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Stability and the Extended Energy Principle of Plasma Physics

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Abstract

The consequences of stress continuity across the plasma–vacuum interface on plasma stability are considered for a system having zero internal magnetic field, and it is shown that neutral stability is obtained at best. For the extended energy principle of plasma physics the viewpoints of Bernstein *et al.* (1958), Rose and Clark (1961) and Schmidt (1966) are reconciled by means of a rigorous and informative treatment alternative to that of Bernstein *et al.* The present work is a closely related sequel to the treatment of the hydromagnetic energy principle given by Seymour and James (1973). Derivations close with a discussion of results, finally specialized for a plasma system having zero internal magnetic field, in direct relation to the stability treatment by James and Seymour (1971) of a field-free constricted plasma between electrodes.

1. Introduction

In an earlier paper (Seymour and James 1973; hereinafter designated SJ) the approach of Van Kampen and Felderhof (1967) was extended to derive the variation of potential energy δW for a finite magnetized plasma contained by a vacuum magnetic field and in contact with conducting electrodes supported by insulators. To find the perturbation ξ least favourable for stability, δW has to be minimized with respect to all possible ξ. As shown by Bernstein *et al.* (1958), continuity of stress at the plasma–vacuum boundary leads, in the notation of SJ, to the constraint relationship

$$-\gamma p_0 \nabla . \xi + \mu_0^{-1} \boldsymbol{B}_0 . (\boldsymbol{Q} + \xi . \nabla \boldsymbol{B}_0) = \mu_0^{-1} \hat{\boldsymbol{B}}_0 . (\nabla \times \delta \hat{\boldsymbol{A}} + \xi . \nabla \hat{\boldsymbol{B}}_0), \tag{1}$$

which restricts the freedom of choice of ξ. To overcome this difficulty Bernstein *et al.* (p. 23) give a short proof which, provided δW is written in an appropriate form, suggests that the energy principle can be extended to displacements ξ that do not satisfy the constraint relation (1).

Rose and Clark (1961) discuss the apparently drastic extension indicated by Bernstein *et al.* (1958) from a physical standpoint, but state towards the end of their discussion that: 'we have seemingly extended the validity of the energy principle'. This remark is then followed by some reservations about the hydromagnetic model assumed. On the other hand, Schmidt (1966), although actually ignoring the constraint condition (1) in his analysis of the linear pinch, does not refer explicitly to the extended energy principle, but states: 'this condition was used in deriving δW_S and is already incorporated in equation (5–44)'.

The viewpoints of Bernstein *et al.* (1958), Rose and Clark (1961) and Schmidt (1966) do not perfectly coincide. In this paper we give a rigorous and conclusive treatment of the extended energy principle which yields results in agreement with those of Bernstein *et al.* and also resolves the attitudes of Rose and Clark and of Schmidt. Prior to this we briefly consider the bearing of equation (1) on plasma stability.

2. Consequences of Stress Continuity across the Plasma–Vacuum Interface S_{pv}

Using the nomenclature of SJ, the special case of a system having zero internal magnetic field proves to be tractable and informative. For this case

$$\nabla p_0 = 0, \tag{2}$$

so that p_0 is constant throughout the plasma, and

$$\boldsymbol{Q} = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_0) = 0. \tag{3}$$

Thus equations (61), (62) and (64) of SJ (hereinafter SJ(61) etc.) reduce to

$$\delta W_{\mathrm{F}} = \tfrac{1}{2}\gamma p_0 \int_{\tau_{\mathrm{p}}(0)} \mathrm{d}\tau_0\,(\nabla.\boldsymbol{\xi})^2, \tag{4}$$

$$\delta W_{\mathrm{S}} = \tfrac{1}{2}\mu_0^{-1} \int_{S_{\mathrm{pv}}(0)} (\mathrm{d}\boldsymbol{S}_0.\boldsymbol{\xi})\{\boldsymbol{\xi}.\nabla(\tfrac{1}{2}\hat{B}_0^2)\}, \tag{5}$$

$$\delta W_{\mathrm{S}} = \tfrac{1}{2}\mu_0^{-1} \int_{S_{\mathrm{pv}}(0)} \mathrm{d}S_0\,(\boldsymbol{n}_0.\boldsymbol{\xi})^2\,\boldsymbol{n}_0.\nabla(\tfrac{1}{2}\hat{B}_0^2), \tag{6}$$

equations SJ(63) and SJ(67) for δW_{E} remain unchanged, and the constraint equation (1) above simplifies to

$$-\gamma\mu_0 p_0\,\nabla.\boldsymbol{\xi} = \hat{\boldsymbol{B}}_0.\nabla\times\delta\hat{\boldsymbol{A}} + \boldsymbol{\xi}.\nabla(\tfrac{1}{2}\hat{B}_0^2), \tag{7}$$

since

$$\hat{\boldsymbol{B}}_0.\boldsymbol{\xi}.\nabla\hat{\boldsymbol{B}}_0 \equiv \boldsymbol{\xi}.\nabla(\tfrac{1}{2}\hat{B}_0^2).$$

From equations SJ(67) and (5) and (7) above,

$$\begin{aligned}\delta W_{\mathrm{S}}+\delta W_{\mathrm{E}} &= -\tfrac{1}{2}\gamma p_0 \int_{S_{\mathrm{pv}}(0)} (\mathrm{d}\boldsymbol{S}_0.\boldsymbol{\xi})\,\nabla.\boldsymbol{\xi} \\ &= -\tfrac{1}{2}\gamma p_0 \int_{\tau_{\mathrm{p}}(0)} \mathrm{d}\tau_0\,\{\boldsymbol{\xi}.\nabla(\nabla.\boldsymbol{\xi}) + (\nabla.\boldsymbol{\xi})^2\}.\end{aligned} \tag{8}$$

Adding equation (4) to this result,

$$\delta W = \delta W_{\mathrm{F}}+\delta W_{\mathrm{S}}+\delta W_{\mathrm{E}} = -\tfrac{1}{2}\gamma p_0 \int_{\tau_{\mathrm{p}}(0)} \mathrm{d}\tau_0\,\boldsymbol{\xi}.\nabla(\nabla.\boldsymbol{\xi}). \tag{9}$$

It follows that displacements $\boldsymbol{\xi}$ in the class of perturbations which satisfies the condition

$$\boldsymbol{\xi}.\nabla(\nabla.\boldsymbol{\xi}) \equiv 0 \tag{10}$$

cause δW to vanish. This class of $\boldsymbol{\xi}$ clearly includes the incompressible perturbation $\nabla.\boldsymbol{\xi} = 0$. Hence for this special plasma configuration it is evidently not possible for δW to be positive for all possible $\boldsymbol{\xi}$. Therefore such a system cannot be completely stable, but at best only neutrally stable.

