

SEPARABLE REPRESENTATION OF THE NUCLEON-NUCLEON INTERACTION IN THE 3S_1 - 3D_1 CHANNEL

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Abstract

A separable representation for the two-nucleon T -matrix in coupled channels has been obtained from a unitary pole expansion for the potential and the method has been tested with the Reid soft-core potential in the 3S_1 - 3D_1 channel. A 3-term expansion provides a T -matrix that is sufficiently accurate for a perturbation theory calculation of the three-nucleon bound state, while a 15-term expansion closely reproduces the exact T -matrix. In addition, a unitary pole approximation is presented that gives a better fit to the phase shifts and coupling parameter than that of Siebert *et al.* (1972).

I. INTRODUCTION

There has been a growing interest in the treatment of the three-nucleon problem and nuclear structure calculations by means of the two-nucleon T -matrix and Brueckner G -matrix in separable representations. This interest has been motivated by the simplifications that separable representations introduce. Thus, in the three-nucleon problem, the separability of the two-body T -matrix reduces the Faddeev equations from an integral equation in two variables to a set of coupled one-dimensional integral equations (Watson and Nuttall 1967). Similarly, in nuclear matter calculations, the separability of the G -matrix enables the Bethe-Faddeev equations to be solved directly, thus facilitating evaluation of the contributions to the binding energy from the three-body correlations (Bhakar and McCarthy 1967; Day *et al.* 1972). In addition, the derivation of effective operators for finite nuclei (Jackson and Lande 1972) and the performance of Brueckner-Hartree-Fock calculations can both be simplified by the use of a separable G -matrix.

Separability of the T - and G -matrices may be achieved through the use of separable potentials that are adjusted to fit the two-nucleon data. Unfortunately, this has the drawbacks that most separable potentials fail to reproduce the one-pion exchange tail of the nucleon-nucleon (N-N) interaction and have a weak tensor force in the 3S_1 - 3D_1 channel (Clement *et al.* 1969), the latter property being important for the saturation of infinite nuclear matter and finite nuclei (Afnan *et al.* 1971).

One method of overcoming these problems, while still retaining the separability of the two-nucleon T - and G -matrices, is to form a unitary pole expansion (hereinafter designated UPE; Harms 1970) of a realistic interaction potential such as the Reid (1968) soft-core potential. The UPE is a separable expansion of the two-nucleon potential in terms of the eigenfunctions of the kernel of the homogeneous Lippmann-Schwinger equation. Upon truncating the expansion to N terms, say, the resulting

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UPE potential may be substituted in the Lippmann–Schwinger equation to obtain a separable T -matrix which satisfies the requirements of two-particle unitarity and reproduces the important features of the actual T -matrix. In the three-nucleon bound state problem and in nuclear matter calculations, the two-body T -matrix is restricted to the negative energy axis, where its most important features are the bound state pole and its residue (the bound state wavefunction), and the UPE potential is specifically designed to reproduce the two-body binding energy and wavefunction for all N .

For central potentials, the accuracy of the UPE is well established and, even for $N = 1$ (the case of the unitary pole approximation, UPA), has been shown to give reliable results (Harms 1970; Harms and Laroze 1970; Jackson and Lande 1972; O'Donoghue and Levinger 1972; Afnan and Read 1973). However, the generalization of the UPE to include the tensor and spin-orbit components of the more realistic N - N interactions has not been previously accomplished, and it is the purpose of the present paper to develop a separable expansion of the two-nucleon T -matrix in the important 3S_1 - 3D_1 channel.

In generalizing the UPE for coupled channels, we impose the conditions that, for any N , the UPE T -matrix has the same position and residue of the two-body pole as does the T -matrix obtained from the original potential and that, in the limit of large N , the two T -matrices are for practical purposes identical. The second condition is important if the UPE is to be used in the calculation of triton binding energies for realistic N - N potentials (Afnan and Read 1973). In Section II, it is shown that the UPA potential is not uniquely determined by the requirement that it yield the same deuteron binding energy and wavefunction as does the original potential, and an additional condition, which unambiguously specifies the UPA in the 3S_1 - 3D_1 channel, is found. In Section III, the UPE is generalized to include the tensor and spin-orbit components of realistic N - N interactions and the convergence of the generalized UPE is tested using the Reid soft-core potential.

II. UNITARY POLE APPROXIMATION

The justification for a separable representation of the nucleon–nucleon T -matrix on the negative energy axis lies in the fact that the T -matrix is almost separable in the vicinity of the bound state pole. This can be demonstrated (Lovelace 1964) by considering the Low equation for the T -matrix,

$$T(E) = V - VG(E)V, \quad (1)$$

where $G(E) = (H - E)^{-1}$ is the Green's function for the Hamiltonian H . If the system has a single bound state, equation (1) can be written in the spectral representation of $G(E)$ as

$$T(E) = V + \frac{V|B\rangle\langle B|V}{E+B} + \int dE' \frac{V|\Psi_E^{(+)}\rangle\langle\Psi_{E'}^{(+)}|V}{(E-E')}, \quad (2)$$

where $|B\rangle$ is the wavefunction and $-B$ is the energy of the bound state. In the vicinity of the bound state pole, the second term on the right-hand side of equation (2)

dominates the behaviour of the T -matrix and we may write

$$T(E) \approx (E+B)^{-1} V|B\rangle\langle B|V \quad \text{as} \quad E \rightarrow -B, \quad (3)$$

thus demonstrating that the T -matrix is separable near the bound state pole.

The aim of the UPA method is to form a separable potential V^A that gives a T -matrix $T^A(E)$ with the property

$$T^A(E) = T(E) \quad \text{for} \quad E \approx -B. \quad (4)$$

However, we now show that equation (4) does not uniquely determine the UPA potential, and that an additional condition has to be introduced.

