# The Appearance of Relativistic Stellar Systems 

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#### Abstract

A formula is derived for the projected density distribution on a photographic plate which would arise from a given static spherically-symmetric relativistic stellar system. For weak gravitational fields, a corresponding post-Newtonian expression is derived which is relatively simple to use once a particular proper stellar number density has been specified for the relativistic system under consideration. For application to the outer regions of such a system, this post-Newtonian formula simplifies considerably, and it is possible to solve von Zeipel's problem explicitly for these outer regions to obtain an expression for the proper number density of stars in such a system from the projected density on a photographic plate. We find that, if Newtonian theory alone were used to calculate the stellar number density in a relativistic system from the projected density, the system so constructed would appear to be less centrally condensed than it really is.


## 1. Introduction

The determination of the three-dimensional density distribution of stars in a globular cluster from the projected density on a photographic plate is a classical problem in astronomy. It was originally solved by von Zeipel (1908) who reduced it to the solution of an Abel type integral equation.

In view of the recent interest in the role that supermassive relativistic clusters might play in astrophysical processes, e.g. as models of quasars or as sources of gravitational radiation, it would seem worth while to re-examine von Zeipel's problem in the context of general relativity. For any system of luminous point masses whose structure is influenced by relativity, the gravitational deflection of light will introduce modifications into the projected density. However, in fully relativistic situations, the problem of determining the proper number density from a simple counting of the number of stars per unit solid angle is indeterminate. The reasons are that in a static spherically-symmetric system there are three unknowns, namely, the two metric coefficients as well as the proper number density, and that the metric coefficients are determined not only by the number of stars present but also by their mass and velocity distributions.

We shall be concerned here initially with the corresponding inverse problem. From a complete solution of the Einstein field equations corresponding to some static spherically-symmetric system of point masses, we shall derive an expression for the projected number density as it would appear to a very distant observer. The proper number density $n(r)$ of the stars in the system, where $r$ is a radial coordinate, is assumed to be given.

An important factor, of course, is the optical depth of the system. Naturally, a system must be optically thin in order that all its stars may be seen. However, high optical depths would be encountered in the majority of relativistic situations. This is especially true for supermassive clusters in the nuclei of certain types of galaxies, where the clusters would probably be obscured by clouds of dust and gas. The stars themselves would also produce large optical depths if they were of the main sequence or giant variety. However, it is unlikely that main sequence stars would ever be found in relativistic systems since elementary estimates of collision times (Fackerell 1968; Zel'dovich and Novikov 1971) show that, in any cluster having a realistic total mass, a main sequence star could not possibly survive for astronomical times at the stellar densities required for relativistic effects to be important. If the system consisted of white dwarfs, neutron stars or black holes then the situation is improved because the optical depth due to the objects themselves would always be small, although the actual detection of such objects could present quite a problem. There is also the problem that, with the resolution obtainable from ground-based telescopes, we cannot expect to be able to distinguish the individual stars in a relativistic stellar system. The only quantity presently measurable is the integrated flux density or surface brightness, although this situation might change in coming years when observatories are established above the atmosphere.

A slightly different phenomenon is a halo of stars surrounding a supermassive black hole. It seems likely (Zel'dovich and Novikov 1971) that, when a stellar system ultimately collapses, a halo of stars would be left surrounding the black hole that is formed in the process. Here the gravitational field can be generated by the hole itself and consequently we do not need a high number density of stars, or even a large number of stars, to produce relativistic effects.

In Section 2, a general expression is derived for the total number of stars per unit solid angle as seen by a distant observer, corresponding to a nonsingular but otherwise arbitrary (with certain restrictions) static spherically-symmetric metric containing an arbitrary (again with certain restrictions) distribution of point masses. In Section 3 this general expression is expanded in powers of $c^{-2}$ to obtain the corresponding post-Newtonian formula for weak gravitational fields. A simplified version of this formula is then derived for the situation where the stars in the outer regions move in the exterior Schwarzschild metric created by the massive inner regions of the system. Finally, in Section 4, this simplified formula is inverted to give the threedimensional number density in the outer regions of the system.

## 2. General Theory

We shall adopt a system of units in which $G=c=1$ and employ the canonical form of the static spherically-symmetric line element

$$
\begin{equation*}
-\mathrm{d} s^{2}=-\exp (2 \Phi(r)) \mathrm{d} t^{2}+r^{2}\left(\mathrm{~d} \sigma^{2}+\sin ^{2} \sigma \mathrm{~d} \phi^{2}\right)+\exp (2 \lambda(r)) \mathrm{d} r^{2} \tag{1}
\end{equation*}
$$

expressed in terms of Schwarzschild coordinates $r, \sigma, \phi, t$. We shall assume throughout that the distribution of stars extends to spatial infinity but that the number density drops off rapidly enough at large $r$ so that the total mass of the system is finite. In this case the metric (1) asymptotically approaches the Schwarzschild exterior solution as $r$ approaches infinity.

