Aust. J. Phys., 1975, **28**, 115–25

Quantization of Fields in Accordance with Modular Statistics

H. S. Green

Department of Mathematical Physics, University of Adelaide, P.O. Box 498, Adelaide, S.A. 5001.

Abstract

A new generalization of quantum statistics is described, which is different from parastatistics, though it includes Fermi statistics and parafermi statistics of order two. It can be applied to the quantization of nonlinear field theories, without violating the correspondence principle. Like parastatistics, it allows the occupation of a given dynamical state by more than one particle of half-odd-integral spin. A special feature is that quark-like particles are naturally associated in modules, which have many of the characteristics of baryons and mesons. The group theoretical properties of the new statistics, and the implied classification of states, are briefly examined.

1. Introduction

Until recently it has been generally assumed that all kinds of elementary particles must satisfy either Fermi or Bose statistics. The possibility of quantization in accordance with a more general statistics (now called parastatistics) was first demonstrated by the author (Green 1953), but at that time there was no obvious physical application for such a generalization. Greenberg (1964) was the first to suggest that certain difficulties associated with Gell-Mann's (1964) quark hypothesis could be resolved most easily by supposing that quarks satisfied parafermi statistics of order three. However, the experimental discovery of two different kinds of neutrinos (ν_β and ν_μ) has provided the strongest indication that two particles thought to be fermions could have the same energy, momentum and spin. The author has pointed out (1972) that this, and other properties of the leptons, are consistent with the hypothesis that the leptons are parafermions of order two. On the other hand, the applications of parastatistics of order three to the quark model by several authors (Greenberg 1964; Govorkov 1968, 1973*a*; Fritzsch and Gell-Mann 1971; Ramakrishnan *et al.* 1971; Bracken and Green 1973) have encountered difficulties which are still unresolved.

The field theories of interacting particles are nonlinear, and so far it has never been shown that such field theories may be quantized except in accordance with Fermi or Bose statistics. The present paper is the result of an effort to remedy this omission. For reasons which will become apparent, the attempt to quantize in accordance with parastatistics of order greater than two was not successful. Instead, another type of generalized statistics appeared to be required, of which Fermi statistics and parafermi statistics of order two are particular and rather trivial examples. The new generalization, here called modular statistics, ensures a natural grouping of the fundamental 'quarks' in modules of a characteristic number m. These modules may represent composite particles satisfying Fermi or Bose statistics. The modular statistics could also find applications to relativistic 'string' models (Nambu 1970; Gervais and Sakita 1973), with similar advantages over the parastring model (Ardalan and Mansouri 1974) which makes use of parastatistics.

In the following, we shall consider only the quantization of fields representing elementary particles of half-odd-integral spin, for which, by a modification of Pauli's (1940) arguments, generalizations of Fermi statistics rather than Bose statistics should be considered. There is a similar generalization of Bose statistics, which will not, however, be considered in this paper.

2. Nonlinear Field Theories and the Correspondence Principle

We begin by considering a classical field theory, with a Lagrangian density L which is explicitly a function of a set of field variables ϕ_α and their derivatives $\phi_{\alpha,\lambda}$ with respect to the space–time coordinates x^λ ($\lambda = 0,1,2,3;\ x^0 = t$). An arbitrary small variation $\delta\phi_\alpha$, $\delta\phi_{\alpha,\lambda}$ of the ϕ_α, $\phi_{\alpha,\lambda}$ results in a change in L of the form

$$\delta L = \pi^\alpha\,\delta\phi_\alpha + \pi^{\alpha\lambda}\,\delta\phi_{\alpha,\lambda}, \tag{1}$$

if we adopt Einstein's convention that repeated Greek affixes are to be summed over all admissible values. The field equations are

$$\pi^{\alpha\lambda}{}_{,\lambda} = \pi^\alpha, \tag{2}$$

and the energy–momentum four-vector is

$$P_\lambda = \int (\pi^{\alpha 0}\,\phi_{\alpha,\lambda} - L\delta^0{}_\lambda)\,\mathrm{d}^3x, \tag{3}$$

where the integration is over all space, or over a closed rectangular region in and on which the field variables are periodic functions of the spatial coordinates.

