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The Propagation of Spherical Vortices

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Abstract

A model based on dimensional arguments is presented in order to predict the propagation velocity of the entrained fluid of a neutrally buoyant spherical vortex. The model vortex is characterized by single length and velocity scales. Its behaviour is considered in the two limiting cases of zero and infinite Reynolds number. Comparison of predictions from the model with experimental observations suggests that all stable vortices can be characterized by these two cases.

Introduction

A vortex is formed in a viscous homogeneous fluid whenever a finite volume of the fluid impulsively attains momentum relative to its surroundings. Provided that the moving fluid is not influenced by any boundary after the initial impulse, the viscous stresses produced by the motion give rise to a localized region of vorticity which causes the volume to entrain ambient fluid and to propagate in the direction of the initial impulse. If the initial aspect ratio of the volume is not too far from unity then the shape of the volume of entrained fluid may be expected to be spheroidal (but when the initial aspect ratio is far from unity, the resulting motion probably is unstable and breaks down to form a family of spherical vortices). Indeed, some observations by Maxworthy (1972) of vortices generated by the injection of a small quantity of water through a hole in one side of a water filled tank suggest that the shape of the entrained volume of fluid is spherical. Moreover, it has been shown that the behaviour of vortices produced by water drops falling into a water tank is described well by a theory involving spherical vortices (Manton 1974).

In the present paper, dimensional arguments are given to predict the radius and propagation velocity of a spherical vortex in the formal limiting cases of zero and infinite Reynolds number. In the former case inertial effects are assumed to be dominated by viscous effects, while the gross properties of the vortex are taken to be independent of the molecular viscosity of the fluid in the latter case. The model equations in this paper include a formulation of the assumption that the momentum of a vortex is conserved. This contradicts the suggestion of Maxworthy (1972) that a vortex systematically loses momentum in the form of a wake. On the other hand, the predictions of the model are found to be consistent with the independent measurements made by Banerji and Barave (1931), Keedy (1967) and Maxworthy. Moreover, the low Reynolds number results are identical to those obtained by Manton (1974) from the asymptotic solution of the Navier–Stokes equations. The existence of a distinct wake behind a vortex has not been demonstrated unambiguously (L. Cubitt and B. R. Morton, personal communication).

Model of Vortex

Because observations suggest that vortices of initial aspect ratio near unity are approximately spherical, we consider a class of vortices whose bulk properties can be characterized by a single length scale r and a single velocity scale v. The length scale is to be identified with the radius of the entrained fluid (or radius of the equivalent sphere if the entrained fluid is not spherical) and the velocity scale is to be identified with the propagation velocity of the entrained fluid. It follows that the momentum and energy of the vortex can be scaled with $\rho r^3 v$ and $\rho r^3 v^2$ respectively, where ρ is the fluid density.

Because there is no net external force acting on the system once the vortex is formed, momentum is conserved and so

$$\mathrm{d}(\rho r^3 v)/\mathrm{d}t = 0, \tag{1}$$

where t is time. A second relation between r and v is given by the assumption that the rate of dissipation of the energy of the vortex depends explicitly upon neither time nor the initial conditions, i.e. the dissipation rate is a function only of the local scales r and v and of the fluid properties ρ and μ, the dynamic viscosity. Thus the Π theorem (see e.g. Sedov 1959) implies

$$\mathrm{d}(\rho r^3 v^2)/\mathrm{d}t = -\rho r^2 v^3 F(\Gamma), \tag{2}$$

where $\Gamma = \rho r v/\mu$ is the local Reynolds number, and F is a dimensionless function.

The system of differential equations (1) and (2) is closed by the specification of the initial conditions and the dissipation function F. The initial conditions are taken to be

$$r = R \qquad \text{and} \qquad v = V \qquad \text{at} \qquad t = 0. \tag{3}$$

We consider the form of F in two limiting cases. As the Reynolds number Γ goes to zero, we expect inertial effects to become unimportant, and so the dissipation rate ought to become independent of ρ. Hence we take

$$F(\Gamma) \sim \alpha/\Gamma \qquad \text{as} \qquad \Gamma \to 0, \tag{4}$$

where α is a dimensionless constant. At very large Reynolds numbers, the rate of energy dissipation should become explicitly independent of the fluid viscosity, this being the case in turbulent flows where the rate of dissipation is governed by the large scale motions. We therefore assume that

$$F(\Gamma) \sim \beta \qquad \text{as} \qquad \Gamma \to \infty, \tag{5}$$

where β is a dimensionless constant.