Since the projected density depends upon the trajectories of photons emitted within the system, the problems at hand may be solved by studying solutions of the eikonal equation, which governs the motion of massless particles in a gravitational field. For a general metric $g^{\alpha \beta}(\alpha, \beta=0, \ldots, 3)$ the eikonal equation is

$$
\begin{equation*}
g^{\alpha \beta}\left(\partial \Psi / \partial x^{\alpha}\right)\left(\partial \Psi / \partial x^{\beta}\right)=0 \tag{2}
\end{equation*}
$$

where $\Psi$ is the eikonal. In a spherically-symmetric gravitational field, photon orbits always lie in a plane, which we take to be the plane $\sigma=\frac{1}{2} \pi$. Landau and Lifshitz (1962) have shown that equation (2) then reduces to

$$
\begin{equation*}
\left(\frac{\mathrm{d} \phi}{\mathrm{~d} r}\right)^{2}=\frac{l^{2} \exp (2 \lambda(r))}{r^{2}\left\{r^{2} \exp (-2 \Phi(r))-l^{2}\right\}} \tag{3}
\end{equation*}
$$

where $l$ is the impact parameter as measured at infinity. In a spherically-symmetric gravitational field, $l$ is the only parameter that characterizes a photon trajectory.

We now consider an observer situated a large distance $x_{0}$ along the $x$ axis where space-time may be assumed flat, as shown in Fig. 1a. Let $\theta$ denote the angle between the $x$ axis (which passes through the centre of the system) and the direction in which the observer looks. It can be seen from the diagram that the impact parameter $l$ is given by $l=x_{0} \sin \theta$. We take the $x$ axis to be the origin of the azimuthal angular coordinate $\phi$ which appears in equation (3). It is our aim to calculate the total number of stars that appear to lie in a unit solid angle $\mathrm{d} \Omega$ at the observer. In the observer's frame of reference, the azimuthal angular coordinate $\psi$ (see Fig. 1b) can have an arbitrary orientation because of the symmetry of the observations about the $x$ axis. The origin of $\psi$ has been chosen to coincide with the $\sigma=\frac{1}{2} \pi$ plane of the metric (1). The element of solid angle at the observer is defined in the usual manner by $\mathrm{d} \Omega=\sin \theta \mathrm{d} \theta \mathrm{d} \psi$.

The four photon trajectories associated with the directions defining $\mathrm{d} \Omega$ form the elongated volume element denoted by $\mathrm{d}^{2} V$ in Fig. $1 a$. Our aim is to calculate the number of stars lying in the region of space-time bounded by these four orbits. The trajectories lie in two planes separated by the angle $\mathrm{d} \psi$ as they leave the neighbourhood of the observer (using the reversibility of photon directions) and, because of the planar nature of the motion, they remain in these planes as they pass through the system.

Since we expect relativistic systems to subtend extremely small solid angles at the Earth, we restrict our discussion to small values of $\theta$. The number of stars in $\mathrm{d}^{2} V$ is expressed as a proper line integral along a photon trajectory, and it is assumed that the number density distribution vanishes sufficiently rapidly with increasing $r$ that the major contribution to the integral comes from regions close to the centre of the system. Explicitly, we assume $n(r)$ to be non-negligible only for values of $r$ satisfying $r \ll r_{0}$.

The calculation of the number of stars in $\mathrm{d}^{2} V$ is facilitated by the fact that, while the region has a finite length (over the region of interest near the centre of the system), it has only an infinitesimal cross sectional area. The coordinate system is oriented so that $\phi$ is the angle between a radial vector from the centre of the system and the $x$ axis. We can now define a proper infinitesimal volume element $\mathrm{d}^{3} v$ lying in $\mathrm{d}^{2} V$ between coordinate 'surfaces' defined by $\phi$ and $\phi+\mathrm{d} \phi$ (see Fig. $1 c$ for $\mathrm{d}^{3} v$ lying in the first quadrant). To third order in small quantities, we have

$$
\begin{equation*}
\mathrm{d}^{3} v=r|\sin \phi| \mathrm{d} \psi r \mathrm{~d} \phi \exp (\lambda(r))(\mathrm{d} r)_{\phi}, \tag{4}
\end{equation*}
$$



Fig. 1. Viewing geometry (not to scale) for an observer at $x_{0}$ and a system of point stellar masses about the origin, showing:
(a) that all stars contained within the volume element $\mathrm{d}^{2} V$ appear to lie within the solid angle element $\mathrm{d} \Omega$ at the observer,
(b) the definition of $\mathrm{d} \Omega$, with $\psi$ measured from the $\sigma=\frac{1}{2} \pi$ plane of metric (1), and
(c) details of the volume element $\mathrm{d}^{3} v$, of which the region $\mathrm{d}^{2} V$ is composed.

