 2^{\frac{1}{2}}(1-x)^{\frac{1}{2}}/\Psi(1) \qquad\qquad x \uparrow 1.$$

Note from equation (6) that $\Psi(1) > 0$.

In the interval $\xi_{1+} < x < \min\{1, g_+\}$,

$$\mathrm{Re}\, f(x) = 0 \qquad \text{if} \qquad \Psi(\xi_{1+}) > 0, \tag{12a}$$

$$= \pi/\{-\Phi(x)\}^{\frac{1}{2}} \qquad\qquad \Psi(\xi_{1+}) < 0, \tag{12b}$$

and

$$\mathrm{Im}\, f(x) = -\tfrac{1}{2}\{-\Phi(x)\}^{-\frac{1}{2}} \ln\left(\frac{\Psi(x) - \{1-x^2\}^{\frac{1}{2}}\{-\Phi(x)\}^{\frac{1}{2}}}{\Psi(x) + \{1-x^2\}^{\frac{1}{2}}\{-\Phi(x)\}^{\frac{1}{2}}}\right), \tag{13}$$

whatever the sign of $\Psi(\xi_{1+})$. From equation (6), $\Psi(x)$ does not change sign in the interval and the argument of the natural logarithm is a positive real number.

Now for $1 < g_+$ we have

$$\mathrm{Re}\, f(x) \sim 2^{\frac{1}{2}}(x-1)^{\frac{1}{2}}/\Psi(1) \qquad \text{for} \qquad x \downarrow 1,$$

if $\Psi(1)$ and $\Psi(\xi_{1+})$ are positive, while

$$\mathrm{Re}\, f(x) \to \pi/\{-\Phi(1)\}^{\frac{1}{2}} \qquad \text{as} \qquad x \to 1$$

from either side, if $\Psi(1)$ and $\Psi(\xi_{1+})$ are negative. Further, whatever the sign of $\Psi(1)$,

$$\mathrm{Im}\, f(x) \sim 2^{\frac{1}{2}}(1-x)^{\frac{1}{2}}/\Psi(1) \qquad \text{for} \qquad x \uparrow 1.$$

Thus $f(x)$ is continuous at $x = 1$.

However, for $g_+ < 1$ with $\Psi(g_+)$ and $\Psi(\xi_{1+})$ positive, $\mathrm{Re}\, f(x)$ is zero throughout the interval $(\xi_{1+}, 1)$ and

$$\mathrm{Im}\, f(x) \to (1-g_+^2)^{\frac{1}{2}}/\Psi(g_+) \qquad \text{as} \qquad x \to g_+ \tag{14}$$

from either side. If we define $\operatorname{Im} f(g_+)$ equal to the right-hand side of (14) then $\operatorname{Im} f(x)$ has continuous derivatives of all orders at each point of $(\xi_{1+}, 1)$. On the other hand, for $g_+ < 1$ but with $\Psi(g_+)$ and $\Psi(\xi_{1+})$ negative, $\operatorname{Re} f(x)$ does not vanish on (ξ_{1+}, g_+) and in this case we have

$$\operatorname{Re} f(x) \sim \pi/(g_+ - g_-)^{\frac{1}{2}}(g_+ - x)^{\frac{1}{2}} \qquad \text{for} \qquad x \uparrow g_+$$

and

$$\operatorname{Im} f(x) \sim \pi/(g_+ - g_-)^{\frac{1}{2}}(x - g_+)^{\frac{1}{2}} \qquad x \downarrow g_+,$$

$$\to (1 - g_+^2)^{\frac{1}{2}}/\Psi(g_+) \qquad x \uparrow g_+.$$

The next interval to consider is $\xi_{1-} < x < \xi_{1+}$. Here

$$\operatorname{Re} f(x) = \pi/2\{-\Phi(x)\}^{\frac{1}{2}}, \tag{15}$$

$$\operatorname{Im} f(x) = -\tfrac{1}{2}\{-\Phi(x)\}^{-\frac{1}{2}} \ln\left(\frac{\{1-x^2\}^{\frac{1}{2}} - \Phi(x)\}^{\frac{1}{2}} - \Psi(x)}{\{1-x^2\}^{\frac{1}{2}}\{-\Phi(x)\}^{\frac{1}{2}} + \Psi(x)}\right).$$