In the 3S_1 - 3D_1 channel, the deuteron wavefunction is a solution of the homogeneous Lippmann-Schwinger equation

$$|\Psi_D\rangle = -G_0(-B) V |\Psi_D\rangle, \quad (5)$$

where $G_0(E) = (H_0 - E)^{-1}$ is the free Green's function. Partial wave expansion of the wavefunction reduces equation (5) to two coupled equations of the form

$$|\mu_l\rangle = - \sum_{l'} G_0(-B) V_{ll'} |\mu_{l'}\rangle, \quad (6)$$

where

$$|\Psi_D\rangle = \eta^{-1} \sum_l |\mu_l\rangle |\alpha l\rangle \quad (7)$$

with the normalization

$$\eta^2 = \sum_l \langle \mu_l | \mu_l \rangle,$$

and $|\alpha l\rangle$, with $\alpha \equiv \{JST\}$, is an eigenstate of the total angular momentum J , spin S , isospin T , and orbital angular momentum l . The sum over l is $l = J \pm 1$, and the potential $V_{ll'} = \langle \alpha l | V | \alpha l' \rangle$. If we define the form factors $|\chi_l\rangle$ by the relation

$$|\mu_l\rangle = G_0(-B) |\chi_l\rangle \quad (8)$$

then the $|\chi_l\rangle$ satisfy the coupled equations

$$|\chi_l\rangle = - \sum_{l'} V_{ll'} G_0(-B) |\chi_{l'}\rangle, \quad (9)$$

with the normalization

$$\sum_l \langle \chi_l | G_0(-B) |\chi_l\rangle = 1.$$

The T -matrix at the pole can now be written in terms of the form factors $|\chi_l\rangle$ using equations (3) and (9) as

$$T_{ll'}(E) \approx \frac{|\chi_l\rangle\langle\chi_{l'}|}{\eta^2(E+B)} \quad \text{for} \quad E \approx -B. \quad (10)$$

To reproduce this T -matrix for $E \approx -B$ with a one-term separable potential, we write the UPA potential as

$$V_{ll'}^A = |\chi_l\rangle C_{ll'} \langle\chi_{l'}|, \quad (11)$$

where $|\chi_l\rangle$ are the form factors obtained from equation (9) and the $C_{ll'}$ are the strengths of the different potential components and satisfy the hermiticity condition $C_{ll'} = C_{l'l}$. For the UPA T -matrix to equal the T -matrix of the original potential at the bound state pole, the strengths $C_{ll'}$ must satisfy the nonlinear equations (see the Appendix)

$$\left. \begin{aligned} C_{02} &= C_{00} + (C_{00} C_{22} - C_{02}^2) \langle \chi_2 | G_0(-B) | \chi_2 \rangle \\ &= C_{22} + (C_{00} C_{22} - C_{02}^2) \langle \chi_0 | G_0(-B) | \chi_0 \rangle, \\ 1 + C_{00} \langle \chi_0 | G_0(-B) | \chi_0 \rangle + C_{22} \langle \chi_2 | G_0(-B) | \chi_2 \rangle \\ &= -(C_{00} C_{22} - C_{02}^2) \langle \chi_0 | G_0(-B) | \chi_0 \rangle \langle \chi_2 | G_0(-B) | \chi_2 \rangle, \end{aligned} \right\} \quad (12)$$

where the angular momentum labels refer to the 3S_1 - 3D_1 channel. Equations (12) do not uniquely determine the strengths $C_{ll'}$, although a possible unique solution may be obtained by imposing the linearizing condition

$$C_{00} C_{22} = C_{02}^2. \quad (13)$$

This condition is satisfied by all Yamaguchi-type potentials (Yamaguchi and Yamaguchi 1954) and allows the strengths to be determined as

$$C_{00} = C_{22} = C_{02} = -1. \quad (14)$$

Equations (11) and (14) define the UPAI potential, which has been shown by Siebert *et al.* (1972) to give the wrong sign for the 3D_1 phase shifts (although the wrong sign also found by them for the coupling parameter is actually due to a factor of $i^{l-l'}$ erroneously included in their T -matrix). The deficiencies of the UPAI potential are due, in part, to the fact that the strengths C_{00} , C_{02} , and C_{22} are not independent, while an angular momentum decomposition performed on a realistic N-N potential with central, tensor, and spin-orbit interactions

$$V = V_C + V_T S_{12} + V_{LS} \mathbf{L} \cdot \mathbf{S} \quad (15)$$

yields the potential components

$$V_{00} = V_C, \quad V_{02} = 2\sqrt{2} V_T, \quad V_{22} = V_C - 3V_{LS} - 2V_T, \quad (16)$$

which in general are independent. This means that the inclusion of central, tensor, and spin-orbit interactions requires three independent strengths in the parameterization of the UPA potential.