An informative alternative form of δW is obtained from equations SJ(63) and (4) and (6) above,

$$\delta W = \tfrac{1}{2}\gamma p_0 \int_{\tau_p(0)} d\tau_0 \, (\nabla \cdot \xi)^2 + \tfrac{1}{2}\mu_0^{-1} \int_{S_{pv}(0)} dS_0 \, (\boldsymbol{n}_0 \cdot \xi)^2 \boldsymbol{n}_0 \cdot \nabla(\tfrac{1}{2}\hat{B}_0^2)$$

$$+ \tfrac{1}{2}\mu_0^{-1} \int_{\hat{\tau}(0)+\tau_i} d\tau_0 \, (\nabla \times \delta \boldsymbol{A})^2 \,. \tag{11}$$

The first and third terms, δW_F and δW_E, in equation (11) are always positive, and if $\boldsymbol{n}_0 \cdot \nabla(\frac{1}{2}\hat{B}_0^2)$ is greater than zero at all points on $S_{pv}(0)$ then the second term, δW_S, is always positive also. Thus $\delta W = 0$ only if $\delta W_F = \delta W_E = \delta W_S = 0$, or when

$$\nabla \cdot \xi \equiv \nabla \times \delta \boldsymbol{A} \equiv \boldsymbol{n}_0 \cdot \xi = 0 \,. \tag{12}$$

The system would therefore have $\delta W > 0$ for all ξ except those displacements satisfying $\boldsymbol{n}_0 \cdot \xi \equiv 0$, for which $\delta W = 0$, in agreement with the neutral stability interpretation of condition (10). However, although the system is not completely stable, but is at best neutrally stable, the least favourable perturbation satisfying $\boldsymbol{n}_0 \cdot \xi = 0$ does not, to first order, physically disturb the plasma surface, and so does not have dire practical consequences.

The system becomes unstable when $\boldsymbol{n}_0 \cdot \nabla(\frac{1}{2}\hat{B}_0^2)$ is negative in some regions of $S_{pv}(0)$, as discussed in Section 4 below.

3. Formal Extension of the Energy Principle

The approach is to minimize δW with respect to ξ, subject to the appropriate boundary conditions. The mathematical difficulties involved in taking account of continuity of stress over the plasma boundary (equation SJ(6)), which in first order yields the constraint equation (1) above, and also in taking account of the vacuum condition

$$\nabla \times \nabla \times \delta \hat{\boldsymbol{A}} = 0 \,, \tag{13}$$

will be avoided by permissibly ignoring these conditions, as will be seen. Using $d\boldsymbol{S}_0' = -d\boldsymbol{S}_0$ and equation SJ(66), the remaining boundary conditions to be satisfied by ξ are

$$d\boldsymbol{S}_0 \times \delta \hat{\boldsymbol{A}} = -(d\boldsymbol{S}_0 \cdot \xi)\hat{\boldsymbol{B}}_0 \tag{14}$$

on the plasma surface $S_{pv}(0)$ and

$$d\boldsymbol{S}_0 \times \delta \boldsymbol{A} = 0 \tag{15}$$

on the conductor surface S_c.

The set of vectors $\tilde{\xi}$ which satisfy equations (1), (13), (14) and (15) is clearly a subset of the set of vectors ξ which satisfy equations (14) and (15) but not necessarily (1) and (13). Therefore the set of $\delta W(\tilde{\xi}, \tilde{\xi})$ is contained in the set of $\delta W(\xi, \xi)$. Hence if $\delta W_{min}(\xi, \xi)$ and $\delta W_{min}(\tilde{\xi}, \tilde{\xi})$ are the potential energy variations obtained by minimizing δW with respect to ξ and $\tilde{\xi}$ respectively then it is concluded that

$$\delta W_{min}(\xi, \xi) \leqslant \delta W_{min}(\tilde{\xi}, \tilde{\xi}) \,, \tag{16}$$

and hence a *sufficient* condition for stability with respect to the actual physical perturbation $\tilde{\xi}$ is that $\delta W_{min}(\xi, \xi) > 0$. While the argument leading to this conclusion is straightforward, it is not so obvious that examination of the sign of $\delta W_{\min}(\xi, \xi)$, with ξ not constrained by equation (1), actually yields a *necessary and sufficient* condition for stability. A detailed mathematical proof of this extended energy principle will now be developed.

(a) $\boldsymbol{\xi}$ ***does not necessarily satisfy the constraint equation (1)***

To establish this important result, consider the perturbation velocity

$$v(r,t) = \frac{\partial \xi(r_0,t)}{\partial t} + \varepsilon \frac{\partial \eta(r_0,t)}{\partial t}, \tag{17}$$

where

$$r - r_0 = \tilde{\xi} = \xi(r_0,t) + \varepsilon\,\eta(r_0,t), \tag{18}$$

ε is a parameter of smallness and η is a finite vector, of zero order in ε on the surface of the plasma, with $|\eta|$ falling rapidly to zero in the distance ε from the surface. η also satisfies the condition

$$\{\partial \eta(r_0,t)/\partial t\} \times \mathrm{d}S(r,t) = 0 \tag{19}$$

and so $\partial \eta/\partial t$ is nonzero only in a volume of order ε, and represents a motion of matter normal to the perturbed fluid surface. η varies only slowly in any direction parallel to the surface, in such a manner that the perturbed pressure and magnetic field satisfy equation SJ(6). The displacement $\xi(r_0, t)$ is of zero order in ε and varies only slowly *in all directions*. The first-order form of equation SJ(6) will now change from equation (1) above, additional terms appearing due to $\varepsilon\eta$ as shown below.