Without loss of generality, we may impose two restrictions on the field theory. Firstly, we may suppose that the field equations, though nonlinear, are of the first-order, so that the $\pi^{\alpha\lambda}$ are functions of the ϕ_α alone and not of the $\phi_{\alpha,\lambda}$. For, if the $\pi^{\alpha\lambda}$ should depend on the $\phi_{\alpha,\lambda}$, we could introduce new field variables $\phi_{\alpha\lambda}$ ($= \phi_{\alpha,\lambda}$) and $\bar{\phi}^{\alpha\lambda}$ ($= -\mathrm{i}\pi^{\alpha\lambda}$) and replace the Lagrangian density $L(\phi_\alpha, \phi_{\alpha,\lambda})$ by

$$L'(\phi_\alpha, \phi_{\alpha\lambda}, \bar{\phi}^{\alpha\lambda}, \phi_{\alpha,\lambda}) = L(\phi_\alpha, \phi_{\alpha\lambda}) + \mathrm{i}\bar{\phi}^{\alpha\lambda}(\phi_{\alpha,\lambda} - \phi_{\alpha\lambda}). \tag{4}$$

The field equations and the energy–momentum vector derived from this Lagrangian density reduce to equations (2) and (3) on elimination of $\phi_{\alpha\lambda}$ and $\bar{\phi}^{\alpha\lambda}$. But the field equations are of the first order, and we may therefore assume that the $\pi^{\alpha\lambda}$ are functions of the field variables only.

Secondly, we may assume that the nonvanishing $\pi^{\alpha 0}$ and the corresponding field variables ϕ_α are functionally independent. For otherwise, we can secure this by effecting a change of field variables. The field equations and energy–momentum vectors are, of course, invariant under a change of field variables, and so is the linear form $\pi^{\alpha 0}\,\delta\phi_\alpha$. By standard processes of analysis we can determine a new set of field variables $\tilde{\phi}_\beta$ such that

$$\pi^{\alpha 0}\,\delta\phi_\alpha = \tilde{\pi}^{\beta 0}\,\delta\tilde{\phi}_\beta + \delta\phi^0, \tag{5}$$

where the nonvanishing $\tilde{\pi}^{\beta 0}$ and the corresponding field variables $\tilde{\phi}_\beta$ are functionally unrelated. Also, the appearance of the term $\delta\phi^0$ in equation (5) can be eliminated by replacing the Lagrangian density L with

$$\tilde{L} = L-(\partial\phi^0/\partial\phi_\alpha)\phi_{\alpha,0}, \tag{6}$$

which makes no difference to the field equations or the energy–momentum vector. Assuming then that the nonvanishing $\pi^{\alpha 0}$ and the corresponding ϕ_α are functionally independent, we identify these ϕ_α as the canonical field variables, and the nonvanishing

$$\bar{\phi}^\alpha = -\mathrm{i}\pi^{\alpha 0} \tag{7}$$

as canonically conjugate variables. Any other independent functions of the field variables are said to be ignorable; they can be expressed in terms of the canonical variables with the help of the field equations.

We now consider the quantization of the classical field theory, reduced if necessary to canonical form by the procedures outlined above. Any valid scheme of quantization must satisfy the requirements of Heisenberg's principle, in the form

$$[\phi_\alpha, P_\lambda] = \mathrm{i}\phi_{\alpha,\lambda}, \qquad [\bar{\phi}^\alpha, P_\lambda] = \mathrm{i}\bar{\phi}^\alpha{}_{,\lambda}. \tag{8a, b}$$

In addition, it is important in a nonlinear field theory to pay attention to the correspondence principle, which requires that the quantized field variables should satisfy field equations identical in form with those of the classical field theory. As the ϕ_α do not necessarily commute or anticommute, except when quantization is in accordance with Bose or Fermi statistics, this requirement is quite exacting. For parafermi statistics of order two, it may be satisfied with the help of a formalism developed by Carey (1972) and K. Skillman (personal communication). But, for parastatistics of order three, it is apparently impossible to satisfy the correspondence principle in this way when the field equations are nonlinear, and a different kind of generalization of quantum statistics is to be desired.

If we are to derive quantized field equations by the variational method, we need a quantal analogue of the formula (1), in which the field variations $\delta\phi_\alpha$ and $\delta\phi_{\alpha,\lambda}$ are right multipliers of the products in which they appear. To facilitate a change in the order of factors which do not commute or anticommute, we introduce a cyclic permutation operator u satisfying

$$u^m = 1 \qquad (u^r \neq 1, \quad 0 < r < m) \tag{9}$$

for *modular* statistics of order m, and

$$\phi_\alpha^{(0)}\phi_\beta^{(r)} = -\phi_\beta^{(r-1)}\phi_\alpha^{(1)}, \qquad \phi_\alpha^{(r)} = u^{-r}\phi_\alpha u^r \qquad (r = 1, 2, ..., m) \tag{10}$$

when ϕ_α and ϕ_β are canonical field variables for the same time t. The superscript of $\phi_\alpha^{(r)}$ may be regarded as an indicator of the number of places the field variable has been moved to the right by interchanging the order of the field variables. We shall see in Section 5 how to define the operator u independently by its effect on the state vectors of the quantized field theory.