where the absolute value of $\sin \phi$ has been used in case the photon orbits cross the $x$ axis, and the quantity $(\mathrm{d} r)_{\phi}$ is the radial coordinate separation (AB in Fig. 1c) between two neighbouring trajectories associated with the impact parameters $l$ and $l+\mathrm{d} l$. Formal integration of equation (3) gives $\phi$ as a function of $r$ and $l$ along a trajectory, that is, $\phi=\phi(r, l)$. In principle this may now be inverted to give $r=r(\phi, l)$ from which it follows that, for a change $\mathrm{d} l=x_{0} \cos \theta \mathrm{~d} \theta$ in $l$, we have

$$
\begin{equation*}
(\mathrm{d} r)_{\phi}=x_{0}(\partial r / \partial l)_{\phi} \cos \theta \mathrm{d} \theta . \tag{5}
\end{equation*}
$$

Since a complete solution of the eikonal equation does not give $r$ explicitly in terms of $\phi$, the calculation of $(\mathrm{d} r)_{\phi}$ is facilitated by using

$$
\begin{equation*}
\left(\frac{\partial r}{\partial l}\right)_{\phi}=-\left(\frac{\partial \phi}{\partial l}\right)_{r} /\left(\frac{\partial \phi}{\partial r}\right)_{l} . \tag{6}
\end{equation*}
$$

The total number of stars $\mathrm{d}^{2} N / \mathrm{d} \Omega$ per unit solid angle at the observer is now obtained by integrating $n(r) \mathrm{d}^{3} v$ over the appropriate range of $\phi$ and dividing the result by $\mathrm{d} \Omega$. We obtain

$$
\begin{equation*}
\frac{\mathrm{d}^{2} N}{\mathrm{~d} \Omega}=x_{0}^{2} l^{-1} \int_{0}^{2 \phi_{m}(l)} n(r) r^{2} \exp \hat{\lambda}(r)\left|\sin \phi\left(\frac{\partial \phi}{\partial l}\right)_{r} /\left(\frac{\partial \phi}{\partial r}\right)_{l}\right| \mathrm{d} \phi . \tag{7}
\end{equation*}
$$

The modulus signs in the integrand are necessary because for very strong gravitational fields the orbits associated with different impact parameters can cross inside the system. At the point of intersection, $(\mathrm{d} r)_{\phi}$ is zero and equation (5) then gives a negative value for $(\mathrm{d} r)_{\phi}$ along a certain section of the orbit.

The quantity $\phi_{m}(l)$ is the value of $\phi$ at the pericentre of the photon orbit. We have used the fact that the orbit is symmetric about its pericentre, a consequence of the static spherical symmetry. If $x_{0}$ is large enough compared with the dimensions of the system, no appreciable errors will be incurred if it is replaced by $\infty$ when the eikonal equation is solved. It then follows from equation (3) that

$$
\begin{equation*}
\phi_{m}(l)=l \int_{r_{m}(l)}^{\infty} u^{-1} \exp \lambda\left\{u^{2} \exp (-2 \Phi)-l^{2}\right\}^{-\frac{1}{2}} \mathrm{~d} u \tag{8}
\end{equation*}
$$

where $r_{m}(l)$ is the solution of

$$
\begin{equation*}
r_{m}^{2}-l^{2} \exp 2 \Phi\left(r_{m}\right)=0 \tag{9}
\end{equation*}
$$

Since $\phi$ is a double-valued function of $r$, it is necessary to distinguish solutions of equation (3) (subject to a given set of initial conditions) for which $\phi \leqslant \phi_{m}$ from those for which $\phi \geqslant \phi_{m}$. We write the former as

$$
\begin{equation*}
\phi_{\mathrm{a}}(r, l)=J(r, l) \quad \text { for } \quad \phi \leqslant \phi_{m} \tag{10}
\end{equation*}
$$

where

$$
\begin{equation*}
J(r, l)=l \int_{r}^{\infty} u^{-1} \exp \lambda\left\{u^{2} \exp (-2 \Phi)-l^{2}\right\}^{-\frac{1}{2}} \mathrm{~d} u \tag{11}
\end{equation*}
$$

and the latter as

$$
\begin{equation*}
\phi_{\mathrm{b}}(r, l)=2 \phi_{m}(l)-J(r, l) \quad \text { for } \quad \phi \geqslant \phi_{m} \tag{12}
\end{equation*}
$$

