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Statistics for Adler–Adler Resonance Parameters

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Abstract

An elementary statistical theory for Adler–Adler resonance parameters is derived and the experimental results for ^{233}U and ^{235}U are analysed. Satisfactory agreement is obtained for all parameters.

1. Introduction

Since the introduction of the single level approximation by Breit and Wigner (1936) much attention has been devoted to the statistical analysis of resonance parameters. As long as levels remain well separated, it matters little whether we are discussing the parameters of the Wigner–Eisenbud (1947) reaction matrix or a version of the Kapur–Peierls (1938) theory, as the parameters in these theories are approximately equal. However, for fissile nuclides, resonance overlap becomes appreciable and we must employ a multilevel theory such as that of Vogt (1958), Reich and Moore (1958) or, more recently, Adler and Adler (1963).

Recent work by Bertram and Cook (1972) has thrown some doubt upon the hitherto accepted assumption that it is the parameters of the Wigner–Eisenbud reaction matrix that possess the elementary statistical properties such as the Wigner distribution of level spacings (Mehta and Gaudin 1960), the Porter–Thomas (1956) distribution of neutron widths and the chi-squared distribution of fission widths. It is the purpose of this paper to investigate some alternative hypotheses in a preliminary way, to ascertain if such studies are worth pursuing and to obtain the practical result of deriving expected distributions for the widely used Adler–Adler resonance parameters. These would be of great assistance in determining the response of reactors to temperature changes, which produce Doppler broadening of resonances, amongst other effects.

2. Adler–Adler Parameters

Extensive neutron cross-section fits to parameters of the Adler–Adler type have been made recently (de Saussure and Perez 1969; de Saussure *et al.* 1970). In this formalism the neutron cross sections are given by

$$\sigma_{\mathrm{nt}} = 2\pi\lambda\!\!\!^{-2} g\{1-\cos(2ka)\} + E^{-\frac{1}{2}} \sum_{\lambda} \frac{\nu_\lambda G_\lambda^{\mathrm{t}} + (\varepsilon_\lambda - E)H_\lambda^{\mathrm{t}}}{(\varepsilon_\lambda - E)^2 + \nu_\lambda^2}, \tag{1}$$

$$\sigma_{\mathrm{na}} = E^{-\frac{1}{2}} \sum_{\lambda} \frac{\nu_\lambda G_\lambda^{\mathrm{a}} + (\varepsilon_\lambda - E)H_\lambda^{\mathrm{a}}}{(\varepsilon_\lambda - E)^2 + \nu_\lambda^2}, \qquad \sigma_{\mathrm{nf}} = E^{-\frac{1}{2}} \sum_{\lambda} \frac{\nu_\lambda G_\lambda^{\mathrm{f}} + (\varepsilon_\lambda - E)H_\lambda^{\mathrm{f}}}{(\varepsilon_\lambda - E)^2 + \nu_\lambda^2}, \tag{2a, b}$$

$$\sigma_{\mathrm{ns}} = \sigma_{\mathrm{nt}} - \sigma_{\mathrm{na}}, \qquad \sigma_{\mathrm{n}\gamma} = \sigma_{\mathrm{na}} - \sigma_{\mathrm{nf}}, \tag{3a, b}$$

where

$$G_\lambda^{\mathrm{t}} = \alpha_\lambda \cos(2ka) + \beta_\lambda \sin(2ka), \qquad H_\lambda^{\mathrm{t}} = \beta_\lambda \cos(2ka) - \alpha_\lambda \sin(2ka). \tag{4a, b}$$

σ_{nt}, σ_{na}, σ_{nf}, σ_{ns} and $\sigma_{\mathrm{n}\gamma}$ are respectively the total, absorption, fission, scattering and capture cross sections, k is the neutron momentum and a is the nuclear radius. The quantities G_λ^c and H_λ^c are the symmetric and asymmetric Adler–Adler parameters for channel c, α and β being the corresponding scattering parameters after hard sphere effects are removed, while ν_λ and ε_λ are respectively the imaginary and real parts of the λth pole in the collision matrix and g is the spin weight factor.

Following Cook (1972), we note that the above forms arise from the expressions for the cross section

$$\sigma_{cc'} = (\pi/k_c^2) \sum_{ljJj'l'} g_J \, |\delta_{clj,c'l'j'} - S_{clj,c'l'j'}|^2, \tag{5}$$

The matrix $\mathbf{S}$ for each partial wave is given by

$$\mathbf{S} = \mathbf{\Omega}[\mathbf{I} + 2\mathrm{i}\mathbf{P}^{\frac{1}{2}}(\mathbf{I} - \mathbf{R}\mathbf{L}_0)^{-1}\,\mathbf{R}\mathbf{P}^{\frac{1}{2}}]\mathbf{\Omega}, \tag{6}$$

where

$$R_{cc'} = \sum_\lambda \gamma_{\lambda c}\gamma_{\lambda c'}/(E_\lambda - E), \tag{7}$$

with E_λ the poles in the reaction matrix and $\gamma_{\lambda c}$ the partial widths; and for s-waves

$$\Omega_{cc'} = \exp(-2\mathrm{i}k_c a)\,\delta_{cc'}, \tag{8}$$

$$\mathbf{P} = P_c\,\delta_{cc'} = k_c a\,\delta_{cc'} \quad \text{for scattering,} \tag{9a}$$