Removal of the condition (13) leaves equations (12) nonlinear and the strengths undetermined, with C_{02} and C_{22} expressed in terms of C_{00} by

$$\left. \begin{aligned} C_{02} &= -\frac{1 + C_{00} \langle \chi_0 | G_0(-B) | \chi_0 \rangle}{\langle \chi_2 | G_0(-B) | \chi_2 \rangle}, \\ C_{22} &= \frac{\langle \chi_0 | G_0(-B) | \chi_0 \rangle - \langle \chi_2 | G_0(-B) | \chi_2 \rangle + C_{00} \langle \chi_0 | G_0(-B) | \chi_0 \rangle^2}{\langle \chi_2 | G_0(-B) | \chi_2 \rangle^2}. \end{aligned} \right\} \quad (17)$$

Our prime concern in finding an additional specifying condition is to ensure that the bound state systems for potentials $V_{ll'}$ and $V_{ll'}^A$ correspond as closely as possible, and so this condition should only involve these potentials and the set of bound state wavefunctions $|\mu_l\rangle$ of equation (7). The relationships

$$\langle \mu_l | V_{ll'} | \mu_{l'} \rangle = \langle \mu_l | V_{ll'}^A | \mu_{l'} \rangle \quad (18)$$

afford such a condition because the equations (17) are insufficient to guarantee this equality. The matrix element $\langle \mu_l | V_{ll'} | \mu_{l'} \rangle$ is the contribution from the ll' component of the potential to the potential energy of the deuteron and so provides a measure of the relative importance assigned to the different components by the potential V . Consequently equation (18) ensures that the same relative importance is maintained in the UPA potential. By means of equations (8) and (11), equation (18) becomes

$$\langle \chi_l | G_0(-B) V_{ll'} G_0(-B) | \chi_{l'} \rangle = \langle \chi_l | G_0(-B) | \chi_l \rangle C_{ll'} \langle \chi_{l'} | G_0(-B) | \chi_{l'} \rangle,$$

which determines the strengths as

$$C_{ll'} = \frac{\langle \chi_l | G_0(-B) V_{ll'} G_0(-B) | \chi_{l'} \rangle}{\langle \chi_l | G_0(-B) | \chi_l \rangle \langle \chi_{l'} | G_0(-B) | \chi_{l'} \rangle}. \quad (19)$$

The strengths (19) satisfy equations (17) and ensure that if $V_{ll'}$ is a one-term separable potential then $V_{ll'}^A \equiv V_{ll'}$. The potential defined by equations (11) and (19) is shown in Section III to be the one-term attractive UPE potential (hereinafter designated 1A).

Figure 1 presents a comparison of calculated bar phase shifts and coupling parameter in the 3S_1 - 3D_1 channel for the Reid (1968) soft-core potential and its UPAI and 1A approximations. The improvement occasioned by substituting the condition (18) for the condition (13) is evident. In particular, the 1A potential yields the correct sign and approximate magnitude for the 3D_1 phase shifts at low energy. This is important in nuclear matter calculations since an attractive contribution from the 3D_1 channel leads to over-binding and an excessive saturation density. The failure of the UPA potential to reproduce the 3D_1 phase shifts at high energy is remedied in the UPE potential which includes both attractive and repulsive terms.

The coupling parameters for the UPAI and 1A potentials are in poor agreement with Reid's (1968) values even at low energies. The coupling parameter is very sensitive to variations in the on-shell T -matrix and the inability of the UPA to approximate it may restrict the usefulness of this type of potential at positive energies. It is interesting that Afnan *et al.* (1971), in fitting separable potentials (with varying D-state probabilities) to the bound state and low-energy data in the 3S_1 - 3D_1 channel, obtained coupling parameters within a few per cent of the 1A values. This suggests that it may not be possible to fit the deuteron observables and the 3S_1 and 3D_1 phase shifts and simultaneously obtain reasonable values for the coupling parameter with a one-term separable potential. The potentials of Afnan *et al.* (1971) give results for infinite nuclear matter that are in excellent agreement with those given by the Reid potential when allowance is made for the difference in D-state probability. This is a strong indication that the failure of the 1A potential to reproduce the coupling