We now effect a formal extension of equations (10) to products involving time derivatives of the canonical field variables. If χ_α represents a canonical field variable, or the time derivative of a canonical field variable, we propose that a product $\chi_\alpha \chi_\beta$ appearing in the classical theory shall be replaced in the quantum theory by the product $(\chi_\alpha \chi_\beta)$ defined as:

$$(\chi_\alpha \chi_\beta) = \chi_\alpha^{(0)} \chi_\beta^{(0)}, \qquad t_\alpha > t_\beta, \tag{11a}$$

$$= -\chi_\beta^{(-1)} \chi_\alpha^{(1)}, \qquad t_\beta > t_\alpha, \tag{11b}$$

$$= \tfrac{1}{2}(\chi_\alpha^{(0)} \chi_\beta^{(0)} - \chi_\beta^{(-1)} \chi_\alpha^{(1)}), \qquad t_\alpha = t_\beta \tag{11c}$$

with

$$\chi_\alpha^{(r)} = u^{-r} \chi_\alpha u^r. \tag{11d}$$

More generally, a product $\chi_\alpha \chi_\beta \ldots \chi_\omega$ in the classical theory is to be replaced by a generalized time-ordered product $(\chi_\alpha \chi_\beta \ldots \chi_\omega)$. The definition of $(\chi_\alpha^{(r)} \chi_\beta^{(s)} \ldots \chi_\omega^{(z)})$ requires the factors $\chi_\alpha^{(r)}$ to be re-ordered in the reverse of the order of their time variables, with a change of sign if an odd permutation is involved; for equal times, the mean of the signed permutations is required, to allow the representation of such products as Fourier transforms with respect to the time variables, in spite of possible discontinuities when the order of the factors is changed. On the understanding that products are always to be interpreted in this way, the parentheses may be omitted from expressions like $(\chi_\alpha^{(r)} \chi_\beta^{(s)} \ldots \chi_\omega^{(z)})$. However, we shall then need to introduce the generalized commutators and anticommutators

$$[\chi_\alpha^{(r)}, \chi_\beta^{(s)} \chi_\gamma^{(t)}] = \chi_\alpha^{(r)}(\chi_\beta^{(s)} \chi_\gamma^{(t)}) - (\chi_\beta^{(s-1)} \chi_\gamma^{(t-1)}) \chi_\alpha^{(r+2)}, \tag{12a}$$

$$\{\chi_\alpha^{(r)}, \chi_\beta^{(s)}\} = \chi_\alpha^{(r)} \chi_\beta^{(s)} + \chi_\beta^{(s-1)} \chi_\alpha^{(r+1)} \tag{12b}$$

to denote expressions which would vanish if the ordering convention were applied to them. We note that, by virtue of these conventions,

$$\chi_\alpha^{(0)} \chi_\beta^{(r)} = -\chi_\beta^{(r-1)} \chi_\alpha^{(1)}, \tag{13}$$

when χ_α and χ_β have the same time t, but it follows from equations (10) that $\{\phi_\alpha^{(0)}, \phi_\beta^{(r)}\} = 0$ when ϕ_α and ϕ_β are canonical field variables with the same time.

For $m = 1$, we have $u = 1$, and the above conventions are consistent with those in common use in quantization in accordance with Fermi statistics; in particular, Heisenberg's convention that a product $\bar{\chi}_\alpha \chi_\beta$ shall be replaced by $\tfrac{1}{2}(\bar{\chi}_\alpha \chi_\beta - \chi_\beta \bar{\chi}_\alpha)$ in quantization is subsumed. For $m = 2$, $u = \pi$, where π is the $\beta - \mu$ conjugation operator introduced by Carey (1972), satisfying $\pi^2 = 1$. For parastatistics of order three or more, there is no operator u with the required properties, but we shall show, nevertheless, that a representation can be found for *modular* statistics with general values of m.

We assume that the Lagrangian density is invariant under cyclic permutation:

$$L = u^{-1} L u. \tag{14}$$

Then a set of self-consistent field equations can be derived by a variational method similar to that of classical field theory. As

$$\chi_\alpha^{(r)} \chi_\beta^{(s)} = -\chi_\beta^{(s-1)} \chi_\alpha^{(r+1)}$$

for variables with the same time, with

$$\delta\chi_\alpha^{(r)}\chi_\beta^{(s)} = -\chi_\beta^{(s-1)}\delta\chi_\alpha^{(r+1)},$$

and the change in the Lagrangian density resulting from the variation of the fields can be expressed in the form

$$\delta L = \sum_{r=1}^{m} (\pi^{(r)\alpha}\delta\phi_\alpha^{(r)} + \pi^{(r)\alpha\lambda}\delta\phi_{\alpha,\lambda}^{(r)}), \tag{15}$$

it follows from equation (14) that

$$u^{-1}\pi^{(r)\alpha}u = \pi^{(r+1)\alpha}, \qquad u^{-1}\pi^{(r)\alpha\lambda}u = \pi^{(r+1)\alpha\lambda}. \tag{16}$$