The integral in equation (7) is still inconvenient to work with unless equations (10) and (11) can be inverted to give $r=r(\phi, l)$. In most applications, e.g. the postNewtonian approximation to be discussed in Section 3, it is more convenient to
change the variable of integration to $r$ by using

$$
\mathrm{d} \phi=(\partial \phi / \partial r)_{l} \mathrm{~d} r,
$$

which holds along a trajectory. We then obtain, in the subscript convention adopted in equations (10) and (12),

$$
\begin{equation*}
\mathrm{d}^{2} N / \mathrm{d} \Omega=x_{0}^{2} l^{-1}\left\{I_{\mathrm{a}}(l)+I_{\mathrm{b}}(l)\right\}, \tag{13}
\end{equation*}
$$

where

$$
\begin{align*}
& I_{\mathrm{a}}=\int_{r_{m}(l)}^{\infty} n(r) r^{2} \exp \lambda(r)\left|\left(\partial \phi_{\mathrm{a}} / \partial l\right)_{\mathrm{r}} \sin \phi_{\mathrm{a}}(r, l)\right| \mathrm{d} r  \tag{14a}\\
& I_{\mathrm{b}}=\int_{r_{m}(l)}^{\infty} n(r) r^{2} \exp \lambda(r)\left|\left(\partial \phi_{\mathrm{b}} / \partial l\right)_{\mathrm{r}} \sin \phi_{\mathrm{b}}(r, l)\right| \mathrm{d} r \tag{14b}
\end{align*}
$$

We note that the distance $x_{0}$ to the system appears only as a simple scaling factor in equation (13).

The preceding analysis would break down if there existed regions of space-time where photons could be trapped by the gravitational field. We shall avoid this situation by restricting our attention to fields which are not strong enough to produce this effect. As it turns out, this is not a particularly harsh restriction since the required fields would be extremely strong. For example, Synge (1966) has shown that all of the photons emitted from the surface of a relativistic star of mass $M$ and coordinate radius $R$ would escape to infinity only when $M / R<\frac{1}{3}$. Although this relation is true only for the Schwarzschild exterior solution, we would expect it to hold approximately for interior metrics as well, and accordingly a criterion for the validity of the above calculations is that

$$
\begin{equation*}
M(r) / r \lesssim \frac{1}{3} \tag{15}
\end{equation*}
$$

be satisfied for all $r$. Here $M(r)$ is the total mass-energy inside the radius $r$.

## 3. Post-Newtonian Approximation

In many astrophysical situations it is not necessary to use the full formalism of the previous section. In many cases the gravitational fields will not be strong enough to warrant the full use of equations (13) and (14), and an analysis of photon trajectories in a post-Newtonian metric should be quite adequate. It is a fact that nearly all models of static spherically-symmetric star clusters which have been constructed to date become unstable when their central redshift reaches the value $Z_{\mathrm{c}} \approx 0 \cdot 5$. It is approximately at this point that the binding energy per unit mass attains a maximum value and the clusters subsequently undergo gravitational collapse once they have evolved quasi-statically to this stage (Fackerell et al. 1969; Ipser 1969). However, cluster models have been constructed which are probably stable for any central redshifts. These are the models constructed by Bisnovatyi-Kogan and Zel'dovich (1969).

In this section we expand the integrals (14) in powers of $c^{-2}$ and retain only the zeroth and first order terms. We adopt the following notation for the expansion of the metric coefficients:

$$
\left.\begin{array}{rl}
\exp 2 \Phi(r) & =1-2 U(r)+\mathcal{O}\left(c^{-4}\right),  \tag{16}\\
\exp 2 \lambda(r) & =1+2 H(r)+\mathcal{O}\left(c^{-4}\right) .
\end{array}\right\}
$$

We assume that $U(r)$ and $H(r)$ are small compared with unity. The function $U(r)$ is the Newtonian gravitational potential defined in terms of the material density $\rho$ by means of Poisson's equation (Chandrasekhar 1965)

$$
\begin{equation*}
\nabla^{2} U=-4 \pi \rho \tag{17}
\end{equation*}
$$

We choose not to expand the proper number density $n(r)$, but leave it intact.
Before expanding the integrals (14), we note the following important fact. It is only the Newtonian value of $g_{00}=-\exp 2 \Phi(r)$ that is required and, as we now demonstrate, this has important consequences for any applications of the postNewtonian analysis. It is clear from equations (9)-(14) and (16) that any expansion depends only upon the quantities $n(r), U(r)$ and $H(r)$. However, in the post-Newtonian approximation both $U(r)$ and $H(r)$ are uniquely determined in terms of the material density $\rho(r)$, which in turn is determined by $n(r)$. If the average rest mass of the objects at a particular point in the system is $\bar{m}_{0}(r)$, we can calculate $\rho(r)$ by using $\rho(r)=\bar{m}_{0}(r) n(r)$. For a spherically symmetric field, the solution of equation (17) is

$$
\begin{equation*}
U(r)=4 \pi r^{-1} \int_{0}^{r} u^{2} \bar{m}_{0}(u) n(u) \mathrm{d} u+4 \pi \int_{r}^{\infty} u \bar{m}_{0}(u) n(u) \mathrm{d} u . \tag{18}
\end{equation*}
$$