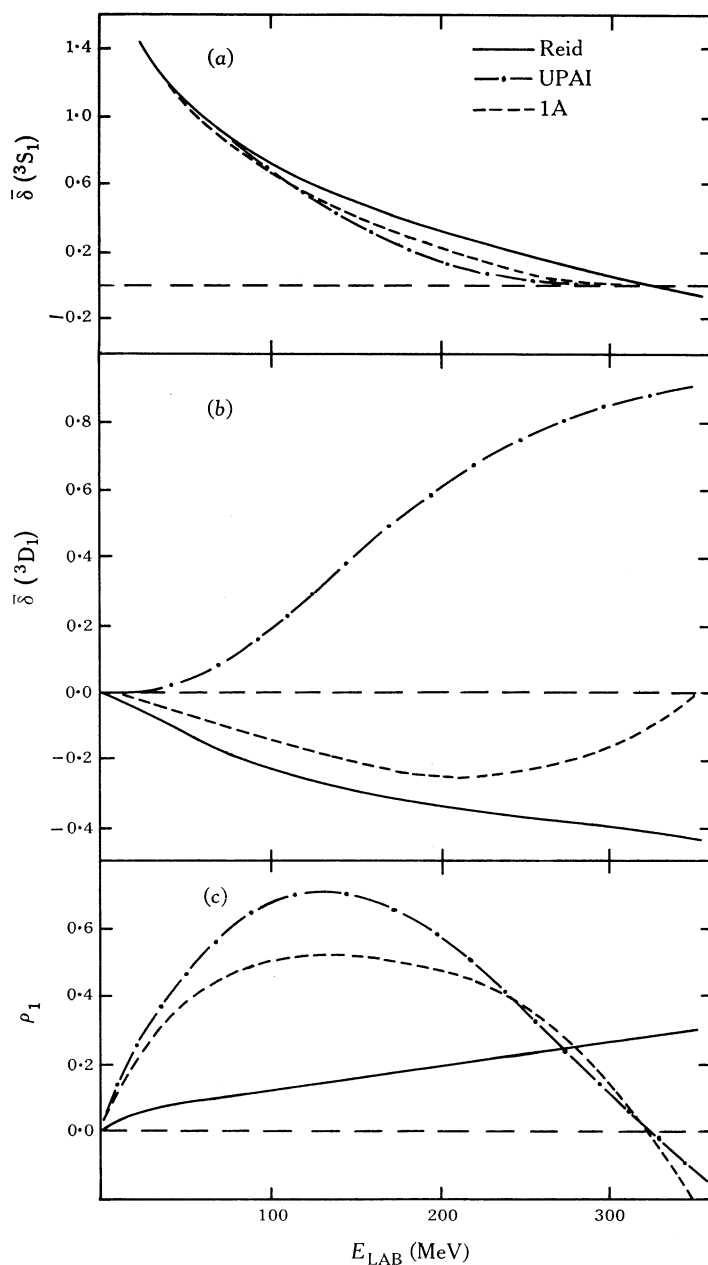


Fig. 1.—Comparison of results for the 3S_1 - 3D_1 channel calculated from the Reid soft-core potential and the UPA potentials UPAI and 1A with strengths given by equations (14) and (19) respectively. Part (a) shows the calculated 3S_1 nuclear bar phase shifts, (b) the 3D_1 nuclear bar phase shifts, and (c) the 3S_1 - 3D_1 coupling parameters as functions of laboratory energy.

parameter is not an important consideration in any calculation which only requires the negative energy two-nucleon T -matrix. It is shown in Section III that a large number of terms must be retained in the UPE to give a good representation of the coupling parameter.

III. UNITARY POLE EXPANSION

In some problems a more accurate separable representation of the T -matrix is required than can be given by the UPA potential. Thus, in the three-nucleon bound state problem, one needs a separable representation of the T -matrix that is sufficiently accurate for use in T -matrix perturbation theory (Afnan and Read 1973) while in the nuclear matter problem, where the UPA and two-term UPE fail at high density ($k \approx 3 \text{ fm}^{-1}$) (Jackson and Lande 1972) one needs a more accurate separable representation of the Brueckner G -matrix for use in calculating three-body correlations. In the present Section, we develop a UPE for coupled channels, with the 3S_1 - 3D_1 partial wave in mind, and test the expansion with the Reid soft-core potential.

The bound state wavefunction in momentum space for the 3S_1 - 3D_1 channel is a solution to the homogeneous integral equation (6). To symmetrize the kernel of this integral equation, we define the states $|\phi_l\rangle$ by

$$|\phi_l\rangle = G_0^{-\frac{1}{2}} |\mu_l\rangle, \quad (20)$$

where $G_0^{-\frac{1}{2}} = (H_0 + B)^{\frac{1}{2}}$, with B the two-body binding energy. The $|\phi_l\rangle$ are then solutions of the equation

$$|\phi_l\rangle = - \sum_{l'} K_{ll'} |\phi_{l'}\rangle, \quad (21)$$

where the kernel $K_{ll'}$ is now hermitian and is given by

$$K_{ll'} = G_0^{\frac{1}{2}} V_{ll'} G_0^{\frac{1}{2}}. \quad (22)$$

Since equation (21) is a homogeneous integral equation with a symmetric kernel, it can be generalized to an eigenvalue problem with real eigenvalues λ_n and eigenvectors $|\phi_l^n\rangle$ to obtain

$$|\phi_l^n\rangle = -\lambda_n \sum_{l'} K_{ll'} |\phi_{l'}^n\rangle, \quad (23)$$

with the normalization chosen such that

$$\sum_l \langle \phi_l^n | \phi_l^m \rangle = \delta_{nm}.$$

Furthermore, the kernel $K_{ll'}$ can be expanded in terms of the eigenfunctions $|\phi_l^n\rangle$ as

$$K_{ll'} = \sum_{n,m=1}^{\infty} |\phi_l^n\rangle C_{ll'}^{nm} \langle \phi_{l'}^m|. \quad (24)$$

To determine the constants $C_{ll'}^{nm}$, we multiply equation (24) by $\langle \phi_l^i|$ from the left and $|\phi_{l'}^j\rangle$ from the right to obtain

$$\langle \phi_l^i | K_{ll'} | \phi_{l'}^j \rangle = \sum_{n,m=1}^{\infty} \langle \phi_l^i | \phi_l^n \rangle C_{ll'}^{nm} \langle \phi_{l'}^m | \phi_{l'}^j \rangle. \quad (25)$$

Equation (25) may be written in matrix form as

$$\mathbf{K}_{ll'} = \mathbf{A}_l \mathbf{C}_{ll'} \mathbf{A}_{l'}, \quad (26)$$

where

$$[\mathbf{A}_l]_{ij} = \langle \phi_l^i | \phi_l^j \rangle, \quad [\mathbf{K}_{ll'}]_{ij} = \langle \phi_l^i | K_{ll'} | \phi_{l'}^j \rangle, \quad [\mathbf{C}_{ll'}]_{ij} = C_{ll'}^{ij}. \quad (27)$$

Since the states $|\phi_l^n\rangle$ are linearly independent, the matrix \mathbf{A}_l has an inverse, which allows us to write the strength matrix $\mathbf{C}_{ll'}$ as

$$\mathbf{C}_{ll'} = \mathbf{A}_l^{-1} \mathbf{K}_{ll'} \mathbf{A}_{l'}^{-1}. \quad (28)$$

Using the definition of the kernel $K_{ll'}$ and equation (24), we can write the potential in a separable form

$$V_{ll'} = \sum_{n,m=1}^{\infty} |\chi_l^n\rangle C_{ll'}^{nm} \langle \chi_{l'}^m|, \quad (29)$$

where the form factors $|\chi_l^n\rangle$ are solutions to the equation

$$|\chi_l^n\rangle = -\lambda_n \sum_{l'} V_{ll'} G_0(-B) |\chi_{l'}^n\rangle, \quad (30)$$

with the normalization

$$\sum_l \langle \chi_l^n | G_0(-B) | \chi_l^m \rangle = \delta_{nm}.$$

We should note that the form factors $|\chi_l^n\rangle$ are related to the states $|\phi_l^n\rangle$ by the relation $|\chi_l^n\rangle = G_0^{-\frac{1}{2}} |\phi_l^n\rangle$. The UPE potential of rank M is then given by

$$V_{ll'}^E = \sum_{n,m=1}^M |\chi_l^n\rangle C_{ll'}^{nm} \langle \chi_{l'}^m|, \quad (31)$$

where the strength matrix is given by equation (28) with the matrices \mathbf{A}_l and $\mathbf{K}_{ll'}$ truncated to $M \times M$ matrices (i.e. in equation (27) $i, j = 1, 2, \dots, M$). This approximation for the UPE potential reduces to the 1A UPA potential for $M = 1$ and $\lambda_1 = 1$.