Consequently, the field equations

$$\pi^{(r)\alpha\lambda}{}_{,\lambda} = \pi^{(r)\alpha} \tag{17}$$

for different values of r are related by a similarity transformation, and are equivalent to equation (1), if $\pi^{(0)\alpha} = \pi^\alpha/m$ and $\pi^{(0)\alpha\lambda} = \pi^{\alpha\lambda}/m$. The energy–momentum vector in the quantized theory is

$$P_\lambda = \int \left(\sum_{r=1}^{m} (\pi^{(r)\alpha 0}\phi_{\alpha,\lambda}^{(r)}) - L\delta_\lambda^0 \right) \mathrm{d}^3x. \tag{18}$$

3. Generalized Anticommutation Relations

The generalized anticommutation relations satisfied by the canonical and conjugate field variables will now be obtained with the help of equations (8). We note first that, when the field variables carry half-odd-integral spin, the Lagrangian density must be an even function of the ϕ_α and their derivatives. Moreover, if $(\chi_\alpha\chi_\beta\cdots\chi_\omega)$ is the product of any even number $2n$ of field variables and/or derivatives with the same time t, and ϕ_σ is a field variable with the time τ, then

$$\lim_{\tau\to t}\,[(\chi_\alpha\chi_\beta\cdots\chi_\omega), \phi_\sigma] = -\lim_{\varepsilon\to 0}\int_{t-\varepsilon}^{t+\varepsilon} \frac{\partial(\chi_\alpha\chi_\beta\cdots\chi_\omega\phi_\sigma)}{\partial\tau}\,\mathrm{d}\tau$$

$$= -\lim_{\varepsilon\to 0}\int_{t-\varepsilon}^{t+\varepsilon}\Bigg[-\left(\chi_\alpha\chi_\beta\cdots\chi_\omega\frac{\partial\phi_\sigma}{\partial\tau}\right) + \chi_\beta^{(-1)}\cdots\chi_\omega^{(-1)}\{\chi_\alpha^{(2n-1)}, \phi_\sigma\}\,\delta(t-\tau)$$

$$+\chi_\alpha\cdots\chi_\omega^{(-1)}\{\chi_\beta^{(2n-2)}, \phi_\sigma\}\,\delta(t-\tau)\ldots+\chi_\alpha\chi_\beta\cdots\{\chi_\omega, \phi_\sigma\}\,\delta(t-\tau)\Bigg]\,\mathrm{d}\tau$$

by virtue of the ordering conventions established in Section 2. So, when $\tau = t$, we obtain

$$[(\chi_\alpha\chi_\beta\cdots\chi_\omega), \phi_\sigma] = \chi_\beta^{(-1)}\cdots\chi_\omega^{(-1)}\{\chi_\alpha^{(2n-1)}, \phi_\sigma\}$$

$$+\chi_\alpha\cdots\chi_\omega^{(-1)}\{\chi_\beta^{(2n-2)}, \phi_\sigma\}\ldots+\chi_\alpha\chi_\beta\cdots\{\chi_\omega, \phi_\sigma\} \tag{19}$$

and

$$[L, \phi_\sigma] = \sum_{r=1}^{m}[\pi^{(r)\alpha}\{\phi_\alpha^{(r)}, \phi_\sigma\} + \pi^{(r)\alpha\lambda}\{\phi_{\alpha,\lambda}^{(r)}, \phi_\sigma\}], \tag{20}$$

with the help of equation (15). Hence

$$[P_\lambda, \phi_\sigma] = -\int \sum_{r=1}^{m} [\pi^{(r)\alpha 0}{}_{,\lambda}\{\phi_\alpha^{(r)}, \phi_\sigma\} + \{\pi^{(r)\alpha 0}, \phi_\sigma^{(-1)}\}\phi_\alpha^{(r+1)}{}_{,\lambda}]\, \mathrm{d}^3x = -\mathrm{i}\phi_{\sigma,\lambda}. \tag{21}$$

Because of the functional independence of the non-vanishing $\pi^{(r)\alpha 0} = \mathrm{i}\bar{\phi}^{(r)\alpha}$, and because $\{\phi_\alpha^{(r)}, \phi_\sigma\} = 0$, we infer that

$$\{\bar{\phi}^{(r)\alpha}, \phi_\sigma^{(-1)}\} = \delta_\sigma^\alpha\, \delta_{r,-1}\, \delta(\boldsymbol{x}_\alpha - \boldsymbol{x}_\sigma)$$

and, more generally,

$$\{\bar{\phi}^{(r)\alpha}, \phi_\beta^{(s)}\} = \delta_\beta^\alpha\, \delta_{r,s}\, \delta(\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta) \tag{22}$$

for variables with equal times. As the generalized anticommutation rules must be invariant under the transformation $r \to r+t$, $s \to s+t$ (mod m), we must adopt the interpretation

$$\{\bar{\phi}^{(r)\alpha}, \phi_\beta^{(s)}\} = \bar{\phi}^{(r)\alpha}\phi_\beta^{(s)} + \phi_\beta^{(s+1)}\bar{\phi}^{(r+1)\alpha}. \tag{23}$$