It then follows readily from the relation

$$
\exp -2 \lambda(r)=1-2 M(r) / r
$$

where $M(r)$ is the total mass-energy inside the radius $r$, that $H(r)$ is given by

$$
\begin{equation*}
H(r)=4 \pi r^{-1} \int_{0}^{r} u^{2} \bar{m}_{0}(u) n(u) \mathrm{d} u, \quad \text { that is, } \quad H(r)=-r \frac{\mathrm{~d} U(r)}{\mathrm{d} r} . \tag{19}
\end{equation*}
$$

We can now proceed to expand the integrals (14) by making a post-Newtonian expansion of equation (9). This gives the following expression for the radial coordinate of the pericentre

$$
\begin{equation*}
r_{m}(l)=l-l U(l)+\mathcal{O}\left(U^{2}\right) \tag{20}
\end{equation*}
$$

The modulus signs in equations (14) would create difficulties if we had to expand the integrals as they stand but fortunately, in the weak field approximation, we may safely dispense with them. The reason is that we do not expect photon orbits to cross the negative $x$ axis. The signs of the $(\partial \phi / \partial l)_{r}$ factors still have to be reckoned with but, since we do not expect orbits having different impact parameters to intersect in the weak field approximation, we have

$$
\left(\partial \phi_{\mathrm{a}} / \partial l\right)_{r}>0 \quad \text { and } \quad\left(\partial \phi_{\mathrm{b}} / \partial l\right)_{r}<0
$$

for all relevant values of $r$.
The most convenient way to expand the integrals (14) is to define two new integrals

$$
\begin{align*}
& K_{\mathrm{a}}(l)=\int_{r_{m}}^{\infty} n(r) r^{2} \exp \lambda(r) \cos \phi_{\mathrm{a}}(r, l) \mathrm{d} r,  \tag{21a}\\
& K_{\mathrm{b}}(l)=\int_{r_{m}}^{\infty} n(r) r^{2} \exp \lambda(r) \cos \phi_{\mathrm{b}}(r, l) \mathrm{d} r, \tag{21b}
\end{align*}
$$

which are related to $I_{\mathrm{a}}(l)$ and $I_{\mathrm{b}}(l)$ by

$$
\begin{aligned}
& I_{\mathrm{a}}(l)=-\left(\mathrm{d} K_{\mathrm{a}}(l) / \mathrm{d} l\right)-n\left(r_{m}\right) r_{m}^{2} \exp \lambda\left(r_{m}\right) \cos \phi_{\mathrm{a}}\left(r_{m}\right)\left(\mathrm{d} r_{m}(l) / \mathrm{d} l\right), \\
& I_{\mathrm{b}}(l)=\left(\mathrm{d} K_{\mathrm{b}}(l) / \mathrm{d} l\right)+n\left(r_{m}\right) r_{m}^{2} \exp \lambda\left(r_{m}\right) \cos \phi_{\mathrm{b}}\left(r_{m}\right)\left(\mathrm{d} r_{m}(l) / \mathrm{d} l\right) .
\end{aligned}
$$

Then, since $\phi_{\mathrm{a}}\left(r_{m}, l\right)=\phi_{\mathrm{b}}\left(r_{m}, l\right)$, equation (13) becomes

$$
\begin{equation*}
\mathrm{d}^{2} N / \mathrm{d} \Omega=x_{0}^{2} l^{-1}\left\{\mathrm{~d} K_{\mathrm{b}}(l) / \mathrm{d} l-\mathrm{d} K_{\mathrm{a}}(l) / \mathrm{d} l\right\} \tag{22}
\end{equation*}
$$

The evaluation of $K_{\mathrm{a}}(l)$ and $K_{\mathrm{b}}(l)$ now proceeds by expansion of $J(r, l)$ defined in equation (11). We start by rewriting

$$
\begin{equation*}
J(r, l)=l \int_{r}^{\infty} \frac{\{1+H(u)\} \mathrm{d} u}{u\left\{u^{2}-l^{2}+2 u^{2} U(u)\right\}^{\frac{1}{2}}} . \tag{23}
\end{equation*}
$$

We note that it is invalid to make a binomial expansion of the integrand in powers of $U$. This might seem a logical procedure to follow because a term by term integration between $r=0$ and $\infty$ of such an expansion could be substituted into $K_{\mathrm{a}}$ and $K_{\mathrm{b}}$, and then integrated term by term again to obtain an expression which in general would be finite and perfectly well behaved for all values of $l$; but unfortunately the expression would also be wrong! The reason is that when we expand the integrand of $J(r, l)$ it is impossible to avoid terms of the form $U(u) /\left(u^{2}-l^{2}\right)^{3 / 2}$ which blows up at $u=l$. Integration of this term from $u=r$ to $\infty$ produces a term proportional to $\left(r^{2}-l^{2}\right)^{-\frac{1}{2}}$, which on integration from $r=r_{m}$ to $\infty$ ultimately produces a term which contains no singularities, and indeed is always small compared with the Newtonian term. But the damage was done right at the start because expansion of the denominator in $J(r, l)$ violates the assumption that the post-Newtonian part of the integrand was small compared with the Newtonian part. This renders the final result invalid even though the integrals obtained are small.