To show that the T -matrix of the UPE potential in equation (31) has the same position for the pole and the same residue as the T -matrix of the original potential, we have to prove that the deuteron binding energy and bound state form factor for the UPE potential of any rank M are the same as those of the local potential. The proof of these results is indicated in the Appendix. Thus, the T -matrix for the UPE potential is equal to that obtained from the original potential in the neighbourhood of the bound state pole. Furthermore, our numerical results show that the size of the neighbourhood, in which this equality holds, increases with increasing M .

To test the above method of constructing a separable representation for the two-nucleon T -matrix, we examined the Reid soft-core potential in the ${}^3\text{S}_1$ - ${}^3\text{D}_1$ channel. To use the UPE potential defined by equation (31) we need to know the form factors $|\chi_l^n\rangle$ and the strength matrix $\mathbf{C}_{ll'}$. The form factors are obtained from the states $|\phi_l^n\rangle$, which are solutions to the two coupled homogeneous integral equations (23). In momentum space, equations (23) may be reduced to an eigenvalue problem for a symmetric matrix by using Gaussian quadratures to represent the

integrals (Brown *et al.* 1969). With a 32-point Gauss–Legendre quadrature formula and the deuteron binding energy given by Reid (1968) for his potential, we obtain an eigenvalue $\lambda_1 = 1.0000$. Since the potential has both attraction and repulsion, the eigenvalues λ_n are either positive (attractive) or negative (repulsive). To construct the UPE potential, we take M_1 form factors corresponding to the smallest positive eigenvalues and M_2 form factors corresponding to the smallest (in absolute value) negative eigenvalues. Such a choice for the UPE T -matrix is consistent with that used for the 1S_0 potential by Harms (1970) and O'Donoghue and Levinger (1972). We label the UPE potentials by $M_1A + M_2R$ to indicate that M_1 attractive and M_2 repulsive terms have been retained in constructing the potential, with $M = M_1 + M_2$ being the rank of the UPE potential. The evaluation of the strength matrix $C_{ll'}$, as given by equation (28) is now straightforward, since all the required quantities have been determined in solving equation (23) and the only additional computing involved is the inversion of the $M \times M$ matrices A_l .

In comparing the T -matrices obtained from the UPE potential and from the Reid (1968) potential by direct solution of the Lippmann–Schwinger equation in momentum space, we have concentrated on the negative energy region for two reasons: (1) In the three-nucleon bound state problem one needs the two-body T -matrix at negative energies. In particular, it is important to know how many terms must be retained in the UPE potential to obtain a good approximation to the Reid T -matrix. At the same time we want to investigate whether the difference between the UPE and UPA T -matrices is small enough to be treated by T -matrix perturbation theory. (2) In evaluating the Brueckner G -matrix in nuclear matter or finite nuclei, the starting energy is usually about -100 MeV. Although the G -matrix is not considered directly here, it should be kept in mind that the reference G -matrix is very closely related to the two-body T -matrix, and that agreement between the UPE and Reid T -matrices at negative energy is a strong indication of agreement between the UPE and Reid G -matrices. This is confirmed by the success of Jackson and Lande (1972) in reproducing the defect wavefunction for the Reid 1S_0 potential at moderate densities with the UPA potential.

In Table 1 we have tabulated the T -matrix elements $\langle p | T_{ll'}(E) | p \rangle$ for the exact Reid soft-core potential, the 9A+6R and 2A+1R UPE potentials, and the 1A and UPAI potentials as functions of the momentum p for three negative energy values. The results are taken from our calculations of the fully off-shell T -matrix elements $\langle p | T_{ll'}(E) | q \rangle$, so that the validity of our conclusions is independent of the condition $q = p$, which has been chosen here because it allows the results to be presented in a compact form.

It can be seen from Table 1 that the 9A+6R UPE reproduces the exact T -matrix very well, even for large negative energies and high momenta. The agreement is good enough to ensure that the three-nucleon properties are identical for these two T -matrices. Furthermore, the trinucleon energy and wavefunction is mainly sensitive to the matrix element $\langle p | T_{00}(E) | q \rangle$ for moderate values of p , q , and E (Siebert *et al.* 1972), and Table 1 shows that the 2A+1R T -matrix provides a very good approximation to this component of the exact T -matrix. These two facts suggest that the properties of the trinucleon, as predicted by the Reid potential, may be calculated by T -matrix perturbation theory (Fuda 1968; Afnan and Read 1973) using the