Consider first the standard fluid mechanics result

$$-\frac{1}{\gamma p}\frac{\mathrm{d}p}{\mathrm{d}t} = \nabla . v = \nabla . \frac{\partial \xi}{\partial t} + \varepsilon \nabla . \frac{\partial \eta}{\partial t}, \tag{20}$$

using equation (17). In view of the assumed properties of $\partial\xi/\partial t$, the term $\nabla.(\partial\xi/\partial t)$ in equation (20) is of zero order in ε. Further, to lowest order in ε

$$\varepsilon \nabla.(\partial\eta/\partial t) \sim |\partial\eta/\partial t|, \tag{21}$$

a zero-order result in ε which is readily obtained by expressing nabla as

$$\nabla = n(n.\nabla) - n \times (n \times \nabla) \tag{22}$$

in equation (21) and, bearing in mind the properties of $\partial\eta/\partial t$ assumed above, permissibly neglecting the term perpendicular to n in the resulting expression. The relation (21) shows that $\varepsilon\,\partial\eta/\partial t$ of equation (17) gives a contribution to $\nabla . v$ and thus from (20) a contribution to $\mathrm{d}p/\mathrm{d}t$ which is of zero order in ε. Changes in p due to $\varepsilon\eta$ are therefore of zero order.

For the magnetic field, equations SJ(4) and SJ(5c) give the familiar infinite electrical conductivity result

$$\partial B/\partial t = \nabla \times (v \times B), \tag{23}$$

which, with $\nabla . B = 0$, permits the convective derivative of B to be expressed as

$$\mathrm{d}B/\mathrm{d}t = \partial B/\partial t + v.\nabla B = B.\nabla v - B\nabla . v. \tag{24}$$

From the foregoing it is concluded immediately that the second term on the right-hand side of equation (24) contains a contribution from $\varepsilon\,\partial\boldsymbol{\eta}/\partial t$ which is of zero order in ε. On the other hand, using equation (17),

$$\begin{aligned}\boldsymbol{B}.\nabla\boldsymbol{v} &= \boldsymbol{B}.\nabla(\partial\boldsymbol{\xi}/\partial t)+\varepsilon\boldsymbol{B}.\nabla(\partial\boldsymbol{\eta}/\partial t)\\ &= \boldsymbol{B}.\nabla(\partial\boldsymbol{\xi}/\partial t)+O(\varepsilon),\end{aligned} \tag{25}$$

since, because of the assumed properties of $\partial\boldsymbol{\eta}/\partial t$ and the fact that $\mathrm{d}\boldsymbol{S}.\boldsymbol{B}=0$ at the interface (see SJ, Section III), $\boldsymbol{B}.\nabla(\partial\boldsymbol{\eta}/\partial t)$ is of zero order in ε. Hence $\varepsilon\,\partial\boldsymbol{\eta}/\partial t$ gives rise only to terms of order ε in equation (25). The net result of these contributions is that in equation (24) terms of zero order in ε arise from $\varepsilon\,\partial\boldsymbol{\eta}/\partial t$. Changes in $\boldsymbol{B}$ due to $\varepsilon\boldsymbol{\eta}$ are of zero order.

From the above considerations it is clear that $\boldsymbol{\xi}=\tilde{\boldsymbol{\xi}}-\varepsilon\boldsymbol{\eta}$ does not necessarily satisfy the constraint equation (1).

(b) Expression of $\delta W(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\xi}})$ in terms of $\delta W(\boldsymbol{\xi},\boldsymbol{\xi})$

We now consider expression of $\delta W(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\xi}})$ in terms of

$$\delta W(\boldsymbol{\xi},\boldsymbol{\xi}) = \delta W_{\mathrm{F}}(\boldsymbol{\xi},\boldsymbol{\xi})+\delta W_{\mathrm{S}}(\boldsymbol{\xi},\boldsymbol{\xi})+\delta W_{\mathrm{E}}(\delta\hat{\boldsymbol{A}},\delta\hat{\boldsymbol{A}}).$$

The perturbation velocity $\boldsymbol{v}$ of equation (17) gives rise to a potential energy variation

$$\begin{aligned}\delta W &= -\int_0^t \mathrm{d}t' \int_{\tau_{\mathrm{p}}(t)} \boldsymbol{v}.(\boldsymbol{j}\times\boldsymbol{B}-\nabla p)\,\mathrm{d}\tau\\ &= -\int_0^t \mathrm{d}t' \int_{\tau_{\mathrm{p}}(t)} \boldsymbol{v}.\left\{\frac{\boldsymbol{B}.\nabla\boldsymbol{B}}{\mu_0}-\nabla\left(p+\frac{B^2}{2\mu_0}\right)\right\}\mathrm{d}\tau,\end{aligned} \tag{26}$$

using equation SJ(5b) and the expansion of $\nabla(\boldsymbol{B}.\boldsymbol{B})$. Further transformation is effected by means of Gauss's theorem and the expansion of $\nabla.\{(p+B^2/2\mu_0)\boldsymbol{v}\}$ to give the form

$$\delta W = -\int_0^t \mathrm{d}t'\left[\int_{\tau_{\mathrm{p}}(t)}\left\{\frac{\boldsymbol{v}.\boldsymbol{B}.\nabla\boldsymbol{B}}{\mu_0}+\left(p+\frac{B^2}{2\mu_0}\right)\nabla.\boldsymbol{v}\right\}\mathrm{d}\tau - \int_{S_{\mathrm{pv}}(t)}\left(p+\frac{B^2}{2\mu_0}\right)\boldsymbol{v}.\mathrm{d}\boldsymbol{S}\right]. \tag{27}$$

Evaluation of the Volume Integral

With the help of equation SJ(5a) it is readily established that

$$\boldsymbol{v}.\boldsymbol{B}.\nabla\boldsymbol{B} = \nabla.(\boldsymbol{v}.\boldsymbol{B}\boldsymbol{B})-\boldsymbol{B}.\boldsymbol{B}.\nabla\boldsymbol{v},$$

and so in equation (27) the volume integral

$$\int_{\tau_{\mathrm{p}}(t)}\frac{\boldsymbol{v}.\boldsymbol{B}.\nabla\boldsymbol{B}}{\mu_0}\,\mathrm{d}\tau = -\int_{\tau_{\mathrm{p}}(t)}\frac{\boldsymbol{B}.\boldsymbol{B}.\nabla\boldsymbol{v}}{\mu_0}\,\mathrm{d}\tau + \int_{S_{\mathrm{pv}}(t)}\frac{(\boldsymbol{v}.\boldsymbol{B})\boldsymbol{B}.\mathrm{d}\boldsymbol{S}}{\mu_0},$$

with the aid of Gauss's theorem. Further, in the sheet-current model $\boldsymbol{B}.\mathrm{d}\boldsymbol{S}$ vanishes at all points on S_{pv}, and hence

$$\int_{\tau_{\mathrm{p}}(t)}\frac{\boldsymbol{v}.\boldsymbol{B}.\nabla\boldsymbol{B}}{\mu_0}\,\mathrm{d}\tau = -\int_{\tau_{\mathrm{p}}(t)}\frac{\boldsymbol{B}.\boldsymbol{B}.\nabla\boldsymbol{v}}{\mu_0}\,\mathrm{d}\tau. \tag{28}$$

