

Static Charged Dust Distribution in the Brans-Dicke Theory

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Abstract

The distribution of static charged dust in the Brans-Dicke theory is considered. It is shown that the ratio of charge density to mass density is related to the scalar interaction ϕ so that for small values of ϕ the charge density will far exceed the mass density. This result suggests that the existence of a finite electron can be realized in the Brans-Dicke theory of gravitation through a static charged dust distribution.

1. Introduction

In classical theory a static charged dust distribution is held in equilibrium under the actions of gravitational attraction and electrostatic repulsion if the charge density is equal to the mass density ($\sigma = \pm \rho$). The same result also holds true in the general theory of relativity (Majumdar 1947; Das 1962; De and Raychaudhuri 1968). However, such a charged dust model cannot be used to represent finite electrons because the charge of an electron far exceeds its mass, and any analogy could only apply to unstable electrons. Thus it is not possible to account for finite electrons with a static charged dust model in the domain of general relativity, and indeed it is generally held that the properties of a finite electron cannot be explained on the basis of known interactions. It is therefore worth while investigating the problem of charged dust in the Brans-Dicke (BD) theory of gravitation (Brans and Dicke 1961), in which a new scalar interaction, namely a ϕ interaction, has been introduced to make the theory Machian. The aim of the present work is to find whether the presence of this ϕ interaction can help in explaining the structure of a finite electron.

A solution to the problem is attempted here based on a relationship obtained by the present author (in unpublished calculations for the BD theory), namely

$$-g_{44} = \phi^{-1}(4\pi\psi^2 + A\psi + B), \quad (1)$$

where g_{44} is the metric component of the static universe, ϕ is the BD scalar field, ψ is the source-free electrostatic field, and the constants A and B are arbitrary. The BD electrostatic field equations have been solved by expressing $4\pi\psi^2 + A\psi + B$ as a perfect square with a suitable choice of the constants A and B such that

$$-g_{44} = \phi^{-1} 4\pi(\psi \pm \sqrt{2})^2. \quad (2)$$

In the present paper we assume the form (2) to be valid within the charged matter. This ultimately leads to the result

$$\sigma/\rho = \pm \phi^{-\frac{1}{2}}, \quad (3)$$

where σ and $(4\pi)^{\frac{1}{2}}\rho' = \rho$ are the charge density and matter density respectively. In Section 3 we discuss the possibility of explaining the existence of a finite electron with the help of a static charged dust model in which the relation (3) holds.

2. Field Equations and Deduction of Relation (3)

The BD Maxwell field equations are given as

$$R_{ij} - \frac{1}{2}Rg_{ij} = -8\pi\phi^{-1}T_{ij} - \omega\phi^{-2}(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}) - \phi^{-1}(\phi_{;ij} - g_{ij}\phi_{;k}^k), \quad (4a)$$

$$(3+2\omega)\phi_{;k}^k = 8\pi T_k^k, \quad (4b)$$

$$F_{ij}^{ij} = \sigma u^i, \quad (4c)$$

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0, \quad (4d)$$

with

$$T_{ij} = \rho' u_i u_j + (g^{ab} F_{ai} F_{bj} - \frac{1}{4} g_{ij} F_{ab} F^{ab}), \quad (5)$$

where a subscript comma or semicolon denotes respectively partial differentiation or covariant differentiation by the index that follows it. The static condition gives $u^x = 0$ and $u^4 = (-g_{44})^{-\frac{1}{2}}$. Greek indices take the values 1, 2, 3 while Latin indices take the values 1, 2, 3, 4.

Equation (4a) can be simplified to yield

$$-R = -8\pi\phi^{-1}T + \omega\phi^{-2}\phi^{,i}\phi_{,i} + 3\phi^{-1}\phi_{;k}^k. \quad (6)$$

Since $u^i u_i = -1$, equation (5) gives

$$T = -\rho'. \quad (7)$$

Using the relations (4b) and (7), we can rewrite equation (6) as

$$-R = +\frac{8\pi\rho'}{\phi} + \frac{\omega\phi^{,i}\phi_{,i}}{\phi^2} - \frac{3}{\phi} \frac{8\pi\rho'}{3+2\omega}$$

or

$$-R = \frac{2\omega}{3+2\omega} \frac{8\pi\rho'}{\phi} + \frac{\omega\phi^{,i}\phi_{,i}}{\phi^2}. \quad (8)$$

From equations (4a) and (8) we have

$$R_{ij} = -\frac{8\pi T_{ij}}{\phi} - \frac{\omega\phi_{,i}\phi_{,j}}{\phi^2} - \frac{\phi_{;ij}}{\phi} - \frac{\omega+1}{3+2\omega} \frac{8\pi\rho' g_{ij}}{\phi}. \quad (9)$$