$$= \delta_{cc'} \quad \text{otherwise} \tag{9b}$$

and

$$\mathbf{L}_0 = \mathbf{S} + \mathrm{i}\mathbf{P}. \tag{10}$$

In the Kapur–Peierls theory we represent the transition matrix

$$\mathbf{I} = (\mathbf{S} - \mathbf{I})/2\mathrm{i}$$

by a sum of poles

$$T_{cc'} = E^{-\frac{1}{2}} \sum_\lambda \frac{g_{\lambda c}\, g_{\lambda c'}}{\varepsilon_\lambda + \mathrm{i}\nu_\lambda - E} + T_{cc'}^0, \tag{11}$$

where the $g_{\lambda c}$ and $(\varepsilon_\lambda, \nu_\lambda)$ are slowly varying functions of energy and $T_{cc'}^0$ is a background term, assumed constant. Except for the energy dependence of the width in the scattering channel, we shall assume all parameters to be constant; this is an excellent approximation to the actual situation in intermediate and heavy isotopes. The problem is, therefore, to find the statistical distributions of the G_λ^c and H_λ^c in equations (1) and (2) given the above information.

The transformations which relate the above theories are reported by Lane and Thomas (1958). They split the $\mathbf{R}$ matrix into a nonresonant and resonant part respectively

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{R}', \tag{12}$$

use the identity

$$(\mathbf{I} - \mathbf{R}\mathbf{L}_0)^{-1}\mathbf{R} = (\mathbf{I} - \mathbf{R}_0\mathbf{L}_0)^{-1}\mathbf{R}_0 + (\mathbf{I} - \mathbf{R}_0\mathbf{L}_0)^{-1}(\mathbf{I} - \mathbf{R}'\mathbf{L}')^{-1}\mathbf{R}'(\mathbf{I} - \mathbf{L}_0\mathbf{R}_0)^{-1} \tag{13}$$

and assume an expansion

$$(\mathbf{I}-\mathbf{R}'\mathbf{L}')^{-1}_{cc'} = \delta_{cc'} + \sum_{\mu,\nu} \gamma_{\mu c'} \beta_{\nu c} A_{\mu\nu}(E), \tag{14}$$

where

$$\beta_{\nu c} = \mathbf{L}'_c \gamma_{\nu c}, \qquad \mathbf{L}' = \mathbf{L}_0(\mathbf{I}-\mathbf{R}_0\mathbf{L}_0)^{-1}.$$

It can then be shown that

$$(\mathbf{I}-\mathbf{R}'\mathbf{L}')^{-1}\mathbf{R}' = \sum_{\lambda,\mu} \gamma_{\lambda c}\gamma_{\mu c'} A_{\lambda\mu}(E), \tag{15}$$

where the eigenvalues of the level matrix

$$\mathbf{A} = (\mathbf{e}-\mathbf{E}-\boldsymbol{\xi})^{-1} \tag{16a}$$

give the complex resonance energies (with $h_\lambda = \varepsilon_\lambda - \mathrm{i}\nu_\lambda$)

$$\mathbf{E} = E\delta_{\lambda\mu}, \tag{16b}$$

$$\boldsymbol{\xi} = \sum_c \beta_{\lambda c}\gamma_{\mu c}, \tag{16c}$$

$$\mathbf{e} = E_\lambda \delta_{\lambda\mu}. \tag{16d}$$

According to (15), equation (6) becomes

$$S_{cc'} = \Omega_c\left[\delta_{cc'} + 2\mathrm{i}P_c^{\frac{1}{2}} \sum_{\lambda,\mu} \alpha_{\lambda c}\alpha_{\mu c'} A_{\lambda\mu}(E) P_{c'}^{\frac{1}{2}}\right]\Omega_{c'} \tag{17}$$

with

$$\gamma_{\lambda c} = (\mathbf{I}-\mathbf{R}_0\mathbf{L}_0)_{cc'}\alpha_{\lambda c'}. \tag{18}$$

Lane and Thomas show that a complex orthogonal matrix $\mathbf{U}$ exists which transforms the symmetric matrix $(\mathbf{e}-\boldsymbol{\xi})$ to a diagonal form $\mathbf{H}$, that is,

$$\mathbf{H} = \boldsymbol{\xi} - \mathrm{i}\boldsymbol{\nu} = \mathbf{U}(\mathbf{e}-\boldsymbol{\xi})\mathbf{U}^{\mathrm{T}}, \tag{19}$$

where the operation T denotes transposition. The same matrix obeys the relations

$$\omega_{\lambda c} = \sum_\nu U_{\lambda\nu}\gamma_{\nu c}, \qquad \theta_{\lambda c} = (\mathbf{I}-\mathbf{R}_0\mathbf{L}_0)^{-1}_{cc'}\omega_{\lambda c'} = \sum_\nu U_{\lambda\nu}\alpha_{\nu c} \tag{20}$$

to yield

$$S_{cc'} = \Omega_c\left[\delta_{cc'} + 2\mathrm{i}\sum_\lambda \frac{P_c^{\frac{1}{2}}\theta_{\lambda c}\theta_{\lambda c'}P_{c'}^{\frac{1}{2}}}{\varepsilon_\lambda - \mathrm{i}\nu_\lambda - E} + S^0_{cc'}\right]\Omega_{c'}, \tag{21}$$

where

$$\mathbf{S}^0 = (\mathbf{I}-\mathbf{R}_0\mathbf{L}_0)^{-1}\mathbf{R}_0,$$

$$\sum_{\lambda,\mu} \alpha_{\lambda c}\alpha_{\mu c'} A_{\lambda\mu}(E) = \sum_\lambda \frac{\theta_{\lambda c}\theta_{\lambda c'}}{\varepsilon_\lambda - \mathrm{i}\nu_\lambda - E} \tag{22}$$

and

$$\mathbf{A} = \mathbf{U}^{\mathrm{T}}(\mathbf{H}-\mathbf{E})^{-1}\mathbf{U}. \tag{23}$$

Putting

$$g_{\lambda c} = \theta_{\lambda c} P_c^{\frac{1}{2}}(E = 1\ \mathrm{eV})/P_c^{\frac{1}{2}}(E), \tag{24}$$

we arrive at the form (11) and obtain approximately energy-independent values for $g_{\lambda c}$.