The solution of the system (1) and (3) with F given by (4) is

$$r/R = (1+t/t_0)^{1/2} \qquad \text{and} \qquad v/V = (1+t/t_0)^{-3/2}, \tag{6a, b}$$

where $t_0 = (3/2\alpha)(R^2/\nu)$ and $\nu = \mu/\rho$ is the kinematic viscosity of the fluid. Equations (6) correspond to the asymptotic (zero Reynolds number) solution of Phillips (1956) and B. R. Morton and J. L. McGregor (personal communication). Equation (6b) is identical to equation (13) of Manton (1974) (except for a linear transformation in time), with the parameter η_0 in the latter work given by

$$\eta_0^2 = \alpha/3\,. \tag{7}$$

Integrating equation (6b) with respect to time, we find the propagation distance of the vortex centre y to be such that

$$y/R = (3/\alpha)\Gamma_0\{1-(1+t/t_0)^{-1/2}\}, \tag{8}$$

where $\Gamma_0 = RV/\nu$ is the initial Reynolds number of the vortex. Hence, from equations (6b) and (8), the velocity of the vortex as a function of position is seen to be

$$v/V = \{1-(\alpha/3)\Gamma_0^{-1}(y/r)\}^3\,. \tag{9}$$

Thus equations (6)–(9) describe the gross behaviour of a vortex in the low Reynolds number case.

For the high Reynolds number case, the system (1)–(3) with F given by (5) has the solution

$$r/R = (1+t/t_1)^{1/4} \qquad \text{and} \qquad v/V = (1+t/t_1)^{-3/4}, \tag{10a, b}$$

where $t_1 = (3/4\beta)(R/V)$. From equation (10b) we find the propagation distance and velocity of the vortex to be given by

$$y/R = (3/\beta)\{(1+t/t_1)^{1/4}-1\} \qquad \text{and} \qquad v/V = \{1+(\beta/3)(y/R)\}^{-3}. \tag{11a, b}$$

Discussion

To compare the predictions of the previous section with experimental results, it is necessary to know the initial velocity V and radius R of each observed vortex. The value of V can be found reasonably well by extrapolating the data back to the time origin. On the other hand, the equivalent radius R cannot be obtained in general. However, it is seen from equations (8), (9) and (11) that the functional form of the relationship between v and y is not affected by the value of R. The low Reynolds number parameter α could be replaced by α/R^2, and the high Reynolds number parameter β by β/R. This implies that choosing incorrect values of R for different sets of data leads to different values of α, although each data set should be consistent with the appropriate equation of the previous section.

A further problem is that the kinematic viscosity ν of the fluid (water) used in each of the experiments discussed here was not stated by the authors. Manton (1974) considered the results of Keedy (1967) in terms of $\nu = 1 \cdot 3 \times 10^{-2}\,\mathrm{cm^2\,s^{-1}}$. This corresponds to a temperature of 10°C, which probably is unrealistically low. In the present work, we therefore take $\nu = 1 \cdot 0 \times 10^{-2}\,\mathrm{cm^2\,s^{-1}}$, corresponding to a temperature of 20°C. Using equation (7) and compensating for the changed value of ν, we find from Manton (1974) that the observations of Keedy for initial Reynolds numbers Γ_0 between 189 and 311 fit equation (9) with $\alpha = 12 \cdot 9$. Keedy's vortices are generated by dyed water drops falling into a tank of clear water, and so the initial radius R is not well defined. The value of R is taken to be the radius of the dyed vortex core extrapolated back to the origin.

Fig. 1 shows the behaviour of the vortex velocity v with propagation distance y observed by Maxworthy (1972) for vortices with initial Reynolds numbers $\Gamma_0 \lesssim 250$. (Two other data sets with Γ_0 of the order of 400 are presented in that work, but they do not align with the data shown here.) It is seen that these results are consistent with the low Reynolds number equation (9) for $\alpha = 16{\cdot}0$. However, as no information was given on the volume of fluid initially injected, the equivalent radius R is not known. Consequently, R is set equal to the radius of the tube through which the injection took place, namely, 0·25 inches.

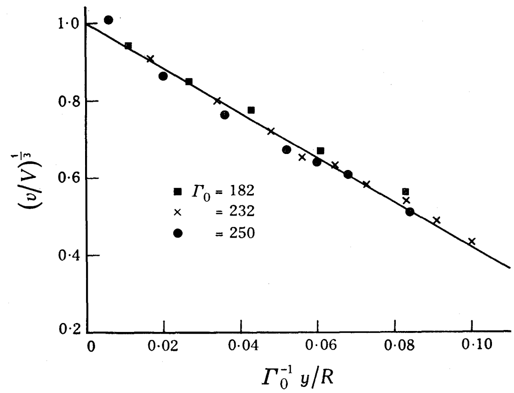

Fig. 1. Comparison of the predicted relationship (straight line) between the normalized velocity v/V and the normalized propagation distance y/R as given by the low Reynolds number equation (9) for $\alpha = 16{\cdot}0$ with the experimental results of Maxworthy (1972) for the initial Reynolds numbers indicated.