We attempt to solve the field equations (4b)–(4d) and (9) in a static universe defined by

$$(ds)^2 = g_{\alpha\beta} dx^\alpha dx^\beta - V^2(dx^4)^2, \quad (10)$$

where $g_{\alpha\beta}$ and g_{44} ($= -V^2$) are independent of x^4 . Equation (4d) is satisfied when

$$F_{ij} = \psi_{j,i} - \psi_{i,j}, \quad (11)$$

where ψ_i is the electromagnetic four-potential. For $\psi_\alpha = 0$ and ψ_4 independent of x^4 , we have only an electrostatic field in the static universe. For convenience we write $\psi_4 \equiv \psi$. Thus

$$F_{\alpha 4} = \psi_{,\alpha} \quad \text{and} \quad F_{\alpha\beta} = 0. \quad (12)$$

From these relations we also obtain

$$F^{\alpha 4} = -V^{-2} g^{\alpha\beta} \psi_{,\beta} \quad \text{and} \quad F^{\alpha\beta} = 0. \quad (13)$$

From equations (12) and (13), the only nonvanishing components of the stress energy tensor T_{ij} given by (5) are

$$T_{\alpha\beta} = V^{-2} (\tfrac{1}{2} g_{\alpha\beta} g^{\sigma\gamma} \psi_{,\sigma} \psi_{,\gamma} - \psi_{,\alpha} \psi_{,\beta}) \quad (14a)$$

and

$$T_{44} = \rho' V^2 + \tfrac{1}{2} g^{\sigma\gamma} \psi_{,\sigma} \psi_{,\gamma}, \quad (14b)$$

where

$$u_\alpha = 0 \quad \text{and} \quad u_4 = (-g_{44})^{\frac{1}{2}}.$$

The relation (9) can be rewritten in terms of its components using equations (14) as

$$R_{\alpha\beta} = -\frac{8\pi}{\phi} \left(\frac{\tfrac{1}{2} g_{\alpha\beta} g^{\sigma\gamma} \psi_{,\sigma} \psi_{,\gamma} - \psi_{,\alpha} \psi_{,\beta}}{V^2} \right) - \frac{\omega \phi_{,\alpha} \phi_{,\beta}}{\phi^2} - \frac{\phi_{;\alpha\beta}}{\phi} - \frac{\omega+1}{3+2\omega} \frac{8\pi\rho' g_{\alpha\beta}}{\phi} \quad (15a)$$

and

$$R_{44} \equiv -V g^{\alpha\beta} V_{;\alpha\beta} = -\frac{8\pi}{\phi} \left(\frac{\omega+2}{3+2\omega} \right) \rho' V^2 - \frac{8\pi g^{\sigma\gamma} \psi_{,\sigma} \psi_{,\gamma}}{2\phi} - \frac{\phi_{;44}}{\phi}, \quad (15b)$$

where $\phi_{,4} = 0$. Also, equations (4b) and (4c) reduce respectively to

$$(3+2\omega)(g^{\alpha\beta} \phi_{;\alpha\beta} + V^{-1} g^{\alpha\beta} V_{,\alpha} \phi_{,\beta}) = -8\pi\rho' \quad (16)$$

and

$$V^{-1} g^{\alpha\beta} \psi_{;\alpha\beta} - V^{-2} g^{\alpha\beta} \psi_{,\alpha} V_{,\beta} = \sigma. \quad (17)$$

In order to arrive at the relation (3) we now proceed as follows. We first assume a functional relationship between V , ϕ and ψ ,

$$V = V(\phi, \psi), \quad (18)$$

where ϕ and ψ are independent of each other. This mutual independence is in the spirit of the BD assumption that the Lagrangian density of matter is not a function of ϕ . With the assumption (18) then, the expression (15b) reduces to

$$\begin{aligned} & V_\phi g^{\alpha\beta} \phi_{;\alpha\beta} + V_{\phi\phi} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + 2V_{\phi\psi} g^{\alpha\beta} \psi_{,\alpha} \phi_{,\beta} + V_\psi g^{\alpha\beta} \psi_{;\alpha\beta} + V_{\psi\psi} g^{\alpha\beta} \psi_{,\alpha} \psi_{,\beta} \\ &= \frac{4\pi g^{\alpha\beta} \psi_{,\alpha} \psi_{,\beta}}{\phi V} - \frac{V_\phi g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta}}{\phi} - \frac{V_\psi g^{\alpha\beta} \psi_{,\alpha} \phi_{,\beta}}{\phi} + \frac{8\pi\rho' V}{\phi} \frac{\omega+2}{3+2\omega}, \end{aligned} \quad (19)$$

where

$$\phi_{;44} = -V g^{\alpha\beta} V_{,\alpha} \phi_{,\beta}$$

and we have used the symbolism

$$\begin{aligned} V_\phi &= \partial V / \partial \phi, \quad V_\psi = \partial V / \partial \psi, \quad V_{\phi\phi} = \partial^2 V / \partial \phi^2, \\ V_{\psi\psi} &= \partial^2 V / \partial \psi^2 \quad \text{and} \quad V_{\phi\psi} = \partial^2 V / \partial \phi \partial \psi. \end{aligned}$$