Neglecting the background terms, we find in the reaction channels

$$\sigma_{cc'} = 4\pi\lambda\!\!\!^{-2} \sum \left(\frac{Z^*_{\lambda cc'}}{h^*_\lambda - E} + \frac{Z_{\lambda cc'}}{h_\lambda - E}\right) E^{\frac{1}{2}}, \tag{25}$$

where

$$Z_{\lambda cc'} = g_{\lambda c} g_{\lambda c'} \sum g^*_{\mu c} g^*_{\mu c'}/(h^*_\mu - h_\lambda). \tag{26}$$

Using the relation

$$\frac{Z^*_{\lambda cc'}}{h^*_\lambda - E} + \frac{Z_{\lambda cc'}}{h_\lambda - E} = \frac{(h_\lambda - E)Z^*_{\lambda cc'} + (h^*_\lambda - E)Z_{\lambda cc'}}{|h_\lambda - E|^2}, \tag{27}$$

we find on comparison with the cross sections (2), after summing over c' channels up to α,

$$E^{-\frac{1}{2}} G^\alpha_\lambda = 8\pi\lambda\!\!\!^{-2} \sum Y_{\lambda cc'} E^{\frac{1}{2}}, \qquad Z = X + \mathrm{i}Y. \tag{28}$$

By removing the penetration-factor dependence upon energy in the definition (24) we have made the $X_{\lambda cc'}$ and $Y_{\lambda cc'}$ approximately constant. Thus we obtain

$$G^\alpha_\lambda = (\hbar^2/2m) \sum Y_{\lambda cc'}, \qquad H^\alpha_\lambda = (\hbar^2/2m) \sum X_{\lambda cc'}. \tag{29}$$

3. Statistics

The information that we normally assume to be given is that the level spacings D in the *reaction matrix* are distributed according to the Wigner formula (Mehta 1967)

$$(\pi D/2\langle D\rangle^2) \exp(-\pi D^2/4\langle D\rangle^2)\, \mathrm{d}D, \tag{30}$$

and that the reduced widths $\gamma^2_{\lambda c}$, again of the reaction matrix, are distributed according to the Porter–Thomas (1956) distribution

$$(2\pi)^{-\frac{1}{2}} \langle\gamma^2\rangle^{-1} \exp(-\gamma^2/2\langle\gamma^2\rangle)\, \mathrm{d}\gamma^2. \tag{31}$$

This distribution together with the expected χ^2 distribution (Lynn 1968) for n degrees of freedom, namely

$$\chi^2_n(x) = \{\Gamma(\tfrac{1}{2}n)\}^{-1} (n/2\bar{x})^{\frac{1}{2}n} x^{\frac{1}{2}n-1} \exp(-nx/2\bar{x}), \tag{32}$$

where $\bar{x}$ is the mean value of x, can be derived by postulating that the reduced widths γ_λ have a normal distribution

$$P(\gamma)\,\mathrm{d}\gamma = (2\pi)^{-\frac{1}{2}} \langle\gamma^2\rangle^{-1} \exp(-\gamma^2/2\langle\gamma^2\rangle)\, \mathrm{d}\gamma \tag{33}$$

with zero mean.

At this stage we make some approximations to remove complications which do not affect the statistics of the Adler–Adler parameters appreciably. In equation (18), the background matrix $\mathbf{R}_0 \mathbf{L}_0$ is usually small and so we put

$$\alpha_{\lambda c} \approx \gamma_{\lambda c}. \tag{34}$$

Using the approximation given by Moldauer (1964), we can write

$$U_{\lambda\mu} \approx \delta_{\lambda\mu} + \mathrm{i}\,\xi^0_{\lambda\mu}/(\varepsilon_\mu - \xi^{\mathrm{D}}_\mu - \varepsilon_\lambda + \mathrm{i}\xi^{\mathrm{D}}_\lambda), \tag{35}$$

where $\xi^0_{\lambda\mu}$ is the matrix defined in equation (16c) with the diagonal components zero and

$$\xi^{\mathrm{D}}_\mu = \xi_{\mu\mu}. \tag{36}$$

Since, under ordinary conditions, the second term in the approximation (35) will be considerably less than the first, we might find, upon investigating the distributions of $g_{\lambda n}$, that the real part of $g_{\lambda n}$ is distributed as a Gaussian according to equation (33), that is,

$$P(g_{nr}) = (2\pi)^{-\frac{1}{2}}\langle g_{nr}^2\rangle^{-1}\exp(-g_{nr}^2/2\langle g_{nr}^2\rangle), \tag{37}$$

where

$$g_{nr} = \mathrm{Re}\, g_n \approx (\Gamma_n^0)^{\frac{1}{2}}. \tag{38}$$

However, it was demonstrated by Bertram and Cook (1972) that there is no justification for assuming the Porter–Thomas distribution for the reaction matrix reduced widths when we have an arbitrary set of boundary conditions.