In terms of this result the entire volume integral in equation (27) becomes

$$\Psi = \int_{\tau_p(t)} \left\{\left(p+\frac{B^2}{2\mu_0}\right)\nabla.\boldsymbol{v}-\frac{\boldsymbol{B}.\boldsymbol{B}.\nabla\boldsymbol{v}}{\mu_0}\right\} \mathrm{d}\tau . \tag{29}$$

From the discussion of the fluid mechanics result (20), $\nabla.\boldsymbol{v}$ is of zero order in ε. Further, from equation (25), $\boldsymbol{B}.\nabla\boldsymbol{v}$ is also of zero order in the smallness parameter. Since p and $\boldsymbol{B}$ are of zero order, it follows that the integrand J of Ψ is also of zero order, and therefore if it is integrated over a volume of order ε then it will yield a result which is of order ε. Changing the domain of the volume integral Ψ to $\tau(t) = \tau_p(t)-\tau_\varepsilon(t)$, where $\tau_\varepsilon(t)$ is the volume in which $\partial\boldsymbol{\eta}/\partial t$ is nonzero, an error of order ε is involved, i.e.

$$\Psi = \int_{\tau_p(t)} J\,\mathrm{d}\tau = \int_{\tau(t)} J\,\mathrm{d}\tau + O(\varepsilon) . \tag{30}$$

In the domain $\tau(t)$, $\boldsymbol{v}(\boldsymbol{r},t) = \partial\boldsymbol{\xi}(\boldsymbol{r}_0,t)/\partial t$, where $\boldsymbol{\xi}(\boldsymbol{r}_0,t)$ has the same properties as in the usual treatments (e.g. Bernstein *et al.* 1958) in which the equation of motion is linearized and perturbed quantities are expressed to first order in the perturbation:

$$p(\boldsymbol{r},t) = p(\boldsymbol{r}_0,t) - \gamma p(\boldsymbol{r}_0,0)\,\nabla.\boldsymbol{\xi} = p_0(1-\gamma\nabla.\boldsymbol{\xi}) , \tag{31}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{B}(\boldsymbol{r}_0,0)+\boldsymbol{Q}+\boldsymbol{\xi}.\nabla\boldsymbol{B}(\boldsymbol{r}_0,0) = \boldsymbol{B}_0+\boldsymbol{Q}+\boldsymbol{\xi}.\nabla\boldsymbol{B}_0 , \tag{32}$$

$$\mathrm{d}\tau = (1+\nabla.\boldsymbol{\xi})\,\mathrm{d}\tau_0 , \tag{33}$$

$$\nabla_r = \nabla_0 - \nabla_0\boldsymbol{\xi}.\nabla_0 , \tag{34}$$

where as usual

$$\boldsymbol{Q} = \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}_0) \tag{35}$$

while, in equations (31)–(33), $\nabla \equiv \nabla_0$.

With the help of equations (31)–(33) the volume integral (30) may, with $\nabla \equiv \nabla_0$, be written to second order in $\boldsymbol{\xi}$ as

$$\begin{aligned}\Psi = \int_{\tau(0)} \mathrm{d}\tau_0 \Bigg[&\left(p_0+\frac{B_0^2}{2\mu_0}\right)\left\{(\nabla.\boldsymbol{\xi})\left(\nabla.\frac{\partial\boldsymbol{\xi}}{\partial t'}\right) - (\nabla\boldsymbol{\xi}).\nabla.\frac{\partial\boldsymbol{\xi}}{\partial t'}\right\} - \gamma p_0(\nabla.\boldsymbol{\xi})\left(\nabla.\frac{\partial\boldsymbol{\xi}}{\partial t'}\right)\\ &+\mu_0^{-1}\left\{(\boldsymbol{B}_0.\boldsymbol{Q})\nabla.\frac{\partial\boldsymbol{\xi}}{\partial t'}+\boldsymbol{B}_0.\boldsymbol{\xi}.(\nabla\boldsymbol{B}_0)\nabla.\frac{\partial\boldsymbol{\xi}}{\partial t'}-\boldsymbol{B}_0.\boldsymbol{B}_0.\left(\nabla\frac{\partial\boldsymbol{\xi}}{\partial t'}\right)\nabla.\boldsymbol{\xi}\right.\\ &+\boldsymbol{B}_0.\boldsymbol{B}_0.(\nabla\boldsymbol{\xi}).\nabla\frac{\partial\boldsymbol{\xi}}{\partial t'}-\boldsymbol{B}_0.\boldsymbol{Q}.\nabla\frac{\partial\boldsymbol{\xi}}{\partial t'}-\boldsymbol{B}_0.\boldsymbol{\xi}.(\nabla\boldsymbol{B}_0).\nabla\frac{\partial\boldsymbol{\xi}}{\partial t'}\\ &\left.-\boldsymbol{\xi}.(\nabla\boldsymbol{B}_0).\boldsymbol{B}_0.\nabla\frac{\partial\boldsymbol{\xi}}{\partial t'}-\boldsymbol{Q}.\boldsymbol{B}_0.\nabla\frac{\partial\boldsymbol{\xi}}{\partial t'}\right\}\Bigg] + O(\varepsilon) .\end{aligned} \tag{36}$$

In this result terms $O(\xi^3)$ and $O(\xi^4)$ have been neglected and first-order terms omitted since, in δW, they must sum to zero because the initial state is an equilibrium state, for which the potential energy function is stationary.