Eliminating $\phi_{;\alpha\beta}$ and $\psi_{;\alpha\beta}$ from equation (19) with the help of the relations (16) and (17), we obtain

$$\begin{aligned} -\frac{8\pi V_\phi \rho'}{3+2\omega} + \left(V_{\phi\phi} - \frac{V_\phi^2}{V} \right) g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + \sigma V_\psi V + \left(V_{\psi\psi} + \frac{V_\psi^2}{V} \right) g^{\alpha\beta} \psi_{,\alpha} \psi_{,\beta} + 2V_{\phi\psi} g^{\alpha\beta} \psi_{,\alpha} \phi_{,\beta} \\ = \frac{4\pi g^{\alpha\beta} \psi_{,\alpha} \psi_{,\beta}}{\phi V} - \frac{V_\phi g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta}}{\phi} - \frac{V_\psi g^{\alpha\beta} \psi_{,\alpha} \phi_{,\beta}}{\phi} + \frac{8\pi \rho' V}{\phi} \frac{\omega+2}{3+2\omega} \end{aligned} \quad (20)$$

and, in matter-free space ($\sigma = \rho = 0$), we thus have

$$\begin{aligned} \left(V_{\phi\phi} - \frac{V_\phi^2}{V} \right) g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + \left(V_{\psi\psi} + \frac{V_\psi^2}{V} \right) g^{\alpha\beta} \psi_{,\alpha} \psi_{,\beta} + 2V_{\phi\psi} g^{\alpha\beta} \psi_{,\alpha} \phi_{,\beta} \\ = \frac{4\pi g^{\alpha\beta} \psi_{,\alpha} \psi_{,\beta}}{\phi V} - \frac{V_\phi g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta}}{\phi} - \frac{V_\psi g^{\alpha\beta} \psi_{,\alpha} \phi_{,\beta}}{\phi}. \end{aligned} \quad (21)$$

It can be shown from equation (21) that the functional form of the relation (18) is given by equation (1), namely

$$-g_{44} = V^2 = \phi^{-1}(4\pi\psi^2 + A\psi + B).$$

It is clear from equation (20) that (1) will also be valid within the charged matter if

$$-\frac{8\pi V_\phi \rho'}{3+2\omega} + \sigma V_\psi V = \frac{8\pi \rho' V}{\phi} \frac{\omega+2}{3+2\omega}$$

or

$$\frac{V_\psi \sigma}{\rho'} = \frac{8\pi}{3+2\omega} \left(\frac{\omega+2}{\phi} + \frac{V_\phi}{V} \right). \quad (22)$$

Since the relation (2) is a special form of equation (1), it will also be valid within the charged matter when equation (22) holds. From equations (22) and (2), we thus arrive at the relation (3), namely

$$\sigma/\rho = \pm \phi^{-\frac{1}{2}},$$

which shows that the ratio of charge density to mass density is related to the scalar interaction ϕ .

3. Concluding Remarks

Unlike the corresponding result in general relativity, the ratio σ/ρ is related to the scalar ϕ which primarily determines the local value of the gravitational constant. Now, if the gravitational constant is taken to be a universal constant $G = 1$ in relativistic units, then the assumption that ϕ varies as G^{-1} also necessitates taking ϕ

to be a constant and equal to unity. For this value of ϕ the relation (3) reduces to $\sigma/\rho = \pm 1$ and, as noted in the Introduction, clearly such a model cannot be used to represent an electron. However, the relation (3) here suggests that for small values of ϕ the charge density will exceed the mass density. Since the value of ϕ at any point is determined by the distribution of matter around that point, different distributions will have different ϕ values. Specifically, in a static spherical shell of mass M and radius R the value of ϕ in its interior is (Brans and Dicke 1961)

$$\phi \sim M/R.$$

Thus, only for small values of M/R will the value of ϕ be small, and M/R will be very small when the radius is very much greater than the mass. Such a model of charged dust distribution with a small value of R could accommodate electrons, and this may point towards the possibility of explaining the existence of an electron on the basis of the Maxwell field equations in the Brans-Dicke framework.

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