In the single level approximation, both the neutron widths and the reaction matrix neutron widths are distributed approximately like the Porter–Thomas function. We shall examine the equivalent situation in the multilevel theory in this paper. We introduce the following two postulates for our investigation of the Adler–Adler parameters:

(I) It is the real part of the Kapur–Peierls reduced width at the corresponding resonance energy which has a Gaussian distribution with zero mean.

(II) The imaginary part also has a Gaussian distribution with zero mean, but has a variance different from and usually much smaller than the variance of the real part.

Unitarity undoubtedly leads to correlations (Lynn 1968), but we should find these to be weak for a very large number of levels and should correlate real with imaginary widths rather than real with real widths.

Let us derive the consequences of axiom I concerning the distribution of G^c and H^c. Making use of equations (26) and (29), we see that each parameter can be written in the fission and capture channels respectively as

$$G_\lambda^f \propto (\Gamma_{\lambda n}^0)^{\frac{1}{2}}\{F_\lambda^f + \Gamma_\lambda^f(\Gamma_n^0)^{\frac{1}{2}}\}, \qquad G_\lambda^\gamma \propto (\Gamma_{\lambda n}^0)^{\frac{1}{2}}\{F_\lambda^\gamma + \Gamma_\lambda^\gamma(\Gamma_n^0)^{\frac{1}{2}}\}, \tag{39a, b}$$

$$H_\lambda^f \propto (\Gamma_{\lambda n}^0)^{\frac{1}{2}} K_\lambda^f, \qquad H_\lambda^\gamma \propto (\Gamma_{\lambda n}^0)^{\frac{1}{2}} K_\lambda^\gamma, \tag{39c, d}$$

where $\Gamma_{\lambda n}^0$ is the neutron width of the resonance, F_λ^c is the background sum of terms in equation (26) with $\mu \neq \lambda$ occurring for G_λ^c, K_λ^c is the background sum of terms in (26) with $\mu \neq \lambda$ occurring for H_λ^c, and Γ_λ^f and Γ_λ^γ are the fission and radiation widths. Here we introduce a third postulate to be tested by experiment:

(III) The quantities

$$X_{\lambda cc'}^N = \mathrm{Re}\left(g_{\lambda c}g_{\lambda c'}\sum_{\mu\neq\lambda} g_{\mu c}^* g_{\mu c'}^*/(h_\mu^* - h_\lambda)\right), \tag{40a}$$

$$Y_{\lambda cc'}^N = \mathrm{Im}\left(g_{\lambda c}g_{\lambda c'}\sum_{\mu\neq\lambda} g_{\mu c}^* g_{\mu c'}^*/(h_\mu^* - h_\lambda)\right) \tag{40b}$$

are ideally infinite sums and the correlation with individual terms in the sum becomes lost when all terms are included.

We therefore postulate that after the summations

$$\sum_c^{\mathrm{f}} X_{\lambda\mathrm{n}c'} = K_\lambda^{\mathrm{f}}, \qquad \sum_c^{\gamma} X_{\lambda\mathrm{n}c'} = K_\lambda^{\gamma}, \tag{41a, b}$$

$$\sum_c^{\mathrm{f}} Y_{\lambda\mathrm{n}c'} = F_\lambda^{\mathrm{f}}, \qquad \sum_c^{\gamma} Y_{\lambda\mathrm{n}c'} = F_\lambda^{\gamma}, \tag{41c, d}$$

the quantities K_λ^{f}, K_λ^{γ}, F_λ^{f} and F_λ^{γ} have normal distributions with zero mean. We also anticipate that, since F_λ^{f} and F_λ^{γ} are sums of terms of alternating sign, whereas the second respective terms in equations (39a) and (39b) are usually large positive quantities, then

$$\langle F_\lambda^{\mathrm{f}}\rangle_\lambda \ll \langle \Gamma_\lambda^{\mathrm{f}}(\Gamma_{\lambda\mathrm{n}}^{0})^{\frac{1}{2}}\rangle, \qquad \langle F_\lambda^{\gamma}\rangle_\lambda \ll \langle \Gamma_\lambda^{\gamma}(\Gamma_{\lambda\mathrm{n}}^{0})^{\frac{1}{2}}\rangle. \tag{42a, b}$$

Therefore, we neglect the effect of the F_λ^{c}'s on the predicted distributions. We also suppose that the Γ_λ^{c}'s are approximately real. It follows from equations (41) and (37) that we expect H_λ^{f} and H_λ^{γ} to be given by the distribution that is expected for the product of two variates with a Gaussian distribution. This is readily shown to be

$$P(H_{\mathrm{f}}) = \frac{1}{\pi\langle H_{\mathrm{f}}^2\rangle^{\frac{1}{2}}}\mathrm{K}_0\left(\frac{|H_{\mathrm{f}}|}{\langle H_{\mathrm{f}}^2\rangle^{\frac{1}{2}}}\right), \qquad P(H_\gamma) = \frac{1}{\pi\langle H_\gamma^2\rangle^{\frac{1}{2}}}\mathrm{K}_0\left(\frac{|H_\gamma|}{\langle H_\gamma^2\rangle^{\frac{1}{2}}}\right), \tag{43}$$

where $\mathrm{K}_0(x)$ is the associated Bessel function (Erdelyi 1953). We shall test this result in the next section. To find the distribution of the G_λ^{c}, we note from equations (39a, b) and (42a, b) that we should find a fairly simple distribution for the quantities

$$\Gamma_\lambda^{\mathrm{f}} \approx G_\lambda^{\mathrm{f}}/\Gamma_{\lambda\mathrm{n}}^{0}, \qquad \Gamma_\lambda^{\gamma} = G_\lambda^{\gamma}/\Gamma_{\lambda\mathrm{n}}^{0}, \tag{44}$$