Once more equation (6) assures us that the argument of the logarithm is a positive real number. It is clear from equations (12a), (12b), and (15) that $\operatorname{Re} f(x)$ has a step discontinuity at $x = \xi_{1+}$, whatever the sign of $\Psi(\xi_{1+})$. For x near ξ_{1+},

$$\operatorname{Im} f(x) \sim \tfrac{1}{2}\{-\Phi(\xi_{1+})\}^{-\frac{1}{2}} \operatorname{sgn}\{-\Psi(\xi_{1+})\} \ln|x - \xi_{1+}|.$$

We come now to the interval $\max\{-1, g_-\} < x < \xi_{1-}$, in which $\Psi(x)$ does not change sign. The required functional forms here are

$$\operatorname{Re} f(x) = \pi/\{-\Phi(x)\}^{\frac{1}{2}} \qquad \text{for} \qquad \Psi(\xi_{1-}) > 0, \tag{16a}$$

$$= 0 \qquad \Psi(\xi_{1-}) < 0, \tag{16b}$$

while $\operatorname{Im} f(x)$ is given by equation (13), whatever the sign of $\Psi(\xi_{1-})$. From equations (15), (16a), and (16b) we see that $\operatorname{Re} f(x)$ has a step discontinuity at $x = \xi_{1-}$, while $\operatorname{Im} f(x)$ has the behaviour

$$\operatorname{Im} f(x) \sim \tfrac{1}{2}\{-\Phi(\xi_{1-})\}^{-\frac{1}{2}} \operatorname{sgn}\{-\Psi(\xi_{1-})\} \ln|x - \xi_{1-}|$$

for x near ξ_{1-}.

For the interval $\min\{-1, g_-\} < x < \max\{-1, g_-\}$, we similarly consider the cases $g_- > -1$ and $g_- < -1$ separately. For $g_- > -1$ we have

$$\operatorname{Re} f(x) = 0, \quad \operatorname{Im} f(x) = -\tfrac{1}{2}\{\Phi(x)\}^{-\frac{1}{2}} \arg\left(\frac{\Psi(x) - i\{1-x^2\}^{\frac{1}{2}}\{\Phi(x)\}^{\frac{1}{2}}}{\Psi(x) + i\{1-x^2\}^{\frac{1}{2}}\{\Phi(x)\}^{\frac{1}{2}}}\right)$$

for $-1 < x < g_-$. For $\Psi(g_-) > 0$ (resp. < 0) the arg function is 2π (resp. 0) at $x = g_-$, while for $\Psi(-1) > 0$ (resp. < 0), the arg function is 2π (resp. 0) at $x = -1$. The function $\Psi(x)$ may change sign in the interval $[-1, g_-]$. When $\Psi(\xi_{1-})$ and

$\Psi(g_-)$ are positive,

$$\operatorname{Re} f(x) \sim \pi/(g_+ - g_-)^{\frac{1}{2}}(x - g_-)^{\frac{1}{2}} \qquad \text{for} \qquad x \downarrow g_-$$

and

$$\operatorname{Im} f(x) \to (1 - g_-^2)^{\frac{1}{2}}/\Psi(g_-) \qquad\qquad x \downarrow g_-,$$

$$\sim -\pi/(g_+ - g_-)^{\frac{1}{2}}(g_- - x)^{\frac{1}{2}} \qquad\qquad x \uparrow g_-.$$

On the other hand, when $\Psi(\xi_{1-})$ and $\Psi(g_-)$ are negative, $\operatorname{Re} f(x)$ is zero throughout the interval $(-1, \xi_{1-})$ and

$$\operatorname{Im} f(x) \to (1 - g_-^2)^{\frac{1}{2}}/\Psi(g_-) \tag{17}$$

as $x \to g_-$ from either side. If we define $\operatorname{Im} f(g_-)$ equal to the right-hand side of (17) then $\operatorname{Im} f(x)$ has continuous derivatives of all orders at each point of $(-1, \xi_{1-})$.

We turn now to the case $g_- < -1$, for which

$$\operatorname{Re} f(x) = \tfrac{1}{2}\{-\Phi(x)\}^{-\frac{1}{2}} \arg\left(\frac{\Psi(x) - i\{x^2 - 1\}^{\frac{1}{2}}\{-\Phi(x)\}^{\frac{1}{2}}}{\Psi(x) + i\{x^2 - 1\}^{\frac{1}{2}}\{-\Phi(x)\}^{\frac{1}{2}}}\right), \quad \operatorname{Im} f(x) = 0$$

for $g_- < x < -1$. For $\Psi(-1) > 0$ (resp. < 0), the arg function is 2π (resp. 0) at $x = -1$, while for $\Psi(g_-) > 0$ (resp. < 0), the arg function is 2π (resp. 0) at $x = g_-$. Thus if $\Psi(\xi_{1-})$ and $\Psi(-1)$ are positive then