TABLE 1
EXACT, UPE, AND UPA T -MATRIX ELEMENTS FOR THE REID SOFT-CORE POTENTIAL

| p fm ⁻¹ | T_{00} (fm ⁻²) | | | | T_{02} (fm ⁻²) | | | | T_{22} (fm ⁻²) | | | | UPAI | | |
|-------------------------|------------------------------|--------|--------|--------|------------------------------|--------|--------|--------|------------------------------|--------|--------|--------|--------|--------|--------|
| | Exact | 9A+6R | 2A+1R | 1A | UPAI | Exact | 9A+6R | 2A+1R | 1A | UPAI | Exact | 9A+6R | | 2A+1R | 1A |
| (a) $E = -10.3675$ MeV | | | | | | | | | | | | | | | |
| 0.2 | -4.352 | -4.347 | -4.350 | -4.344 | -4.368 | -0.120 | -0.115 | -0.092 | -0.088 | -0.084 | 0.002 | 0.001 | -0.001 | -0.001 | -0.002 |
| 0.5 | -3.453 | -3.451 | -3.454 | -3.455 | -3.474 | -0.425 | -0.431 | -0.432 | -0.426 | -0.408 | 0.009 | 0.008 | -0.026 | -0.031 | -0.048 |
| 1.0 | -1.555 | -1.555 | -1.529 | -1.528 | -1.537 | -0.660 | -0.665 | -0.733 | -0.776 | -0.744 | -0.146 | -0.148 | -0.213 | -0.234 | -0.360 |
| 1.5 | -0.256 | -0.255 | -0.273 | -0.300 | -0.302 | -0.464 | -0.466 | -0.478 | -0.497 | -0.476 | -0.444 | -0.446 | -0.480 | -0.488 | -0.750 |
| 2.0 | 0.170 | 0.170 | 0.107 | -0.001 | -0.001 | -0.056 | -0.059 | -0.047 | 0.031 | 0.030 | -0.641 | -0.643 | -0.634 | -0.614 | -0.945 |
| 3.0 | -0.098 | -0.097 | -0.169 | -0.390 | -0.392 | 0.442 | 0.440 | 0.478 | 0.549 | 0.526 | -0.512 | -0.513 | -0.476 | -0.459 | -0.705 |
| (b) $E = -41.47$ MeV | | | | | | | | | | | | | | | |
| 0.2 | -1.958 | -1.952 | -1.969 | -1.951 | -2.029 | -0.085 | -0.081 | -0.052 | -0.045 | -0.039 | 0.003 | 0.002 | -0.000 | -0.000 | -0.001 |
| 0.5 | -1.537 | -1.536 | -1.553 | -1.552 | -1.614 | -0.248 | -0.254 | -0.234 | -0.221 | -0.190 | 0.030 | 0.029 | -0.003 | -0.007 | -0.022 |
| 1.0 | -0.682 | -0.682 | -0.670 | -0.686 | -0.714 | -0.308 | -0.313 | -0.366 | -0.402 | -0.345 | -0.024 | -0.021 | -0.038 | -0.051 | -0.167 |
| 1.5 | -0.059 | -0.058 | -0.086 | -0.135 | -0.140 | -0.225 | -0.228 | -0.238 | -0.257 | -0.221 | -0.069 | -0.071 | -0.107 | -0.106 | -0.348 |
| 2.0 | 0.194 | 0.194 | -0.120 | -0.000 | -0.000 | -0.075 | -0.079 | -0.061 | 0.016 | 0.014 | -0.148 | -0.150 | -0.150 | -0.133 | -0.439 |
| 3.0 | 0.132 | 0.133 | 0.047 | -0.175 | -0.182 | 0.151 | 0.149 | -0.198 | 0.284 | 0.244 | -0.124 | -0.125 | -0.090 | -0.099 | -0.327 |
| (c) $E = -124.41$ MeV | | | | | | | | | | | | | | | |
| 0.2 | -1.202 | -1.196 | -1.240 | -1.201 | -1.385 | -0.080 | -0.075 | -0.043 | -0.036 | -0.027 | 0.003 | 0.002 | 0.000 | -0.000 | -0.001 |
| 0.5 | -0.915 | -0.914 | -0.957 | -0.955 | -1.102 | -0.220 | -0.225 | -0.193 | -0.176 | -0.129 | 0.034 | 0.034 | 0.002 | -0.000 | -0.015 |
| 1.0 | -0.357 | -0.357 | -0.369 | -0.423 | -0.487 | -0.243 | -0.248 | -0.286 | -0.320 | -0.236 | -0.060 | -0.057 | 0.002 | -0.004 | -0.114 |
| 1.5 | 0.064 | 0.065 | 0.010 | -0.083 | -0.096 | -0.178 | -0.181 | -0.183 | -0.205 | -0.151 | 0.017 | 0.015 | -0.020 | -0.008 | -0.238 |
| 2.0 | 0.250 | 0.250 | 0.148 | -0.000 | -0.000 | -0.087 | -0.090 | -0.064 | 0.013 | 0.010 | -0.026 | -0.028 | -0.032 | -0.009 | -0.300 |
| 3.0 | 0.226 | 0.227 | 0.117 | -0.108 | -0.124 | 0.065 | 0.063 | -0.128 | 0.226 | 0.167 | -0.012 | -0.013 | 0.022 | -0.007 | -0.224 |