since these are given by the sums of squares of quantities drawn from a Gaussian distribution in our approximation. Then we should find

$$P(\Gamma_{\mathrm{f}}) = \{\Gamma(\tfrac{1}{2}N)\}^{-1}(N/2\langle\Gamma_{\mathrm{f}}\rangle)^{\frac{1}{2}N}(\Gamma_{\mathrm{f}})^{\frac{1}{2}N-1}\exp(-N\Gamma_{\mathrm{f}}/2\langle\Gamma_{\mathrm{f}}\rangle), \tag{45}$$

which is the chi-squared distribution for N degrees of freedom and N is the number of fission channels (Lynn 1968). Since there are usually a large number of capture channels, we expect the $\Gamma_{\lambda\gamma}$ to be distributed as a Gaussian distribution with mean $\langle\Gamma_\gamma\rangle$ and very small variance.

Finally, we shall test the hypothesis that $\Gamma_{\lambda\mathrm{n}}^{0}$ indeed follows a Porter–Thomas distribution. Using the Adler–Adler parameters G_λ^{t} and H_λ^{t} defined in the total cross section (1) and making use of equations (25) and (26), we find that for the true complex $g_{\lambda\mathrm{n}}$,

$$H_\lambda^{\mathrm{t}} - \mathrm{i}G_\lambda^{\mathrm{t}} = g_{\lambda\mathrm{n}}^2 \approx \Gamma_{\lambda\mathrm{n}}^{0}. \tag{46}$$

This yields

$$g_{\lambda\mathrm{nr}} = [\tfrac{1}{2}H_\lambda^{\mathrm{t}} + \tfrac{1}{2}\{(H_\lambda^{\mathrm{t}})^2 + (G_\lambda^{\mathrm{t}})^2\}^{\frac{1}{2}}]^{\frac{1}{2}}, \qquad g_{\lambda\mathrm{ni}} = -\tfrac{1}{2}G_\lambda^{\mathrm{t}}[\tfrac{1}{2}H_\lambda^{\mathrm{t}} + \tfrac{1}{2}\{(H_\lambda^{\mathrm{t}})^2 + (G_\lambda^{\mathrm{t}})^2\}^{\frac{1}{2}}]^{-\frac{1}{2}} \tag{47}$$

for the real and imaginary parts of $g_{\lambda\mathrm{n}}$. We note that in the single level approximation the quantity usually designated as $\Gamma_{\lambda\mathrm{n}}^{0}$ is really

$$\Gamma_{\lambda\mathrm{n}}^{\prime 0} = g_{\lambda\mathrm{nr}}^2 - g_{\lambda\mathrm{ni}}^2 = \mathrm{Re}(g_{\lambda\mathrm{n}}^2). \tag{48}$$

If $P(g_{nr})$ has variance σ_1^2 and $P(g_{ni})$ has variance σ_2^2 and each are Gaussian distributions with zero mean, then we can readily show that

$$P(\Gamma_n'^0)\,\mathrm{d}\Gamma_n'^0 = \frac{1}{\pi\sigma_1\sigma_2}\exp\left\{\frac{\Gamma_n'^0}{2}\left(\frac{1}{2\sigma_2^2}-\frac{1}{2\sigma_1^2}\right)\right\}\mathrm{K}_0\left\{\frac{\Gamma_n'^0}{2}\left(\frac{1}{2\sigma_1^2}+\frac{1}{2\sigma_2^2}\right)\right\}\mathrm{d}\Gamma_n'^0\,. \tag{49}$$

If σ_2^2 is very small, as we normally assume, then for $\sigma_2 \to 0$

$$\begin{aligned} P(\Gamma_n'^0) &\to \frac{1}{\pi\sigma_1\sigma_2}\exp\left\{\frac{\Gamma_n'^0}{2}\left(\frac{1}{2\sigma_2^2}-\frac{1}{2\sigma_1^2}\right)\right\}\left\{\frac{\pi^{\frac{1}{2}}\sigma_2}{(\Gamma_n'^0)^{\frac{1}{2}}}\exp\left(-\frac{\Gamma_n'^0}{4\sigma_2^2}\right)\right\} \\ &\to (\pi\Gamma_n'^0)^{-\frac{1}{2}}\sigma_1^{-1}\exp(-\Gamma_n'^0/4\sigma_1^2)\,, \end{aligned} \tag{50}$$

which is a Porter–Thomas distribution with mean

$$\langle\Gamma_n'^0\rangle = 2\sigma_1^2\,. \tag{51}$$

In equation (50) we have used the asymptotic form for the Bessel function $\mathrm{K}_0(x)$. On the other hand, should σ_2 become of order σ_1 we would expect a distribution like

$$P(\Gamma_n'^0) = (\pi\sigma_1^2)^{-1}\mathrm{K}_0(\Gamma_n'^0/2\sigma_1^2)\,, \qquad \sigma_2 \approx \sigma_1\,. \tag{52}$$

We shall now test the Adler–Adler parameters to see whether the distribution behaves like (50) or more like (52).

4. Experimental Results

In examining a statistical distribution $P(x)$ satisfying

$$\int_0^\infty P(x)\,\mathrm{d}x = 1\,, \tag{53}$$

we have found that the cumulative distribution function

$$D(y) = \int_0^\infty P(x)\,\mathrm{d}x \tag{54}$$

provides a better means of comparing theory with experiment than the usual histogram analysis because, for fissile isotopes, usually only of the order of 100 levels are available, and the large errors on the histogram values make comparison difficult. We obtained our Adler–Adler parameters from de Saussure and Perez (1969) and de Saussure *et al.* (1970). A program was written for an IBM 360/50 computer to perform the necessary analysis.