$$\operatorname{Re} f(x) \to \pi/\{-\Phi(-1)\}^{\frac{1}{2}} \qquad \text{for} \qquad x \to -1$$

from either side, while

$$\operatorname{Im} f(x) \sim 2^{\frac{1}{2}}(x+1)^{\frac{1}{2}}/\Psi(-1) \qquad\qquad x \downarrow -1,$$

so that $f(x)$ is continuous at $x = -1$ if we define $f(-1) = \pi/\{-\Phi(-1)\}^{\frac{1}{2}}$. If, on the other hand, $\Psi(\xi_{1-})$ and $\Psi(-1)$ are negative then

$$\operatorname{Re} f(x) \sim -2^{\frac{1}{2}}(-1-x)^{\frac{1}{2}}/\Psi(-1) \qquad \text{for} \qquad x \uparrow -1,$$

$$\operatorname{Im} f(x) \sim 2^{\frac{1}{2}}(x+1)^{\frac{1}{2}}/\Psi(-1) \qquad\qquad x \downarrow -1.$$

The next interval to consider is $-(2EF)^{-1}(E^2+F^2) < x < \min\{-1, g_-\}$. For brevity we write

$$\xi_0 = -(2EF)^{-1}(E^2+F^2) \leqslant -1.$$

In what follows we assume $E \neq F$, so that $\xi_0 < -1$, and also $A \neq B$, so that $\xi_0 < g_-$. In the interval under consideration,

$$\operatorname{Re} f(x) = -\tfrac{1}{2}\{\Phi(x)\}^{-\frac{1}{2}} \ln\left(\frac{\Psi(x) + \{x^2 - 1\}^{\frac{1}{2}}\{\Phi(x)\}^{\frac{1}{2}}}{\Psi(x) - \{x^2 - 1\}^{\frac{1}{2}}\{\Phi(x)\}^{\frac{1}{2}}}\right) \tag{18}$$

and

$$\operatorname{Im} f(x) = -\pi/\{\Phi(x)\}^{\frac{1}{2}} \qquad \text{for} \qquad \Psi(\min\{-1, g_-\}) > 0, \tag{19a}$$

$$= 0 \qquad\qquad \Psi(\min\{-1, g_-\}) < 0. \tag{19b}$$

Note from equation (6) that $\Psi(x)$ does not change sign in the interval. For $g_- > -1$,

$$\mathrm{Re}\, f(x) \sim -2^{\frac{1}{2}}(-1-x)^{\frac{1}{2}}/\Psi(-1) \qquad \text{for} \qquad x \uparrow -1,$$

whatever the sign of $\Psi(-1)$, while

$$\mathrm{Im}\, f(x) \to -\pi/\{\Phi(-1)\}^{\frac{1}{2}} \qquad \text{as} \qquad x \to -1$$

from either side, if $\Psi(-1)$ is positive, and

$$\mathrm{Im}\, f(x) \sim 2^{\frac{1}{2}}(x+1)^{\frac{1}{2}}/\Psi(-1) \qquad \text{for} \qquad x \downarrow -1,$$

if $\Psi(-1)$ is negative. On the other hand, for $g_- < -1$ we have $\mathrm{Im}\, f(x) = 0$ throughout the interval $\xi_0 < x < -1$ if $\Psi(g_-)$ is negative. In this case, if we define

$$\mathrm{Re}\, f(g_-) = -(g_-^2-1)^{\frac{1}{2}}/\Psi(g_-)$$

then $\mathrm{Re}\, f(x)$ has continuous derivatives of all orders at each point of $(\xi_0, -1)$. However, for $g_- < -1$ and $\Psi(g_-)$ positive we have

$$\begin{aligned} \mathrm{Re}\, f(x) &\sim \pi/(g_+-g_-)^{\frac{1}{2}}(x-g_-)^{\frac{1}{2}} && \text{for} \quad x \downarrow g_-, \\ &\to -(g_-^2-1)^{\frac{1}{2}}/\Psi(g_-) && \phantom{\text{for}} \quad x \uparrow g_-, \end{aligned}$$

and

$$\mathrm{Im}\, f(x) \sim -\pi/(g_+-g_-)^{\frac{1}{2}}(g_--x)^{\frac{1}{2}} \qquad x \uparrow g_-.$$

Finally, for $x < \xi_0$, $\mathrm{Re}\, f(x)$ is given by (18) while

$$\mathrm{Im}\, f(x) = -\pi/2\{\Phi(x)\}^{\frac{1}{2}}. \tag{20}$$

From equations (19a), (19b), and (20), $\mathrm{Im}\, f(x)$ has a step discontinuity at $x = \xi_0$ whatever the sign of $\Psi(\xi_0)$. From equation (18), $\mathrm{Re}\, f(x)$ has the behaviour

$$\mathrm{Re}\, f(x) \sim \tfrac{1}{2}\{\Phi(\xi_0)\}^{-\frac{1}{2}}\,\mathrm{sgn}\{-\Psi(\xi_0)\}\ln|x-\xi_0|$$

for x near ξ_0.