This problem may be circumvented by introducing the following change of variable in equation (23)

$$
\begin{equation*}
v=u+u U(u) \tag{24}
\end{equation*}
$$

and noting that the inverse relation is $u=v-v U(v)$ through order $U$. In terms of $v$, we can now write

$$
\begin{equation*}
J(r, l)=\operatorname{cosec}^{-1}\left(r l^{-1}+r l^{-1} U(r)\right)+\Theta(r, l) \tag{25}
\end{equation*}
$$

where

$$
\begin{equation*}
\Theta(r, l)=l \int_{r}^{\infty} \frac{H(v)-v U^{\prime}(v)}{v\left(v^{2}-l^{2}\right)^{\frac{1}{2}}} \mathrm{~d} v \tag{26}
\end{equation*}
$$

is purely a post-Newtonian term. Here $U^{\prime}(v)$ denotes the derivative of $U$ with respect to $v$, and henceforth a prime will always denote the derivative of a function with respect to its argument.

It now follows readily from equations (10) and (25) that

$$
\begin{equation*}
\cos \phi_{\mathrm{a}}(r, l)=r^{-1}\left\{r^{2}-l^{2}+2 r^{2} U(r)\right\}^{\frac{1}{2}}-r^{-1}\left(r^{2}-l^{2}\right)^{\frac{1}{2}} U(r)-r^{-1} l \Theta(r, l) \tag{27}
\end{equation*}
$$

through order $U$ and $\Theta$. The $\left\{r^{2}-l^{2}+2 r^{2} U(r)\right\}^{\frac{1}{2}}$ term must be left intact at this stage since expansion of it still leads to a singular post-Newtonian term. Neglecting squared and higher terms in $U$ and $\Theta$, equations (16), (20), (21a) and (27) may now be combined to give

$$
\begin{equation*}
K_{\mathrm{a}}(l)=Q(l)-l P(l)-\int_{l}^{\infty} r n(r)\{U(r)-H(r)\}\left(r^{2}-l^{2}\right)^{\frac{1}{2}} \mathrm{~d} r \tag{28}
\end{equation*}
$$

where

$$
\begin{equation*}
Q(l)=\int_{l-l U(l)}^{\infty} r n(r)\left\{r^{2}-l^{2}+2 r^{2} U(r)\right\}^{\frac{1}{2}} \mathrm{~d} r \tag{29}
\end{equation*}
$$

and

$$
\begin{equation*}
P(l)=\int_{l}^{\infty} r n(r) \Theta(r, l) \mathrm{d} r . \tag{30}
\end{equation*}
$$

The only term in equation (28) that is not 'cleanly' divided into a Newtonian or post-Newtonian part is $Q(l)$, but we are now in a position to obtain a valid expansion of this by employing a similar transformation to (24). Accordingly we set

$$
\begin{equation*}
s=r+r U(r)=r+s U(s)+\mathcal{O}\left(U^{2}\right) \tag{31}
\end{equation*}
$$

and note that the lower limit of $Q(l)$ now becomes simply $s=l$. This contains no post-Newtonian part, and it is this fact which enables us to expand $Q(l)$ in a valid fashion. In addition, we make a Taylor series expansion of the number density

$$
n(r)=n(s)-s U(s) n^{\prime}(s)+\mathcal{O}\left(U^{2}\right) .
$$

Consequently, to post-Newtonian order, $Q(l)$ becomes

$$
\begin{equation*}
Q(l)=\int_{l}^{\infty} r n(r)\left(r^{2}-l^{2}\right)^{\frac{1}{2}} \mathrm{~d} r-\int_{l}^{\infty} r F_{0}(r)\left(r^{2}-l^{2}\right)^{\frac{1}{2}} \mathrm{~d} r, \tag{32}
\end{equation*}
$$

where

$$
F_{0}(r)=2 n(r) U(r)+r n(r) U^{\prime}(r)+r U(r) n^{\prime}(r)
$$

The second integral in $Q(l)$ is a post-Newtonian term and so we have used $r$ as the variable of integration there. The justification for also using $r$ in place of $s$ in the first integral is that in applications we shall specify $n(r)$ as an explicit function of $r$. It is this functional dependence that is to be used in the integral, a fact which reduces $s$ (or $r$ ) to a dummy variable of integration.