First consider the parameters H_λ^{f} and H_λ^{γ} which have predicted approximate distributions given by (43). Table 1*a* gives the results for ^{235}U for the distributions with unit normalization and unit variance. We can see that computing the variance directly from the experiments does not give a good fit even after renormalization, since too much weight is given to the few large values of H_λ^{f} which have poor statistics. In Table 1*b* we have a similar fit to the ^{235}U data, but with the variance adjusted to give a better fit to the region where there are most levels and the statistics are good. We see that the Gaussian product distribution is a significantly better fit over the range where the statistics are good, but both theoretical distributions tend to underestimate the number of large levels. This, however, could well be due to the poor

Table 1. Distribution of Adler–Adler asymmetry parameters for ^{235}U

Cumulative experimental statistics over 142 levels are compared with corresponding results for a Porter–Thomas distribution (PT), a single Gaussian (G) and a two-Gaussian product (2G). An asterisk indicates a renormalized distribution

(a) H^f distribution, $\langle(H^f)^2\rangle = 742\ b^2\,eV^2$ | (c) H^γ distribution, $\langle(H^\gamma)^2\rangle = 101\ b^2\,eV^2$

y	Exp.	PT	PT*	2G	2G*	y	Exp.	PT	PT*	2G	2G*
0	1·00	1·00	0·49	1·00	0·44	0	1·00	1·00	0·86	1·00	0·78
0·1	0·65	0·68	0·33	0·78	0·34	0·1	0·62	0·68	0·58	0·78	0·61
0·2	0·45	0·56	0·27	0·65	0·29	0·2	0·51	0·56	0·48	0·65	0·51
0·3	0·37	0·47	0·23	0·55	0·24	0·3	0·41	0·47	0·41	0·55	0·42
0·4	0·27	0·41	0·20	0·47	0·21	0·4	0·35	0·41	0·35	0·47	0·37
0·5	0·22	0·35	0·17	0·41	0·18	0·5	0·32	0·35	0·30	0·41	0·32
0·6	0·18	0·31	0·15	0·36	0·16	0·6	0·25	0·31	0·27	0·36	0·27
0·7	0·14	0·27	0·13	0·31	0·14	0·7	0·23	0·27	0·23	0·31	0·24
0·8	0·13	0·24	0·12	0·27	0·12	0·8	0·21	0·24	0·21	0·27	0·21
0·9	0·11	0·21	0·10	0·24	0·10	0·9	0·18	0·21	0·18	0·24	0·18
1·0	0·092	0·19	0·092	0·21	0·092	1·0	0·16	0·19	0·16	0·21	0·16
1·2	0·070	0·15	0·073	0·16	0·071	1·2	0·12	0·15	0·13	0·16	0·13
1·4	0·056	0·12	0·058	0·13	0·056	1·4	0·11	0·12	0·10	0·13	0·10
1·6	0·056	0·096	0·047	0·099	0·044	1·6	0·08	0·10	0·08	0·10	0·08
1·8	0·056	0·077	0·038	0·078	0·034	1·8	0·08	0·08	0·07	0·08	0·06
2·0	0·056	0·063	0·031	0·062	0·028	2·0	0·04	0·06	0·05	0·06	0·05
2·5	0·042	0·037	0·018	0·035	0·015	2·5	0·04	0·04	0·03	0·03	0·03
3·0	0·035	0·023	0·011	0·020	0·008	3·0	0·04	0·02	0·02	0·02	0·02
3·5	0·021	0·014	0·0067	0·011	0·005	3·5	0·02	0·01	0·01	0·01	0·01
4·0	0·021	0·0085	0·0041	0·0065	0·003	4·0	0·02	0·008	0·007	0·006	0·005
5·0	0·007	0·0033	0·0016	0·0022	0·001	5·0	0·00	0·003	0·003	0·002	0·002
6·0	0·007	0·0013	0·0006	0·0008	0·0003	6·0	0·00	0·001	0·001	0·0008	0·0006

(b) Fitted H^f distribution, $\langle(H^f)^2\rangle = 185\ b^2\,eV^2$

y	Exp.	PT	2G	y	Exp.	PT	2G
0	1·00	1·00	1·00	1·6	0·13	0·10	0·10
0·2	0·65	0·56	0·65	1·8	0·11	0·07	0·08
0·4	0·45	0·41	0·47	2·0	0·09	0·06	0·06
0·6	0·37	0·31	0·36	2·5	0·07	0·04	0·04
0·8	0·27	0·24	0·27	3·0	0·06	0·02	0·02
1·0	0·22	0·19	0·21	4·0	0·05	0·008	0·007
1·2	0·18	0·15	0·16	6·0	0·02	0·001	0·001
1·4	0·14	0·12	0·13				