We have considered the above detailed behaviour of $f(x)$ in order to establish its properties at each of the points ± 1, $g_\pm$, $\xi_{1\pm}$, and ξ_0, which we have assumed to be all distinct. Certain of these seven points (but never g_+ and g_-) will coincide in special cases, but the conclusions we now draw will not be altered in such cases. The property (iii) of the function $f(z)$ defined in Section II can be established by the method described near the end of that section. The integral round the large semicircle of radius R in the upper half-plane will approach zero as $R \to \infty$, since $|f(z)| \to 0$ as $|z| \to \infty$, uniformly in the interval $0 \leqslant \arg z \leqslant \pi$. The integration contour along the real axis will also need to be indented by small semicircles in the upper half-plane centred on each of the seven special points above. The behaviour detailed in the preceding paragraphs at each of these points is such that, when each small semicircle is shrunk to its centre, the corresponding integral will approach zero. The properties (i)–(iii) of $f(z)$ have therefore been established and the theorem of Section II applies. When there is a normal threshold (with $(a+b) \geqslant 0$ or alternatively $(\alpha+\beta) \leqslant \pi$), the singularities of $F(z)$ are thus $z = 1$ and $z = \infty$.

For the case of an anomalous threshold ($a+b < 0$ or $\alpha+\beta > \pi$), $F(z)$ is the sum of two functions, $F_1(z)$ the Cauchy integral over the interval $[1, +\infty)$ and $F_2(z)$ the Cauchy integral over $[\xi_{1+}, 1]$. The function $F_1(z)$ behaves exactly as in the normal threshold case (though now the case $g_+ \leqslant 1$ cannot occur) and its singularities are just $z = 1$ and $z = \infty$. For the function $F_2(z)$, the corresponding function $f_2(z)$ is defined as

$$f_2(z) = \pi/\{(z-g_-)(g_+-z)\}^{\frac{1}{2}},$$

the function having cuts along the real axis from $-\infty$ to g_- and from g_+ to $+\infty$ and its value on (g_-, g_+) being

$$f_2(x) = \pi/\{(x-g_-)(g_+-x)\}^{\frac{1}{2}}.$$

In this case the contour of integration will be indented at $x = g_\pm$. It is clear that $f_2(z)$ satisfies conditions (i)–(iii) of Section II and thus the singularities of $F_2(z)$ are just $z = \xi_{1+}$ and $z = 1$.

It remains to study the positions of the singularities of $F(z)$ in relation to the physical regions of z, for which

$$x \leqslant g_- \quad \text{and} \quad x \geqslant g_+, \qquad y = 0,$$

as pointed out early in this section. In Section 4 of VF it is shown that, for an anomalous threshold case,

$$(A-B)^2 < (E+F)^2 < (A+B)^2 \qquad \text{or} \qquad g_- < 1 < g_+.$$

From the inequalities (8) and (9) we can extend this relationship to

$$g_- < \xi_{1+} < 1 < g_+,$$

so that the finite singularities $z = 1$ and $z = \xi_{1+}$ of $F(z)$ lie outside the physical regions of z. This is the conjecture of Rechenberg and Sudarshan (1972), and it was already proved in VF. We conclude further that if a singularity of $F(z)$ occurs in a physical region of z, it must be a normal threshold singularity (and thus $z = 1$). A careful examination of Figure 2 and the accompanying text in Section 7 of VF shows that, if the quantities a and b satisfy the inequalities (7), the case $g_- > 1$ cannot occur. The case $g_+ < 1$ can occur, however. It is not difficult to show that the regions in the (a,b) plane satisfying (7) and the region satisfying $g_+(a,b) < 1$ have a non-void intersection for

$$G < \tfrac{1}{2}(\sqrt{2}+1)(E+F).$$

This result is consistent with the conclusions of Norton (1964) and Coleman and Norton (1965), as remarked at the end of Section 1.

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