In its effect on the superscripts, the conjugate variable $\bar{\phi}^{(r)\alpha}$ is equivalent to a product of $2m-1$ canonical variables. We verify further that the relation (8b) yields

$$[P_\lambda, \bar{\phi}^\sigma] = -\int \sum_{r=1}^{m} [\pi^{(r)\alpha 0}{}_{,\lambda}\{\phi_\alpha^{(r)}, \bar{\phi}^\sigma\} + \{\pi^{(r)\alpha 0}, \bar{\phi}^{(-1)\sigma}\}\phi_{\alpha,\lambda}^{(r-1)}]\, \mathrm{d}^3x = -\mathrm{i}\bar{\phi}^\sigma{}_{,\lambda}$$

and hence that

$$\{\bar{\phi}^{(r)\alpha}, \bar{\phi}^{(s)\beta}\} = 0, \tag{24}$$

where

$$\{\bar{\phi}^{(r)\alpha}, \bar{\phi}^{(s)\beta}\} = \bar{\phi}^{(r)\alpha}\bar{\phi}^{(s)\beta} + \bar{\phi}^{(s-1)\beta}\bar{\phi}^{(r+1)\alpha}. \tag{25}$$

It is possible to formulate commutation rules which do not involve the permutation operator u explicitly. From the rules already given, it follows that

$$\phi_\alpha\phi_\beta \ldots \phi_\chi\phi_\omega = -\phi_\omega\phi_\beta \ldots \phi_\chi\phi_\alpha \qquad (m+1 \text{ factors}), \tag{26a}$$

$$\bar{\phi}_\alpha\bar{\phi}_\beta \ldots \bar{\phi}_\chi\bar{\phi}_\omega = -\bar{\phi}_\omega\bar{\phi}_\beta \ldots \bar{\phi}_\chi\bar{\phi}_\alpha \qquad (m+1 \text{ factors}), \tag{26b}$$

$$\phi_\alpha\bar{\phi}_0^\beta\phi_\gamma + \phi_\gamma\bar{\phi}^\beta\phi_\alpha = \delta_\alpha^\beta\,\delta(\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta)\phi_\gamma + \delta_\gamma^\beta\,\delta(\boldsymbol{x}_\beta - \boldsymbol{x}_\gamma)\phi_\alpha, \tag{26c}$$

$$\bar{\phi}^\alpha\phi_\beta\bar{\phi}_0^\gamma + \bar{\phi}^\gamma\phi_\beta\bar{\phi}^\alpha = \delta_\beta^\alpha\,\delta(\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta)\bar{\phi}^\gamma + \delta_\beta^\gamma\,\delta(\boldsymbol{x}_\beta - \boldsymbol{x}_\gamma)\bar{\phi}^\alpha \tag{26d}$$

for variables with the same time t, where of course $\bar{\phi}^\alpha = \bar{\phi}^{(0)\alpha}$. These reduce to standard results for $m = 1$ (Fermi statistics) and $m = 2$ (parafermi statistics of order two), but are not consistent with parafermi statistics for $m = 3$ or more. Starting with the relations (26), the operator u may be defined as effecting a cyclic permutation of the factors of a product of m field variables:

$$u\phi_\beta \ldots \phi_\chi\phi_\alpha = (-1)^{m-1}\phi_\alpha\phi_\beta \ldots \phi_\chi u \qquad (m+1 \text{ factors}), \tag{27}$$

or, more generally, of any product of $m-r$ field variables and r identical factors u, written in any order. There is a change of sign if an odd permutation of the field variables is involved.

We now state an ansatz which enables us to construct matrix representations of the ϕ_α and $\bar{\phi}^\alpha$, and the permutation operator u, and so demonstrate the existence of operators satisfying the relations (26) and (27). We first introduce the set of m-dimensional matrices ε_r $(r = 1, 2, \ldots, m)$, given by

$$(\varepsilon_r)_{ij} = \delta_{i,j+r}\,\omega^j\,, \qquad (i,j = 1, 2, \ldots, m\ (\mathrm{mod}\, m))\,,$$

where $\omega = \exp(2\pi i/m)$ is a complex mth root of unity. The matrices satisfy

$$\varepsilon_r^m = 1\,, \qquad \varepsilon_r\varepsilon_s = \omega^{s-r}\varepsilon_s\varepsilon_r\,, \qquad \left(\sum_{r=1}^{m} c_r\varepsilon_r\right)^m = \sum_{r=1}^{m} c_r^m + \delta_m \tag{28}$$