(d) K^f distribution, $\langle K^f\rangle = 6{\cdot}21$ | (e) K^γ distribution, $\langle K^\gamma\rangle = 0{\cdot}969$

y	Exp.	G	2G	y	Exp.	G	2G
0	1·00	1·00	1·00	0	1·00	1·00	1·00
0·1	0·77	0·68	0·78	0·1	0·74	0·68	0·78
0·2	0·63	0·56	0·65	0·2	0·64	0·56	0·65
0·3	0·53	0·47	0·55	0·3	0·58	0·47	0·55
0·4	0·38	0·41	0·47	0·4	0·46	0·41	0·47
0·5	0·33	0·35	0·41	0·5	0·37	0·35	0·41
0·6	0·30	0·31	0·36	0·6	0·32	0·31	0·36
0·7	0·25	0·27	0·31	0·7	0·27	0·27	0·31
0·8	0·23	0·24	0·27	0·8	0·23	0·24	0·27
0·9	0·21	0·21	0·24	0·9	0·20	0·21	0·24
1·0	0·17	0·19	0·21	1·0	0·19	0·19	0·21
1·2	0·11	0·15	0·16	1·2	0·13	0·15	0·16
1·4	0·08	0·12	0·13	1·4	0·11	0·12	0·13
1·6	0·17	0·10	0·10	1·6	0·10	0·10	0·10
1·8	0·06	0·08	0·08	1·8	0·07	0·08	0·08
2·0	0·06	0·06	0·06	2·0	0·06	0·06	0·06
2·5	0·06	0·04	0·03	2·5	0·04	0·04	0·03
3·0	0·04	0·02	0·02	3·0	0·03	0·02	0·02
3·5	0·02	0·01	0·01	3·5	0·03	0·01	0·01
4·0	0·01	0·008	0·006	4·0	0·007	0·008	0·006
5·0	0·00	0·001	0·007	5·0	0·000	0·003	0·002
6·0	0·00	0·001	0·007	6·0	0·000	0·001	0·0007

Table 2. Distribution of Adler–Adler asymmetry parameters for ^{233}U

Cumulative experimental statistics over 69 levels are compared with corresponding results for a Porter–Thomas distribution (PT) and a two-Gaussian product (2G). An asterisk indicates a renormalized distribution

(a) H^{f} distribution, $\langle(H^{\mathrm{f}})^2\rangle = 1660\ \mathrm{b}^2\,\mathrm{eV}^2$

y	Exp.	PT	PT*	2G	2G*
0	1·00	1·00	1·31	1·00	1·18
0·1	0·88	0·68	0·89	0·78	0·92
0·2	0·78	0·56	0·73	0·65	0·77
0·3	0·70	0·47	0·62	0·55	0·65
0·4	0·62	0·41	0·53	0·47	0·56
0·5	0·48	0·35	0·46	0·41	0·48
0·6	0·39	0·31	0·40	0·36	0·42
0·7	0·33	0·27	0·35	0·31	0·37
0·8	0·30	0·24	0·31	0·27	0·32
0·9	0·26	0·21	0·28	0·24	0·28
1·0	0·25	0·19	0·25	0·21	0·25
1·2	0·17	0·15	0·20	0·16	0·20
1·4	0·14	0·12	0·16	0·13	0·15
1·6	0·13	0·10	0·13	0·10	0·12
1·8	0·12	0·08	0·10	0·08	0·10
2·0	0·07	0·06	0·08	0·06	0·07
2·5	0·01	0·04	0·05	0·03	0·04
3·0	0·01	0·02	0·03	0·02	0·02
3·5	0·0	0·01	0·02	0·01	0·01
4·0	0·0	0·008	0·01	0·006	0·008
5·0	0·0	0·003	0·004	0·002	0·003
6·0	0·01	0·001	0·002	0·0007	0·0009

(b) H^{γ} distribution, $\langle(H^{\gamma})^2\rangle = 37\cdot1\ \mathrm{b}^2\,\mathrm{eV}^2$

y	Exp.	PT	2G
0	1·00	1·00	1·00
0·1	0·93	0·68	0·78
0·2	0·72	0·56	0·65
0·3	0·62	0·47	0·55
0·4	0·55	0·41	0·47
0·5	0·43	0·35	0·41
0·6	0·43	0·31	0·36
0·7	0·35	0·27	0·31
0·8	0·28	0·24	0·27
0·9	0·23	0·21	0·24
1·0	0·20	0·19	0·21
1·2	0·16	0·15	0·16
1·4	0·13	0·12	0·13
1·6	0·12	0·10	0·10
1·8	0·09	0·08	0·08
2·0	0·06	0·06	0·06
2·5	0·04	0·04	0·03
3·0	0·01	0·02	0·02
3·5	0·01	0·01	0·01
4·0	0·00	0·008	0·006
5·0	0·00	0·003	0·002
6·0	0·00	0·001	0·0007

(c) K^{f} distribution, $\langle K^{\mathrm{f}}\rangle = 24\cdot3$

y	Exp.	PT	2G
0	1·00	1·00	1·00
0·1	0·85	0·68	0·78
0·2	0·65	0·56	0·65
0·3	0·52	0·47	0·55
0·4	0·43	0·41	0·47
0·5	0·41	0·35	0·41
0·6	0·37	0·31	0·36
0·7	0·31	0·27	0·31
0·8	0·22	0·24	0·27
0·9	0·17	0·21	0·24
1·0	0·15	0·19	0·21
1·2	0·15	0·15	0·16
1·4	0·11	0·12	0·12
1·6	0·11	0·096	0·10
1·8	0·093	0·077	0·078
2·0	0·074	0·063	0·062
2·5	0·019	0·037	0·035
3·0	0·019	0·023	0·020
3·5	0·019	0·014	0·011
4·0	0·019	0·008	0·006
5·0	0·000	0·003	0·002
6·0	0·000	0·001	0·0008