It is now important to show that the time integration of the expression (36) (and of the corresponding second-order expression for the surface integral of equation (27)) can be accomplished without requiring ξ to satisfy the first-order constraint (1). Since r_0 is independent of time, the integrations with respect to time and volume to be performed when Ψ of (36) is substituted into δW of (27) may be commuted: the first time integral to be evaluated is therefore

$$I_1 = \int_0^t \mathrm{d}t' \, (\nabla . \xi)\left(\nabla . \frac{\partial \xi}{\partial t'}\right) = \tfrac{1}{2}(\nabla . \xi)^2 . \tag{37}$$

The next integration is

$$I_2 = -\int_0^t \mathrm{d}t' \, (\nabla \xi) . \nabla . \frac{\partial \xi}{\partial t'} = -\int_0^t \mathrm{d}t' \, (\partial_i \xi_j)(\partial_j \dot{\xi}_i)$$

and, because of the symmetry in i and j on the right-hand side, integration by parts leads to

$$I_2 = -\tfrac{1}{2}(\partial_i \xi_j)(\partial_j \xi_i) = -\tfrac{1}{2}(\nabla \xi) . \nabla . \xi . \tag{38}$$

The terms involving $\boldsymbol{B}_0$ and $\boldsymbol{Q} = \nabla \times (\xi \times \boldsymbol{B}_0)$ in equation (36) can be integrated if $\boldsymbol{Q}$ is first expanded and the terms rearranged to give

$$I_3 = -\mu_0^{-1} \int_0^t \mathrm{d}t' \left(\boldsymbol{B}_0 . \nabla \xi - \boldsymbol{B}_0 \nabla . \xi\right) . \left(\boldsymbol{B}_0 . \nabla \frac{\partial \xi}{\partial t'} - \boldsymbol{B}_0 \nabla . \frac{\partial \xi}{\partial t'}\right),$$

so that

$$I_3 = -\tfrac{1}{2}\mu_0^{-1} \, | \boldsymbol{B}_0 . \nabla \xi - \boldsymbol{B}_0 \nabla . \xi |^2 . \tag{39}$$

Using Ψ as given by equation (36), and the results (37)–(39), the first integral in equation (27) becomes the following volume integral over the region τ_0,

$$I(\tau_0) = -\int_{\tau(0)} \mathrm{d}\tau_0 \, \{(I_1 + I_2)(p_0 + B_0^2/2\mu_0) - \gamma p_0 I_1 + I_3\} + O(\varepsilon), \tag{40}$$

the terms $O(\varepsilon)$ arising from equation (30). If the domain of integration of the volume integral in equation (40) is now changed from $\tau(0)$ to the equilibrium volume $\tau_{\mathrm{p}}(0)$ of the plasma, the error which arises is of order ε. Except for an error of this order, the part of equation (40) of second order in ξ is precisely the $\delta \overline{W}$ expression derived by Van Kampen and Felderhof (1967, p. 75, equation (20)) for a system comprising fluid only. It can be transformed to give the more familiar form, equation SJ(11), without requiring ξ to satisfy equation (1) of this treatment. Thus

$$I(\tau_0) = \delta \overline{W} + O(\varepsilon), \tag{41}$$

where $\delta \overline{W}$ is given by equation SJ(11).

Evaluation of the Surface Integral

Since p and $\boldsymbol{B}$ are such that continuity of stress over the boundary (equation SJ(6)) is satisfied, the surface term in equation (27) may be written

$$\Sigma = \int_0^t \mathrm{d}t' \int_{S_{\mathrm{pv}}(t)} \tfrac{1}{2}\mu_0^{-1} \hat{B}^2 \boldsymbol{v} . \mathrm{d}\boldsymbol{S}, \tag{42}$$

with $\mathrm{d}\boldsymbol{S}$ directed out of the plasma.

Using the chain rule operator

$$\nabla_r \equiv \nabla_0 - \nabla_r \xi . \nabla_0 ,$$

the change occurring in $\mathrm{d}\boldsymbol{S}$ as the perturbation develops is obtained from equation SJ(30) as

$$\mathrm{d}(\mathrm{d}\boldsymbol{S})/\mathrm{d}t = (\nabla_0 . \boldsymbol{v})\mathrm{d}\boldsymbol{S}_0 - (\nabla_0 \boldsymbol{v}) . \mathrm{d}\boldsymbol{S}_0 \tag{43}$$

to first order in ξ. Hence, using equation (17) and the usual dot notation to indicate a time derivative, equation (43) becomes

$$\mathrm{d}(\mathrm{d}\boldsymbol{S})/\mathrm{d}t = (\nabla_0 . \dot{\xi})\mathrm{d}\boldsymbol{S}_0 - \nabla_0 \dot{\xi} . \mathrm{d}\boldsymbol{S}_0 + \nabla_0 . (\varepsilon\dot{\eta})\mathrm{d}\boldsymbol{S}_0 - \nabla_0(\varepsilon\dot{\eta}) . \mathrm{d}\boldsymbol{S}_0 . \tag{44}$$

Recalling the prescribed properties of η, equation (44) can be integrated to obtain the first-order result

$$\mathrm{d}\boldsymbol{S} = \mathrm{d}\boldsymbol{S}_0 + (\nabla_0 . \xi)\mathrm{d}\boldsymbol{S}_0 - (\nabla_0 \xi) . \mathrm{d}\boldsymbol{S}_0 + O(\varepsilon), \tag{45}$$

as shown in Appendix 1.

To obtain an expression for $\hat{\boldsymbol{B}}$ on the perturbed surface to first order in ξ, two cases must be considered, corresponding firstly to ξ at the surface directed out of the plasma, and secondly to ξ directed into the plasma. In each case, as shown in Appendix 2, the result has the common first-order form

$$\hat{\boldsymbol{B}}(\boldsymbol{r}, t) = \hat{\boldsymbol{B}}_0 + \xi . \nabla_0 \hat{\boldsymbol{B}}_0 + \delta\hat{\boldsymbol{B}} + O(\varepsilon), \tag{46}$$

where the first-order quantity

$$\delta\hat{\boldsymbol{B}} = \hat{\boldsymbol{B}}(\boldsymbol{r}, t) - \hat{\boldsymbol{B}}(\boldsymbol{r}, 0). \tag{47}$$

Using equations (45) and (46), the quantity Σ of equation (42) becomes, with retention of second-order terms only,

$$\Sigma' = \mu_0^{-1} \int_0^t \mathrm{d}t' \int_{S_{\mathrm{pv}}(0)} \Big\{ \tfrac{1}{2}\hat{B}_0^2 \frac{\partial \xi}{\partial t'} . \Big((\nabla . \xi)\mathrm{d}\boldsymbol{S}_0 - (\nabla \xi) . \mathrm{d}\boldsymbol{S}_0 \Big)$$

$$+ \hat{\boldsymbol{B}}_0 . \delta\hat{\boldsymbol{B}} \frac{\partial \xi}{\partial t'} . \mathrm{d}\boldsymbol{S}_0 + \hat{\boldsymbol{B}}_0 . \xi . \nabla \hat{\boldsymbol{B}}_0 \frac{\partial \xi}{\partial t'} . \mathrm{d}\boldsymbol{S}_0 \Big\} + O(\varepsilon), \tag{48}$$

where $\nabla \equiv \nabla_0$.