TABLE 2
EXACT AND UPE BAR PHASE SHIFTS AND COUPLING PARAMETERS FOR REID SOFT-CORE POTENTIAL

| M | $E_{\text{LAB}} = 24$ MeV | | | $E_{\text{LAB}} = 48$ MeV | | | $E_{\text{LAB}} = 144$ MeV | | | $E_{\text{LAB}} = 304$ MeV | | |
|-------|---------------------------|------------|----------|---------------------------|------------|----------|----------------------------|------------|----------|----------------------------|------------|----------|
| | δ_0 | δ_2 | ρ_1 | δ_0 | δ_2 | ρ_1 | δ_0 | δ_2 | ρ_1 | δ_0 | δ_2 | ρ_1 |
| 1A | 1.421 | -0.017 | 0.236 | 1.093 | -0.054 | 0.377 | 0.437 | -0.202 | 0.522 | 0.004 | -0.149 | 0.117 |
| 2A+1R | 1.423 | -0.023 | 0.200 | 1.102 | -0.067 | 0.290 | 0.515 | -0.206 | 0.347 | 0.059 | -0.312 | 0.324 |
| 7A+6R | 1.424 | -0.049 | 0.071 | 1.105 | -0.116 | 0.085 | 0.519 | -0.273 | 0.181 | 0.057 | -0.391 | 0.306 |
| 9A+6R | 1.425 | -0.050 | 0.066 | 1.105 | -0.115 | 0.086 | 0.520 | -0.279 | 0.161 | 0.055 | -0.395 | 0.296 |
| Exact | 1.426 | -0.050 | 0.064 | 1.105 | -0.115 | 0.081 | 0.521 | -0.281 | 0.152 | 0.057 | -0.403 | 0.269 |

2A + 1R T -matrix to determine the unperturbed system and the 9A + 6R T -matrix to approximate the exact T -matrix.

A comparison of the 1A, UPAI, and exact T -matrix elements $\langle p | T_{22}(E) | p \rangle$ for the 3D_1 channel again provides support for our removal of the constraint (13) in the UPA case. If the UPAI and 1A T -matrices were to be considered as reference G -matrices then the former would give considerably more attraction in nuclear matter. In fact, the binding energy and saturation density are highly dependent on the sign of the 3D_1 contribution, which is repulsive for all realistic potentials (Haftel and Tabakin 1970) but attractive for most one-term separable potentials satisfying the condition (13).

The domain of agreement between the exact and UPE T -matrices as a function of the rank M of the expansion can be roughly estimated from Table 1. It is evident that, with increasing M , one can move further from the pole and still reproduce the exact T -matrix to reasonable accuracy. This is clearly evident in the values of T_{02} where, for $E = -10.3675$ MeV, the agreement between the 1A and exact T -matrices is reasonably good but, at $E = -124.41$ MeV, more than one term is required in the UPE for a comparable degree of accuracy.

A more detailed comparison of the exact and UPE T -matrices is presented in Table 2, in which the bar phase shifts and coupling parameter are given for the exact and UPE potentials for different values of M . We find that, for the 1A potential, the coupling parameter ρ_1 and 3D_1 phase shifts change sign at high energy but that this effect is removed by the addition of one repulsive term (in the 2A + 1R potential). As more terms are included in the UPE potential, the UPE values for the phase shifts and coupling parameter approach the exact values. The agreement is reasonably good for the 9A + 6R UPE potential although there are still discrepancies between the coupling parameters at high energies.

The Reid soft-core potential was used here because it provides the best fit to the two-nucleon data of all the current phenomenological two-nucleon potentials (Reid 1968). However, the main feature of the N-N interaction which makes the UPE approach so attractive is the importance of the deuteron wavefunction in determining the negative energy T -matrix. The pole dominance argument (Lovelace 1964) is applicable to any N-N potential and this suggests that the UPE will have more general application to potentials other than the Reid soft-core potential. Preliminary results for the one-boson exchange potential (Bryan and Scott 1969) and the Hamada-Johnston potential (Hamada and Johnston 1962) strengthen this conclusion.

IV. ACKNOWLEDGMENTS

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APPENDIX

Derivation of Equation (12) for the UPA Potential

The T -matrix corresponding to the one-term separable potential (11) can be written as

$$T_{ll'}^A(E) = |\chi_l\rangle \{N_{ll'}(E)/D(E)\} \langle\chi_{l'}|, \quad (\text{A1})$$

where, in the 3S_1 - 3D_1 channel, $N_{ll'}(E)$ and $D(E)$ are given by

$$\left. \begin{aligned} N_{00}(E) &= C_{00} + (C_{00} C_{22} - C_{02}^2) \langle\chi_2| G_0(E) |\chi_2\rangle, \\ N_{22}(E) &= C_{22} + (C_{00} C_{22} - C_{02}^2) \langle\chi_0| G_0(E) |\chi_0\rangle, \\ N_{02}(E) &= N_{20}(E) = C_{02}, \\ D(E) &= 1 + C_{00} \langle\chi_0| G_0(E) |\chi_0\rangle + C_{22} \langle\chi_2| G_0(E) |\chi_2\rangle \\ &\quad + (C_{00} C_{22} - C_{02}^2) \langle\chi_0| G_0(E) |\chi_0\rangle \langle\chi_2| G_0(E) |\chi_2\rangle. \end{aligned} \right\} \quad (\text{A2})$$

To determine the strengths $C_{ll'}$ from the condition

$$T_{ll'}^A(-B) = T_{ll'}(-B), \quad (\text{A3})$$

we consider the neighbourhood of the bound state pole, for which $E = -B + \varepsilon$ with ε small. To first order in ε , we have

$$\left. \begin{aligned} G_0(E) &= G_0(-B) + \varepsilon G_0^2(-B), \\ D(E) &= D(-B) + \varepsilon d(-B), \\ N_{ll'}(E) &= N_{ll'}(-B) + \varepsilon (C_{00} C_{22} - C_{02}^2) \langle\chi_{l'}| G_0^2(-B) |\chi_{l'}\rangle, \end{aligned} \right\} \quad (\text{A4})$$

where $l' = |l-2|$ and

$$\begin{aligned} d(-B) &= C_{00} \langle\chi_0| G_0^2(-B) |\chi_0\rangle + C_{22} \langle\chi_2| G_0^2(-B) |\chi_2\rangle \\ &\quad + (C_{00} C_{22} - C_{02}^2) \{ \langle\chi_2| G_0(-B) |\chi_2\rangle \langle\chi_0| G_0^2(-B) |\chi_0\rangle \\ &\quad + \langle\chi_0| G_0(-B) |\chi_0\rangle \langle\chi_2| G_0^2(-B) |\chi_2\rangle \}. \end{aligned} \quad (\text{A5})$$