for arbitrary numerical constants c_r. We shall require also a set of Fermi creation and annihilation operators b_{rj}, b_{rj}^* (where the asterisk denotes a hermitian conjugate), satisfying

$$\{b_{rj}, b_{sk}\} = 0\,, \qquad \{b_{rj}, b_{sk}^*\} = \delta_{rs}\,\delta_{jk}\,, \tag{29}$$

and a set of functions $f_{\alpha j}, f_j^\alpha$ of the coordinates, satisfying the orthogonal condition

$$\sum_j f_j^\alpha f_{\beta j} = \delta_\beta^\alpha\,\delta(\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta) \tag{30}$$

within the region of space considered. Then the variables defined by

$$\phi_\alpha = \sum_j a_j f_{\alpha j}\,, \qquad a_j = \sum_{r=1}^{m} \varepsilon_r\, b_{rj}/m^{\frac{1}{2}}\,, \tag{31}$$

$$\bar{\phi}^\alpha = \sum_j a_j^* f_j^\alpha, \qquad a_j^* = \sum_{r=1}^{m} \varepsilon_r^*\, b_{rj}^*/m^{\frac{1}{2}} \tag{32}$$

satisfy the required commutation rules. The properties required of the operator u are realized by taking $u = \varepsilon_m$, though this assignment is clearly not unique.

The representation of the field variables obtained in this way is reducible in general, and applies only to those variables with the same fixed time t. Values at other times can be found, as in the classical theory, by solving the field equations. When this is done, any field variable ϕ_α or $\bar{\phi}^\alpha$ can be separated into parts containing only positive or negative frequencies respectively. Let us suppose that the affix j in equations (31) takes positive values associated with negative frequencies and negative values associated with positive frequencies. Then a_j can be interpreted as an annihilation operator, and a_j^* as a creation operator, for particles when $j > 0$; but a_j will then be a creation operator, and a_j^* an annihilation operator for antiparticles when $j < 0$. We can regard the actual value of j as corresponding to the dynamical observables, i.e. the momentum, energy and spin of the particle or antiparticle created or annihilated.

4. Groups of Transformations with Generalized Statistics

It is well known that, if the Lagrangian density is invariant under a group of transformations, there exists an associated set of operators, constructed from generators of the group, whose eigenvalues can be used to label the states of the quantized fields. Thus, the energy, momentum and spin constructed from the generators of the Poincaré group are the dynamical operators whose eigenvalues determine the important dynamical properties of the system. A significant feature of theories quantized in accordance with generalized statistics which has been pointed out by various authors (Govorkov 1968; Druhl *et al.* 1970) is that there may exist other operators, i.e. statistical operators, whose eigenvalues determine additional properties of the system.

It is easy to show that any representation of the field variables ϕ_α and $\bar{\phi}^\alpha$ and the cyclic permutation operator u provides a corresponding representation of the unitary group $U(m)$. For, if

$$A_{pq} = \int (\bar{\phi}^{(p)} \phi^{(q)})\, \mathrm{d}^3x = \sum_j (a_j^{(p)*} a_j^{(q)}), \tag{33}$$

it is readily verified with the help of equation (36) below that

$$[A_{pq}, A_{rs}] = \delta_{rq} A_{ps} - \delta_{ps} A_{rq}, \tag{34}$$

so that the A_{pq} $(p,q = 1,2,\dots,m)$ are generators of $U(m)$ (cf. Hammermesh 1962). If the Lagrangian density is invariant under the corresponding group of transformations, the commutators $[P_\lambda, A_{pq}]$ will vanish, and the eigenvalues of the commuting operators A_{rr} $(r = 1,2,\dots,m)$ will be good quantum numbers. In the application to the quark model, where $m = 3$, it would be natural to define the isospin I_3 and the hypercharge Y by $I_3 = \frac{1}{2}(A_{22} - A_{11})$ and $Y = \frac{1}{3}(2A_{33} - A_{11} - A_{22})$. The result would be a field theory, entirely consistent with Gell-Mann's (1964) theory, with three different kinds of quarks each satisfying Fermi statistics among themselves. However, the quarks, if they exist, do not appear to satisfy Fermi statistics and there are no physical grounds for requiring invariance of the Lagrangian density under the full group $U(m)$.