Since $\hat{\boldsymbol{E}} = -\partial\hat{\boldsymbol{A}}/\partial t$, the boundary condition SJ(24) on $S_{\mathrm{pv}}(t)$ becomes, with $\mathrm{d}\boldsymbol{S} = -\mathrm{d}\boldsymbol{S}'$,

$$(\partial\hat{\boldsymbol{A}}/\partial t) \times \mathrm{d}\boldsymbol{S} = (\mathrm{d}\boldsymbol{S} . \boldsymbol{v})\hat{\boldsymbol{B}}. \tag{49}$$

With $\delta\hat{\boldsymbol{B}} = \nabla \times \delta\hat{\boldsymbol{A}}$, where $\delta\hat{\boldsymbol{A}}(\boldsymbol{r}, t)$ is the first-order perturbation $\hat{\boldsymbol{A}}(\boldsymbol{r}, t) - \hat{\boldsymbol{A}}(\boldsymbol{r}, 0)$, equations (17), (45) and (46) may be substituted into equation (49) to obtain the first-order result

$$\mathrm{d}\boldsymbol{S}_0 \times \frac{\partial(\delta\hat{\boldsymbol{A}})}{\partial t} = -\left(\mathrm{d}\boldsymbol{S}_0 . \frac{\partial \xi}{\partial t}\right)\hat{\boldsymbol{B}}_0 + O(\varepsilon), \tag{50}$$

remembering that $\partial\hat{\boldsymbol{A}}(\boldsymbol{r}, 0)/\partial t = 0$.

Equation SJ(33) becomes, with retention of terms in the integrand to second order in ξ,

$$\mu_0\,\delta W_{\mathrm{BE}} = \int_{S_{\mathrm{pv}}(0)} \mathrm{d}\boldsymbol{S}_0 \cdot \int_0^t \mathrm{d}t' \left\{ \tfrac{1}{2}\hat{B}_0^2 \left(\frac{\partial \xi}{\partial t'} + (\nabla \cdot \xi)\frac{\partial \xi}{\partial t'} - \frac{\partial \xi}{\partial t'} \cdot \nabla \xi \right) + \xi \cdot \nabla(\tfrac{1}{2}\hat{B}_0^2)\frac{\partial \xi}{\partial t'} + \hat{\boldsymbol{B}}_0 \cdot \nabla \times \delta \hat{A}\frac{\partial \xi}{\partial t'} \right\}, \tag{51}$$

where W_{BE} is the magnetic energy external to the plasma.

Comparing equation (48) with (51), and SJ(38) with (50) above, shows that the evaluation of Σ' to zero order in ε reduces to the procedure followed in SJ, Section III*b*, for obtaining the second-order part of δW_{BE}, which did not involve the use of the constraint equation (1) but where, of course, ξ was constrained by that equation. Thus Σ' is given by the second-order part of equation SJ(59), plus terms of order ε, that is,

$$\Sigma' = \delta W_{\mathrm{BE}} + O(\varepsilon) = \tfrac{1}{2}\mu_0^{-1} \int_{S_{\mathrm{pv}}(0)} \mathrm{d}\boldsymbol{S}_0 \cdot \{ \tfrac{1}{2}\hat{B}_0^2(\xi\, \nabla \cdot \xi - \xi \cdot \nabla \xi) + \xi\, \xi \cdot \nabla(\tfrac{1}{2}\hat{B}_0^2) \} + \tfrac{1}{2}\mu_0^{-1} \int_{\hat{\tau}(0)+\tau_{\mathrm{i}}} (\nabla \times \delta \boldsymbol{A})^2 \,\mathrm{d}\tau_0 + O(\varepsilon). \tag{52}$$

In Section III*c* of SJ the derivation of equation SJ(60) for δW in terms of the functionals δW_{F}, δW_{S} and δW_{E} under the conditions cited resulted from combination of $\delta \overline{W}$ and δW_{BE}. Under the conditions given at the beginning of Section III*c* of SJ, $\delta \overline{W}$ of equation SJ(11) assumes the explicit form

$$\delta \overline{W} = \tfrac{1}{2} \int_{\tau_{\mathrm{p}}(0)} \mathrm{d}\tau_0 \,\{ \mu_0^{-1} |\boldsymbol{Q}|^2 - \boldsymbol{j}_0 \cdot (\boldsymbol{Q} \times \xi) + \gamma p_0 (\nabla \cdot \xi)^2 + (\nabla \cdot \xi)\xi \cdot \nabla p_0 \} - \tfrac{1}{2}\mu_0^{-1} \int_{S_{\mathrm{pv}}(0)} \mathrm{d}\boldsymbol{S}_0 \cdot \{ \tfrac{1}{2}\hat{B}_0^2(\xi\, \nabla \cdot \xi - \xi \cdot \nabla \xi) + (\xi \cdot \boldsymbol{B}_0 \cdot \nabla \boldsymbol{B}_0)\xi \}. \tag{53}$$

Combination of this result with the second-order form of δW_{BE} obtainable from equation SJ(59) leads to the form of δW required here, namely

$$\delta W = \tfrac{1}{2} \int_{\tau_{\mathrm{p}}(0)} \mathrm{d}\tau_0 \,\{ \mu_0^{-1} |\boldsymbol{Q}|^2 - \boldsymbol{j}_0 \cdot (\boldsymbol{Q} \times \xi) + \gamma p_0 (\nabla \cdot \xi)^2 + (\nabla \cdot \xi)\xi \cdot \nabla p_0 \} + \tfrac{1}{2}\mu_0^{-1} \int_{S_{\mathrm{pv}}(0)} \mathrm{d}\boldsymbol{S}_0 \cdot \xi \{ \xi \cdot \nabla(\tfrac{1}{2}\hat{B}_0^2) - \xi \cdot \boldsymbol{B}_0 \cdot \nabla \boldsymbol{B}_0 \} + \tfrac{1}{2}\mu_0^{-1} \int_{\hat{\tau}(0)+\tau_{\mathrm{i}}} (\nabla \times \delta \boldsymbol{A})^2 \,\mathrm{d}\tau_0 . \tag{54}$$