Substituting equation (32) in (28) and differentiating with respect to $l$ results in the following expression

$$
\begin{equation*}
K_{\mathrm{a}}^{\prime}(l)=-l L_{1}(l)+l L_{2}(l)-l W(l), \tag{33}
\end{equation*}
$$

where

$$
\begin{gather*}
L_{1}(l)=\int_{l}^{\infty} r n(r)\left(r^{2}-l^{2}\right)^{-\frac{1}{2}} \mathrm{~d} r, \quad L_{2}(l)=\int_{l}^{\infty} r F(r)\left(r^{2}-l^{2}\right)^{-\frac{1}{2}} \mathrm{~d} r,  \tag{34}\\
W(l)=l^{-1} P(l)+P^{\prime}(l) \quad \text { and } \quad F(r)=3 n(r) U(r)+2 r n(r) U^{\prime}(r)+r n^{\prime}(r) U(r) .
\end{gather*}
$$

The evaluation of $K_{\mathrm{b}}(l)$ now proceeds in an analogous fashion to that of $K_{\mathrm{a}}(l)$, the only additional information required being the expansion of $\phi_{m}(l)$. However, $\phi_{m}(l)$ is just $J(r, l)$ evaluated at $r=r_{m}(l)$ and it thus follows readily from equation (25) that the post-Newtonian expansion of $\phi_{m}(l)$ is

$$
\begin{equation*}
\phi_{m}(l)=\frac{1}{2} \pi+\Theta(l), \tag{35}
\end{equation*}
$$

where $\Theta(l)=\Theta(l, l)$ in equation (26). Evaluating $K_{\mathrm{b}}^{\prime}(l)$ and combining it with $K_{\mathrm{a}}^{\prime}(l)$ in equation (22), permits us to write the number of stars per unit solid angle at the observer in the form

$$
\begin{equation*}
\mathrm{d}^{2} N / \mathrm{d} \Omega=2 x_{0}^{2}\left\{L_{1}(l)+\Delta_{\mathrm{PN}}(l)\right\} \tag{36}
\end{equation*}
$$

to post-Newtonian order. Here

$$
\begin{gather*}
\Delta_{\mathrm{PN}}(l)=W(l)+\ln (l) \Theta(l)-L_{2}(l)-G(l) L_{3}(l),  \tag{37}\\
L_{\mathbf{3}}(l)=\int_{r}^{\infty} r n(r) \mathrm{d} r \quad \text { and } \quad G(l)=l^{-1} \Theta(l)+\Theta^{\prime}(l) .
\end{gather*}
$$

A question which naturally comes to mind is whether the post-Newtonian correction to the apparent density is positive or negative. We cannot make a general statement about this, however, since the post-Newtonian term must undergo a change in sign for some value of the impact parameter $l$. This is necessary to conserve the total number of stars observed in both cases.

It is possible, however, to prove that the post-Newtonian term always has a definite sign when we observe the outer regions of a stellar system. If we make the reasonable assumption that the stars in the outer extremities do not contribute significantly to the gravitational field, the metric (1) will be given approximately by the Schwarzschild line element in these regions. In the post-Newtonian limit, this gives $U(r)=H(r)=M / r$, where $M$ is the total mass of the system. The postNewtonian theory developed in this section is thus valid provided $r, l \gg M$. The condition on $l$ comes from equation (20), which for this case is

$$
r_{m}(l)=l(1-M / l)
$$

The integrals in equation (37) now simplify considerably and we find that

$$
\begin{gather*}
L_{2}(l)=M \int_{l}^{\infty} n(r)\left(r^{2}-l^{2}\right)^{-\frac{1}{2}} \mathrm{~d} r+M \int_{l}^{\infty} r n^{\prime}(r)\left(r^{2}-l^{2}\right)^{-\frac{1}{2}} \mathrm{~d} r  \tag{38}\\
G(l)=4 M \int_{l}^{\infty} r^{-2}\left(r^{2}-l^{2}\right)^{-\frac{1}{2}} \mathrm{~d} r-6 M l^{2} \int_{l}^{\infty} r^{-4}\left(r^{2}-l^{2}\right)^{-\frac{1}{2}} \mathrm{~d} r \equiv 0  \tag{39}\\
W(l)+\ln (l) \theta(l)=2 M \int_{l}^{\infty} n(r)\left(r^{2}-l^{2}\right)^{-\frac{1}{2}} \mathrm{~d} r \tag{40}
\end{gather*}
$$

Combining equations (38), (39) and (40) in (37) gives

$$
\begin{equation*}
\Delta_{\mathrm{PN}}=M \int_{l}^{\infty} n(r)\left(r^{2}-l^{2}\right)^{-\frac{1}{2}} \mathrm{~d} r-M \int_{l}^{\infty} r n^{\prime}(r)\left(r^{2}-l^{2}\right)^{-\frac{1}{2}} \mathrm{~d} r \tag{41}
\end{equation*}
$$

which is always positive since $n(r)$ is always a decreasing function of $r$. This is a result of the bending of light by the gravitational field.