In view of the above, we content ourselves with the much weaker invariance under cyclic permutations implied by equation (14), and adopt the hypothesis that operators whose eigenvalues are good quantum numbers must be cyclic invariants. This does not exclude the invariants

$$\sigma_1 = \sum_r A_{rr}, \quad \sigma_2 = \sum_{r,s} A_{rs} A_{sr}, \quad \sigma_3 = \sum_{r,s,t} A_{rs} A_{st} A_{tr}, \quad \dots \tag{35}$$

of $U(m)$, so the states may still be classified in $U(m)$ multiplets. Other cyclic invariants can be constructed from the generators

$$L_{pq} = A_{pq} - A_{qp} \tag{36}$$

of $SO(m)$ within $U(m)$, by multiplying together operators of the type

$$L_s^{(r)} = \sum_{p=1}^{m} \omega^{pr} L_{p\,p+s}/m^{\frac{1}{2}}, \tag{37}$$

which satisfy $u^{-1} L_s^{(r)} u = \omega^{-r} L_s^{(r)}$. The $U(m)$ multiplets may be labelled by a commuting set of such cyclic invariants, together with certain invariants of $SO(m)$ within $U(m)$.

Thus, in the application to the quark model, where again $m = 3$, we may define the isospin component I_3 by

$$I_3 = \tfrac{1}{2}\mathrm{i}L_1^{(0)}$$

and the other components by

$$I_1 + \mathrm{i}I_2 = N(I_3)(L_1^{(1)})^2, \qquad I_1 - \mathrm{i}I_2 = (L_1^{(2)})^2 N(I_3), \tag{38}$$

where the normalization factor $N(I_3)$ is determined as shown by Bracken and Green (1973, 1974). With these definitions, it can be demonstrated that I_3 is the generator of cyclic permutations, and that the requirement of cyclic invariance is therefore sufficient (together with invariance under charge conjugation) to secure conservation of isospin and charge. The hypercharge Y is related to the Z-spin, as defined by Green and Bracken (1974). An apparent advantage of these assignments over those usually adopted is that the quarks do not have to satisfy Fermi statistics and that, indeed, the states of highest symmetry maximize the isospin and hypercharge. It is also apparent that this result is obtained more simply and naturally with the present generalization of quantum statistics than with parafermi statistics of order three.

5. Classification of States

We now proceed to a discussion of the vector space corresponding to any representation of the field variables ϕ_α and $\bar{\phi}^\alpha$ and the cyclic permutation operator u. We assume that the vacuum state, defined as the state of lowest energy, is invariant under cyclic permutation, and its state vector $| V_0\rangle$ therefore satisfies

$$a_j | V_0\rangle = 0, \qquad j > 0, \tag{39a}$$

$$a_j^* | V_0\rangle = 0, \qquad j < 0, \tag{39b}$$

$$u | V_0\rangle = | V_0\rangle. \tag{39c}$$

Other states may be constructed from $| V_0\rangle$ by applying creation operators a_j^* $(j > 0)$ or a_j $(j < 0)$ and the operator u. The order of these operators can be changed only to the extent permitted by the identities

$$a_i a_j \dots a_k a_l \equiv -a_l a_j \dots a_k a_i, \tag{40a}$$

$$a_i^* a_j^* \dots a_k^* a_l^* \equiv -a_l^* a_j^* \dots a_k^* a_i^*, \tag{40b}$$

$$a_i a_j^* a_k + a_k a_j^* a_i \equiv \delta_{ij} a_k + \delta_{jk} a_i, \tag{40c}$$

$$a_i^* a_j a_k^* + a_k^* a_j a_i^* \equiv \delta_{ij} a_k^* + \delta_{jk} a_i^*, \tag{40d}$$

and similar relations involving u, which can be deduced from equations (26) or (10), (22) and (24). However, it will be noticed that a product of creation operators and factors u can be factorized (though not uniquely in general) into *modules*, which commute or anticommute with one another. The basic modules, involving only creation operators, are of the form

$$b_{ij\ldots v} = a_i a_j \ldots a_v \qquad (m \text{ factors}), \tag{41a}$$

$$b^*_{ij\ldots v} = a^*_i a^*_j \ldots a^*_v \qquad (m \text{ factors}), \tag{41b}$$

$$c_{ij} = a^*_i a_j \qquad \text{or} \qquad c^*_{ij} = a_i a^*_j \,. \tag{41c}$$

Other modules are derived from these by substituting u for any factor a_k, or $u^* = u^{-1}$ for any factor a^*_k, as, for example,

$$b_{i0\ldots v} = a_i u \ldots a_v \qquad (m \text{ factors}), \tag{42a}$$

$$b^*_{00\ldots v} = u^{*2} \ldots a^*_v \qquad (m \text{ factors}), \tag{42b}$$

$$c_{0j} = u^* a_j \qquad \text{or} \qquad c^*_{i0} = a_i u^* \,. \tag{42c}$$

In the physical applications, modules or particular products of modules may represent composite particles, such as baryons or mesons, which will satisfy Fermi or Bose statistics. However, it should be noticed that the simple products defined in equations (41) and (42) do not change the eigenvalues of the statistical operators by fixed amounts, and they must therefore be decomposed with respect to $U(m)$ and $SO(m)$ to obtain creation operators corresponding to definite statistical quantum numbers. The method of decomposition is well known and will not be discussed here.