On the basis of the rigorous derivation given, we now conclude that equations (41) and (52) permit the second-order variation in potential energy to be obtained from equation (27) as

$$\delta W(\tilde{\xi}, \tilde{\xi}) = \delta \overline{W} + \delta W_{\mathrm{BE}} + O(\varepsilon) = \delta W(\xi, \xi) + O(\varepsilon) = \delta W_{\mathrm{F}}(\xi, \xi) + \delta W_{\mathrm{S}}(\xi, \xi) + \delta W_{\mathrm{E}}(\delta \boldsymbol{A}, \delta \boldsymbol{A}) + O(\varepsilon), \tag{55}$$

where $\delta W(\xi,\xi)$ is given by equation (54), and δW_F, δW_S and δW_E are the same functionals as appear in equation SJ(60), but here it has been shown that ξ need not satisfy the constraint equation (1).

4. Discussion

From the foregoing it is quite clear that for a given functional $\delta W(\xi,\xi)$ of the small, slowly varying function ξ, which does not necessarily satisfy equation (1) but which is such that equations (13), (14) and (15) are satisfied, there is a physically realizable small perturbation $\tilde{\xi}$ such that equation SJ(6) is satisfied and which makes the second-order variation in potential energy arbitrarily close to $\delta W(\xi,\xi)$. Thus

$$\delta W_{\min}(\tilde{\xi},\tilde{\xi}) = \delta W_{\min}(\xi,\xi) + O(\varepsilon), \tag{56}$$

and so a *necessary and sufficient* condition for stability is obtained by examining the sign of $\delta W_{\min}(\xi,\xi)$, while in minimizing $\delta W(\xi,\xi)$ the boundary condition (1) may be ignored.

As pointed out by Bernstein *et al.* (1958, p. 24) a *sufficient* condition for instability can be obtained by using the same form of functional, but without requiring $\delta\hat{A}$ to satisfy equation (13). This arises because (13) is the Euler equation for minimizing δW_E subject to equations (14) and (15). Hence if $\delta\hat{A}$ appearing in δW_E given by equation SJ(63) does not satisfy equation (13) above, another function, $\delta\hat{A}^*$ say, which does satisfy (13) would certainly decrease δW_E without changing δW_F or δW_S. Summarizing, if functions ξ and $\delta\hat{A}$ are found which satisfy equations (14) and (15), but not necessarily equations (1) and (13), and which make the functional $\delta W(\xi,\xi)$ negative, then there is a physically realizable perturbation $\tilde{\xi}$ for which δW is certainly negative and the system is unstable.

To conclude, for a system with zero internal magnetic field the potential energy variation corresponding to equation (11) becomes, from the extended energy principle result (55) and use of equations (4), (6) and SJ(63),

$$\delta W(\tilde{\xi},\tilde{\xi}) = \tfrac{1}{2}\gamma p_0 \int_{\tau_p(0)} d\tau_0\,(\nabla\cdot\xi)^2 + \tfrac{1}{2}\mu_0^{-1}\int_{S_{pv}(0)} dS_0\,(\boldsymbol{n}_0\cdot\xi)^2\boldsymbol{n}_0\cdot\nabla(\tfrac{1}{2}\hat{B}_0^2)$$
$$+\tfrac{1}{2}\mu_0^{-1}\int_{\hat{\tau}(0)+\tau_i} d\tau_0\,(\nabla\times\delta A)^2 + O(\varepsilon). \tag{57}$$

However, unlike ξ in equation (11), in this result ξ is not constrained by equation (1). Hence, reverting to the discussion following equation (11), there is freedom here to choose an incompressible perturbation ξ, where $\boldsymbol{n}_0\cdot\xi$ is nonzero only within the surface fluting region R. Introducing $\boldsymbol{K}$, the vector curvature of the magnetic lines of force, given by

$$\hat{B}_0^2\boldsymbol{K} = \boldsymbol{n}_0\,\boldsymbol{n}_0\cdot\nabla(\tfrac{1}{2}\hat{B}_0^2)$$

for the field-free system (James and Seymour 1971, equation (34)), and using equation (57) with ξ solenoidal,

$$\delta W(\tilde{\xi},\tilde{\xi}) = \tfrac{1}{2}\mu_0^{-1}\int_{S_{pv}(0)} dS_0\,(\boldsymbol{n}_0\cdot\xi)^2\hat{B}_0^2 K + \tfrac{1}{2}\mu_0^{-1}\int_{\hat{\tau}(0)+\tau_i} d\tau_0\,(\nabla\times\delta A)^2 + O(\varepsilon), \tag{58}$$

where $\boldsymbol{K} = n_0 K$. As shown by Bernstein *et al.* (1958, p. 31), it is possible to make the magnitude of the volume integral in equation (58) arbitrarily small compared with that of the surface integral, and therefore a necessary and sufficient condition for instability can be obtained by examining the sign of the surface integral alone.

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Appendix 1. Derivation of Equation (45)

In a local cartesian coordinate system with $\boldsymbol{e}_z \times \mathrm{d}\boldsymbol{S}_0 = 0$ and $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ tangential to the plasma surface at a point, equation (44) can be expressed as

$$\begin{aligned}
\mathrm{d}(\mathrm{d}\boldsymbol{S})/\mathrm{d}t &= (\nabla_0 \cdot \dot{\boldsymbol{\xi}})\mathrm{d}\boldsymbol{S}_0 - (\nabla_0 \dot{\boldsymbol{\xi}}) \cdot \mathrm{d}\boldsymbol{S}_0 + \varepsilon(\partial_i \dot{\eta}_i)\mathrm{d}\boldsymbol{S}_0 - \boldsymbol{e}_i\, \varepsilon(\partial_i \dot{\eta}_j)\mathrm{d}S_{0j} \\
&= (\nabla_0 \cdot \dot{\boldsymbol{\xi}})\mathrm{d}\boldsymbol{S}_0 - (\nabla_0 \dot{\boldsymbol{\xi}}) \cdot \mathrm{d}\boldsymbol{S}_0 + \varepsilon(\partial_z \dot{\eta}_z)\mathrm{d}\boldsymbol{S}_0 - \boldsymbol{e}_z\, \varepsilon(\partial_z \dot{\eta}_j)\mathrm{d}S_{0j} + O(\varepsilon) \\
&= (\nabla_0 \cdot \dot{\boldsymbol{\xi}})\mathrm{d}\boldsymbol{S}_0 - (\nabla_0 \dot{\boldsymbol{\xi}}) \cdot \mathrm{d}\boldsymbol{S}_0 + \varepsilon(\partial_z \dot{\eta}_z)\mathrm{d}\boldsymbol{S}_0 - \varepsilon(\partial_z \dot{\eta}_z)\boldsymbol{e}_z\, \mathrm{d}S_0 + O(\varepsilon)\,, \qquad \text{(A1)}
\end{aligned}$$