(d) K^{γ} distribution, $\langle K^{\gamma}\rangle = 0\cdot0068$

y	Exp.	PT	2G
0	1·0	1·0	1·0
0·1	0·65	0·68	0·78
0·2	0·50	0·56	0·65
0·3	0·41	0·47	0·55
0·4	0·39	0·41	0·47
0·5	0·30	0·35	0·41
0·6	0·28	0·31	0·36
0·7	0·24	0·27	0·31
0·8	0·19	0·24	0·27
0·9	0·15	0·21	0·24
1·0	0·15	0·19	0·21
1·2	0·11	0·15	0·16
1·4	0·056	0·12	0·13
1·6	0·056	0·096	0·099
1·8	0·056	0·077	0·078
2·0	0·037	0·063	0·062
2·5	0·037	0·037	0·034
3·0	0·037	0·023	0·020
3·5	0·037	0·014	0·011
4·0	0·037	0·008	0·006
5·0	0·000	0·003	0·002
6·0	0·000	0·001	0·0008

statistics since at $y = 6\cdot0$ the expected error in the prediction is some several hundred per cent.

The results for H^{γ}_{λ} were similar to those for H^{f}_{λ}. Table 1*c* shows that the renormalized statistics for the Porter–Thomas distribution are good for $y \geqslant 0\cdot3$ but that the renormalized Gaussian product distribution is comparable in this range and far better for $0\cdot1 \leqslant y \leqslant 0\cdot3$. Our assertion in Section 3, that K^{f} and K^{γ}_{λ} are distributed approximately as a Gaussian distribution, though crude, does work in practice. We have tested this assertion further by examining the statistics of K_λ^{f} and K_λ^{γ}. We can see from Table 1*d* that for K_λ^{f} either distribution is acceptable for $y \geqslant 0 \cdot 4$ but for $y \leqslant 0 \cdot 4$ the Gaussian product is better than the single Gaussian. This was expected as K_λ^{f}, from equation (40a), is such that the leading term in the sum for K_λ^{f} contains, in fact, a product of two variables, each of which we have postulated has a Gaussian distribution. In practice, however, it appears to make little difference to the distribution for H_λ^{f}. Table 1*e* contains an analysis of the statistics for K_λ^{γ}. Once again the results favour the Gaussian product for $y \leqslant 0 \cdot 4$, but the single Gaussian is an excellent fit for larger values.

Table 3. Distribution of multilevel fission widths for ^{235}U and ^{233}U

Cumulative experimental statistics are compared with the predicted cumulative distributions for the two most favoured cases

y	Exp.	2 channels	3 channels	y	Exp.	3 channels	4 channels
	(*a*) ^{235}U, $\langle \Gamma_{\mathrm{f}} \rangle = 0 \cdot 062$ eV				(*b*) ^{233}U, $\langle \Gamma_{\mathrm{f}} \rangle = 0 \cdot 177$ eV		
0·2	0·87	0·90	0·90	0	1·00	1·00	1·00
0·4	0·71	0·67	0·75	0·1	0·93	0·96	0·98
0·6	0·60	0·55	0·61	0·2	0·93	0·90	0·94
0·8	0·52	0·45	0·49	0·3	0·87	0·83	0·88
1·0	0·39	0·37	0·39	0·4	0·83	0·75	0·81
1·2	0·30	0·30	0·31	0·5	0·74	0·68	0·74
1·4	0·23	0·25	0·24	0·6	0·63	0·65	0·66
1·6	0·20	0·20	0·19	0·7	0·63	0·55	0·59
1·8	0·16	0·17	0·14	0·8	0·52	0·49	0·52
2·0	0·14	0·14	0·11	0·9	0·46	0·44	0·46
2·5	0·10	0·08	0·06	1·0	0·43	0·39	0·41
3·0	0·07	0·05	0·03	1·2	0·30	0·31	0·31
4·0	0·042	0·02	0·007	1·4	0·15	0·24	0·23
6·0	0·028	0·002	0·0004	1·6	0·13	0·18	0·17
				1·8	0·11	0·14	0·13
				2·0	0·074	0·11	0·092
				2·5	0·037	0·057	0·040
				3·0	0·019	0·029	0·017
				3·5	0·019	0·015	0·007
				4·0	0·019	0·007	0·003
				5·0	0·019	0·002	0·0005
				6·0	0·000	0·0004	0·00008

We have also examined the same parameters for ^{233}U contained in the same data source. In Table 2*a* the cumulative Gaussian product is seen to give an acceptable fit even before renormalization in the case of H_λ^{f}, while the fit for H^{γ} (Table 2*b*) is not as good for small values of y, but still significantly better than that of the Porter–Thomas distribution. The analyses of K_λ^{f} and K_λ^{γ} for ^{233}U are given in Tables 2*c* and 2*d*. Notice that there appear to be rather more smaller levels missed than we would expect from other sources (Musgrove 1967). The fit to K_λ^{f} is quite good, and we interpret the much poorer fit to K_λ^{γ} as arising from the large experimental errors for this parameter. Nevertheless, the trend of the K_λ^{γ} values is reproduced satisfactorily.

Table 3*a* shows the predictions for the distribution of multilevel fission widths in ^{235}U for the two cases which produce by far the best fit. The results favour the presence of either two or three fission channels, in agreement with previous results (Schmidt 1966). Similarly, Table 3*b* shows the results for ^{233}U, but this time the statistics favour either three or four fission channels.

5. Conclusions

The aim of this work has been to produce a simple set of statistical rules which will allow us to analyse Adler–Adler resonance parameters in a meaningful way and, in addition, to generate sets of such parameters for calculating multilevel fission cross sections such as might be required in a reactor physics calculation. We feel that the rules given should prove adequate in practice and that the resultant statistics will be only slightly more complicated than those normally used in single-level parameter analysis.

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