With modular statistics of order m, an arbitrary state vector $| V_m \rangle$ can be expressed in the form

$$| V_m \rangle = P_{ij\ldots} | V_{m-1} \rangle, \quad | V_{m-1} \rangle = P^{(1)}_{kl\ldots} | V_{m-2} \rangle, \quad \ldots \quad | V_1 \rangle = P^{(m-1)}_{xy\ldots} | V_0 \rangle, \tag{43}$$

where $| V_r \rangle$ represents a state which may contain no more than r particles with the same dynamical observables, and $| V_0 \rangle$ accordingly represents the vacuum state. The operator $P_{ij\ldots}$ is a product of basic modules, and $P^{(r)}_{ij\ldots}$ is a product of modules each consisting of $m-r$ creation operators and r factors u or u^*. A canonical form can be found in which it is not possible, with the help of the commutation rules, to move any factor u or u^* nearer to the vacuum state vector. The vector $| V_{m-1} \rangle$ is then analogous to the reservoir state vectors defined by Bracken and Green (1973) and Govorkov (1973*b*).

It will be noticed that the permutation operator u may be defined, if desired, by its effect on the state vectors of the system. If $b_{ij\ldots v}$ and c_{ij} are basic modules as defined by equations (41), we have, for example,

$$u b_{ij\ldots v} = (-1)^{m-1} b_{vij\ldots} u \,, \qquad u c_{ij} = -c^*_{ji} u \,,$$

and similar but sometimes more complicated relations can be found for other modules.

The factorization of products of creation operators into modules which may be associated with composite particles is a natural consequence of the generalized anti-commutation relations (10), (22) and (24). These relations also ensure the 'cluster property', which is a source of troublesome restrictions with parastatistics (see Ohnuki and Kamefuchi 1968, 1969, 1970; Gray 1973) but is needed to guarantee the statistical independence of spatially separated events. This and previously noticed features of modular statistics suggest that it is potentially the most useful generalization of ordinary quantum statistics in its applications to particle physics. It is necessary to say 'potentially' because there are obviously dynamical problems still to be solved in formulating a Lagrangian density which will guarantee the stability of the composite particles and still permit the observed interactions between them. However, the fact that the quantization in accordance with modular statistics can be applied to *any* Lagrangian density with the requisite symmetries has allowed us to bypass such problems in the present paper.

References

Ardalan, F., and Mansouri, F. (1974). *Phys Rev.* D **9**, 3341.
Bracken, A. J., and Green, H. S. (1973). *J. Math. Phys.* **14**, 1784.
Carey, A. L. (1972). *Progr. Theor. Phys.* **49**, 658.
Druhl, K., Haag, R., and Roberts, J. F. (1970). *Commun. Math. Phys.* **18**, 204.
Fritzsch, H., and Gell-Mann, M. (1971). Proc. Conf. on Duality and Symmetry, Tel-Aviv 1971, p. 317 (Weizmann: Jerusalem).
Gell-Mann, M. (1964). *Phys. Rev. Lett.* **8**, 214.
Gervais, J. L., and Sakita, B. (1973). *Phys. Rev. Lett.* **30**, 716.
Govorkov, A. B. (1968). *Sov. Phys. JETP* **27**, 960.
Govorkov, A. B. (1973*a*). *Int. J. Theor. Phys.* **7**, 49.
Govorkov, A. B. (1973*b*). 'Sequences of Parastatistics' (Joint Institute for Nuclear Research: Dubna, U.S.S.R.).
Gray, D. A. (1973). *Progr. Theor. Phys.* **49**, 1027.
Green, H. S. (1953). *Phys. Rev.* **90**, 270.
Green, H. S. (1972). *Progr. Theor. Phys.* **47**, 1400.
Green, H. S., and Bracken, A. J. (1974). *Int. J. Theor. Phys.* **11**, 157.
Greenberg, O. W. (1964). *Phys. Rev. Lett.* **13**, 598.
Hammermesh, M. (1962). 'Group Theory' (Addison-Wesley: Reading, Mass.).
Nambu, Y. (1970). Proc. Summer Symp. on Symmetries and the Quark Model (Gordon & Breach: New York).
Ohnuki, Y., and Kamefuchi, S. (1968). *Phys. Rev.* **170**, 1270.
Ohnuki, Y., and Kamefuchi, S. (1969). *Nucl. Phys.* B **9**, 539.
Ohnuki, Y., and Kamefuchi, S. (1970). *Ann. Phys. (New York)* **51**, 337.
Pauli, W. (1940). *Phys. Rev.* **58**, 716.
Ramakrishnan, A., Vasuderan, R., and Chandrasekaran, P. S. (1971). *J. Math. Analysis Applic.* **35**, 249.

Manuscript received 7 November 1974