The condition that $T_{II'}^A(E)$ have the same position for the bound state pole and the same residue as $T_{II'}(E)$ requires that

$$D(-B) = 0 \quad \text{and} \quad \eta^2 N_{II'}(-B) = d(-B), \quad (\text{A6})$$

where the normalization constant η of equations (7) and (10) is determined from

$$\eta^2 = \sum_I \langle \chi_I | G_0^2(-B) | \chi_I \rangle.$$

The conditions (A6) can now be rewritten as the set of three nonlinear equations (12) for the strengths $C_{II'}$ of the potential V^A .

Deuteron Binding Energy and Bound State Form Factor for UPE Potential

We may write the UPE potential (31) in the form

$$V_{II'}^E = |\chi\rangle \Delta_I \mathbf{C} \Delta_{I'} \langle \chi|, \quad (\text{A7})$$

where $|\chi\rangle$ represents a row vector of form factors

$$|\chi\rangle = (|\chi_0^1\rangle, |\chi_0^2\rangle, \dots, |\chi_0^M\rangle, |\chi_2^1\rangle, |\chi_2^2\rangle, \dots, |\chi_2^M\rangle)$$

and $\langle \chi|$ is the corresponding column vector. The $2M \times 2M$ matrices \mathbf{C} and Δ_I are given by

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{00} & \mathbf{C}_{02} \\ \mathbf{C}_{20} & \mathbf{C}_{22} \end{bmatrix} \quad \text{and} \quad \Delta_I = \begin{bmatrix} \mathbf{I} \delta_{I0} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \delta_{I2} \end{bmatrix}, \quad (\text{A8})$$

which are formed from the $M \times M$ matrices $C_{II'}$ of equation (31) and the M -dimensional unit \mathbf{I} and zero \mathbf{O} matrices respectively. The Lippmann-Schwinger equation for the T -matrix corresponding to the potential (A7) may be solved in a closed form and the solution $T_{II'}^E(E)$ written as

$$T_{II'}^E(E) = |\chi\rangle \Delta_I \mathbf{C} \mathbf{M}^{-1}(E) \Delta_{I'} \langle \chi|, \quad (\text{A9})$$

where

$$\mathbf{M}(E) = \mathbf{I} + \sum_L \Delta_L \langle \chi | G_0(E) | \chi \rangle \Delta_L \mathbf{C}. \quad (\text{A10})$$

The binding energy of the deuteron E_D for the potential (A7) is determined from the equation

$$\det[\mathbf{M}(-E_D)] = 0. \quad (\text{A11})$$

If we multiply both sides of the equation conjugate to equation (30) by $G_0(-B) |\chi_i^{(m)}\rangle$ we obtain the relationships

$$-\lambda^{-1} = \mathbf{A}_0 \mathbf{C}_{00} + \mathbf{A}_2 \mathbf{C}_{20} = \mathbf{A}_0 \mathbf{C}_{02} + \mathbf{A}_2 \mathbf{C}_{22} \quad (\text{A12})$$

between the $M \times M$ matrices \mathbf{A}_I and $\mathbf{C}_{II'}$ defined in equation (27) and the matrix λ^{-1} defined as

$$[\lambda^{-1}]_{ij} = \delta_{ij} / \lambda_i \quad \text{for} \quad i, j = 1, \dots, M.$$

Using the relationships (A12) and the fact that $\lambda_1 = 1$, it can be shown that, for $E = -B$, the first and $(M+1)$ th rows of $\mathbf{M}(E)$ in equation (A10) differ only by a multiplicative constant of minus one. Thus

$$\det[\mathbf{M}(-B)] = 0 \quad (\text{A13})$$

and the UPE potential (31) produces the same deuteron binding energy as the original local potential V .

To prove that the UPE T -matrix has the same residue at $E = -B$ as the T -matrix for the potential V we must show that the bound state form factors for the UPE potential are identical to those for the potential V . Consider the equation for the UPE bound state form factors $|\chi_I^E\rangle$, where

$$\begin{aligned} |\chi_I^E\rangle &= -\sum_{I'} V_{II'}^E G_0(-B) |\chi_{I'}^E\rangle \\ &= -\sum_{I'} \sum_{n,m=1}^M |\chi_I^n\rangle C_{II'}^{nm} \langle \chi_{I'}^m | G_0(-B) |\chi_I^E\rangle. \end{aligned} \quad (\text{A14})$$

The solution of equation (A14) may be written as

$$|\chi_I^E\rangle = \sum_{n=1}^M a_n |\chi_I^n\rangle, \quad (\text{A15})$$

where

$$a_n = -\sum_{I'} \sum_{m=1}^M C_{II'}^{nm} \langle \chi_{I'}^m | G_0(-B) |\chi_I^E\rangle. \quad (\text{A16})$$

From equations (A15), (A16), (28), and (30) we obtain

$$a_n(1 - \lambda_n^{-1}) = 0. \quad (\text{A17})$$

Since $\lambda_n = 1$ implies $n = 1$, equation (A17) requires that $a_n = 0$ for $n = 2, \dots, M$. From the normalization condition

$$\sum_I \langle \chi_I^E | G_0(-B) |\chi_I^E\rangle = 1$$

we determine that $a_1^2 = 1$ and thus

$$|\chi_I^E\rangle \equiv |\chi_I^1\rangle, \quad (\text{A18})$$

where the $|\chi_I^1\rangle$ are the bound state form factors for the potential V .