bearing in mind the properties assumed for the finite vector $\boldsymbol{\eta}$ and that $\mathrm{d}\boldsymbol{S}_0 = \boldsymbol{e}_z\, \mathrm{d}S_0$. Cancellation of terms gives to first order in $\boldsymbol{\xi}$ the perturbation

$$\mathrm{d}(\mathrm{d}\boldsymbol{S})/\mathrm{d}t = (\nabla_0 \cdot \dot{\boldsymbol{\xi}})\mathrm{d}\boldsymbol{S}_0 - (\nabla_0 \dot{\boldsymbol{\xi}}) \cdot \mathrm{d}\boldsymbol{S}_0 + O(\varepsilon)\,, \qquad \text{(A2)}$$

which can then be integrated with respect to time to yield the first-order result (45) of Section 3*b*.

Appendix 2. Derivation of Equation (46)

Case (*a*). Initially choose a physical perturbation $\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi} + \varepsilon\boldsymbol{\eta}$ at the surface which is directed out of the plasma. In this case it is first necessary to consider the effect of the change in time occurring at the point $\boldsymbol{r}$ to which the fluid element is displaced,

$$\hat{\boldsymbol{B}}(\boldsymbol{r}, t) = \hat{\boldsymbol{B}}(\boldsymbol{r}, 0) + \delta\hat{\boldsymbol{B}}(\boldsymbol{r}, t)\,. \qquad \text{(A3)}$$

It is now permissible to express $\hat{\boldsymbol{B}}(\boldsymbol{r}, 0)$ in terms of a Taylor expansion, considered to be made at $t = 0$, to account for the spatial displacement from $\boldsymbol{r}_0$ to $\boldsymbol{r}$. Then to first order

$$
\begin{aligned}
\hat{\boldsymbol{B}}(\boldsymbol{r},t) &= \hat{\boldsymbol{B}}(\boldsymbol{r}_0,0)+\tilde{\boldsymbol{\xi}}\,.\,\nabla_0\,\hat{\boldsymbol{B}}(\boldsymbol{r}_0,0)+\delta\hat{\boldsymbol{B}}(\boldsymbol{r},t)\\
&= \hat{\boldsymbol{B}}(\boldsymbol{r}_0,0)+\boldsymbol{\xi}\,.\,\nabla_0\,\hat{\boldsymbol{B}}(\boldsymbol{r}_0,0)+\varepsilon\boldsymbol{\eta}\,.\,\nabla_0\hat{\boldsymbol{B}}(\boldsymbol{r}_0,0)+\delta\hat{\boldsymbol{B}}(\boldsymbol{r},t)\\
&= \hat{\boldsymbol{B}}_0+\boldsymbol{\xi}\,.\,\nabla_0\,\hat{\boldsymbol{B}}_0+\delta\hat{\boldsymbol{B}}(\boldsymbol{r},t)+O(\varepsilon)\,.
\end{aligned}
\tag{A4}
$$

Case (*b*). Next choose a perturbation $\tilde{\boldsymbol{\xi}}$ at the surface which is directed into the plasma. In this case an equation analogous to (A3) cannot be written because the function $\hat{\boldsymbol{B}}(\boldsymbol{r}_0,0)$ does not exist. However, it is physically meaningful to consider the spatial effect initially, and so application of Taylor's expansion gives, to first order,

$$
\hat{\boldsymbol{B}}(\boldsymbol{r},t) = \hat{\boldsymbol{B}}(\boldsymbol{r}_0,t)+\tilde{\boldsymbol{\xi}}\,.\,\nabla_0\,\hat{\boldsymbol{B}}(\boldsymbol{r}_0,t)\,.
\tag{A5}
$$

Expanding the vacuum Maxwell equations $\nabla\times\hat{\boldsymbol{B}} = \nabla\,.\,\hat{\boldsymbol{B}} = 0$ in a local cartesian coordinate system having $\boldsymbol{e}_z\times \mathrm{d}\boldsymbol{S}_0 = 0$ as before, and recalling that perturbed quantities have been assumed to vary slowly in directions parallel to the surface, it is found that $\hat{\boldsymbol{B}}$ must vary slowly *in all directions.* Therefore equation (A5) becomes

$$
\hat{\boldsymbol{B}}(\boldsymbol{r},t) = \hat{\boldsymbol{B}}(\boldsymbol{r}_0,t)+\boldsymbol{\xi}\,.\,\nabla_0\,\hat{\boldsymbol{B}}(\boldsymbol{r}_0,t)+O(\varepsilon)\,,
$$

and then consideration of the change of $\hat{\boldsymbol{B}}(\boldsymbol{r}_0,t)$ with time leads to the first-order result

$$
\begin{aligned}
\hat{\boldsymbol{B}}(\boldsymbol{r},t) &= \hat{\boldsymbol{B}}(\boldsymbol{r}_0,0)+\delta\hat{\boldsymbol{B}}(\boldsymbol{r}_0,t)+\boldsymbol{\xi}\,.\,\nabla_0\,\hat{\boldsymbol{B}}(\boldsymbol{r}_0,0)+O(\varepsilon)\\
&= \hat{\boldsymbol{B}}_0+\boldsymbol{\xi}\,.\,\nabla_0\,\hat{\boldsymbol{B}}_0+\delta\hat{\boldsymbol{B}}(\boldsymbol{r}_0,t)+O(\varepsilon)\,,
\end{aligned}
\tag{A6}
$$

which is consistent with the result (A4) for case (*a*).

Putting $\delta\hat{\boldsymbol{B}}(\boldsymbol{r},t)\approx\delta\hat{\boldsymbol{B}}(\boldsymbol{r}_0,t) = \delta\hat{\boldsymbol{B}}$, we have for cases (*a*) and (*b*) above the common result (46) of Section 3*b*.

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