## 4. Calculation of Number Density

In regions of space-time where they are valid, the expressions (36) and (41) for the projected density allow us to derive an expression for the number density $n(r)$ in the post-Newtonian approximation. In Newtonian theory equation (36) reduces to

$$
\begin{equation*}
\mathrm{d}^{2} N / \mathrm{d} \Omega=2 x_{0}^{2} \int_{l}^{\infty} r n(r)\left(r^{2}-l^{2}\right)^{-\frac{1}{2}} \mathrm{~d} r \tag{42}
\end{equation*}
$$

which has the solution (von Zeipel 1908)

$$
\begin{equation*}
n(r)=-2 \pi^{-1} \int_{r}^{\infty} D^{\prime}(l)\left(l^{2}-r^{2}\right)^{-\frac{1}{2}} \mathrm{~d} l \equiv T(r), \quad \text { say } \tag{43}
\end{equation*}
$$

where

$$
D(l)=\frac{1}{2} x_{0}^{-2} \mathrm{~d}^{2} N(l) / \mathrm{d} \Omega
$$

is the observed quantity.
Combining equation (36) with (41) and defining

$$
f(r)=\left(1+r^{-1} M\right) n(r)-M n^{\prime}(r)
$$

results in

$$
\mathrm{d}^{2} N / \mathrm{d} \Omega=2 x_{0}^{2} \int_{l}^{\infty} r f(r)\left(r^{2}-l^{2}\right)^{-\frac{1}{2}} \mathrm{~d} r
$$

which has exactly the same form as equation (42). Consequently, the solution for $f(r)$ is, from equation (43),

$$
f(r)=T(r)
$$

or

$$
\begin{equation*}
\mathrm{d} n(r) / \mathrm{d} r-\left(M^{-1}+r^{-1}\right) n(r)=-M^{-1} T(r) \tag{44}
\end{equation*}
$$

Equation (44) is a linear first-order differential equation for $n(r)$ whose solution is

$$
\begin{equation*}
n(r)=M^{-1} r \exp \left(M^{-1} r\right) \int_{r}^{\infty} \exp \left(-M^{-1} u\right) u^{-1} T(u) \mathrm{d} u \tag{45}
\end{equation*}
$$

This formula for $n(r)$ is the exact solution of (44) but, since $M / r \ll 1$, we are at liberty to discard all terms of order $(M / r)^{2}$ and smaller to obtain the correct post-Newtonian expression.

Successive integration by parts in equation (45) allows us to expand $n(r)$ as a power series in $M / r$ and we subsequently arrive at

$$
n(r)=T(r)-(M / r)\left\{T(r)-r R^{\prime}(r)\right\}+\mathcal{O}(M / r)^{2} .
$$

We see immediately that the post-Newtonian term here is negative and we are thus led to the following conclusion. If the number density of stars in a relativistic stellar system were calculated using only Newtonian theory, it would lead to an overestimate of the density in the outer regions where the above formulae would most likely hold to a good approximation.

## 5. Conclusions

We have derived an expression for the projected density distribution that would arise from a given relativistic stellar system. In the post-Newtonian approximation
this expression reduces to a formula which is relatively simple to use once a particular proper number density has been specified for the relativistic stellar system under consideration. It is possible in the post-Newtonian approximation to solve von Zeipel's problem explicitly in the outer regions of such a system, i.e. to obtain the proper number density of stars in the outer regions of such a system from the projected density on a photographic plate. We find that if Newtonian theory alone were used to deduce the stellar number density in a relativistic stellar system from the projected density, the system so constructed would appear to be less centrally condensed than it is in reality. In other words, the system would appear to be less relativistic than it really is since high central condensations are usually synonymous with more highly evolved systems, which in turn are more likely to be relativistic (Fackerell et al. 1969).

The present result is a consequence of the bending of light by the gravitational field. As a result of this bending, some of the stars which appear in the outer regions of the system are actually a little closer in towards the centre than their positions on the photographic plate would indicate (i.e. using Newtonian theory).

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## References

Bisnovatyi-Kogan, G. S., and Zel'dovich, Ya. B. (1969). Astrofizika 5, 223.
Chandrasekhar, S. (1965). Astrophys. J. 142, 1488.
Fackerell, E. D. (1968). Astrophys. J. 153, 643.
Fackerell, E. D., Ipser, J. R., and Thorne, K. S. (1969). Comments Astrophys. space Phys. 1, 134. Ipser, J. R. (1969). Astrophys. J. 158, 17.
Landau, L. D., and Lifshitz, E. M. (1962). 'Classical Theory of Fields' (Pergamon: London).
Synge, J. L. (1966). Mon. Not. R. astr. Soc. 131, 463.
von Zeipel, H. (1908). Annls Obs. Paris Mem. 25, F29.
Zel'dovich, Ya. B., and Novikov, I. D. (1971). 'Relativistic Astrophysics' (Univ. Chicago Press).