Some careful observations of vortices generated by injecting a known volume of water into a tank were displayed by Banerji and Barave (1931). In Fig. 2*a*, their results for $\Gamma_0 \lesssim 350$ are compared with the low Reynolds number equation (8) for $\alpha = 16{\cdot}1$. The initial equivalent radius R differs by a factor of 2·0 in the observations, although the radius of the inlet tube was unchanged. The lack of scatter in the data suggests that R is indeed the relevant length scale.

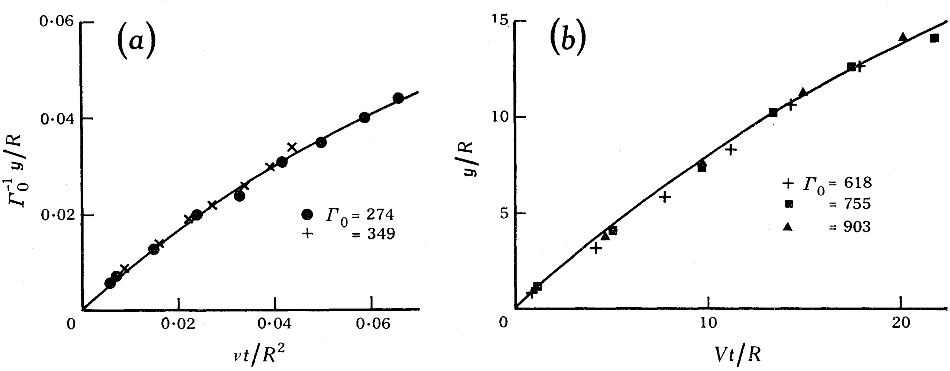

Fig. 2. Comparison of the predicted relationships (curves) between the normalized propagation distance y/R and the time t as given by

(*a*) the low Reynolds number equation (8) for $\alpha = 16{\cdot}1$ and

(*b*) the high Reynolds number equation (11a) for $\beta = 0{\cdot}0567$

with the experimental results of Banerji and Barave (1931) for the initial Reynolds numbers indicated.

Thus three independent studies suggest that the gross behaviour of vortices with initial Reynolds numbers less than about 350 is described adequately by the present model in the low Reynolds number case. Although the experimental value of α varies between 12·9 and 16·1, there is a large degree of uncertainty associated with the experimental values of R and ν. The observations are therefore not inconsistent with the dissipation parameter α being a universal constant equal to approximately 16.

Banerji and Barave (1931) presented data for three vortices with $\Gamma_0 > 600$. These data do not appear to conform with the low Reynolds number equation (8). On the other hand, they do coincide well with the high Reynolds number equation (11a) for $\beta = 0 \cdot 0567$ (see Fig. 2*b*). Now Maxworthy (1972) asserted that vortices with Reynolds numbers in the range $300 \lesssim \Gamma_0 \lesssim 500$ are inherently unstable, while stable vortices with $\Gamma_0 \gtrsim 500$ can exist. This result and the observations of Banerji and Barave suggest that all stable vortices can be characterized by the present model: the high Reynolds number case is applicable to vortices with $\Gamma_0 \gtrsim 500$ and the low Reynolds number case to those with $\Gamma_0 \lesssim 300$.

From equations (4) and (5), we see that the rate of dissipation of energy in a vortex for the low Reynolds number case equals that in an equivalent vortex for the high Reynolds number case when the Reynolds number is Γ_t, where

$$\Gamma_t = \alpha/\beta. \tag{12}$$

Thus, if a stable vortex exists at a Reynolds number Γ and if the vortex tends to maximize the rate of dissipation of its energy, then $F(\Gamma)$ ought to be given by equation (4) for $\Gamma < \Gamma_t$ and by equation (5) for $\Gamma > \Gamma_t$. Inserting the value of α and β found from the results of Banerji and Barave (1931) into equation (12), we obtain $\Gamma_t = 284$. This can be compared with the observation of Maxworthy (1972) that vortices with $\Gamma_0 \gtrsim 300$ are unstable. Hence Γ_t appears to give an upper limit on the low Reynolds number case. Any vortex with $\Gamma < \Gamma_t$ ought to be laminar and to remain stable as Γ decreases with time.

On the other hand, the behaviour of vortices which approach Γ_t from above has apparently not been studied. The Reynolds number of a vortex in the high Reynolds number case decreases continuously; in particular, equations (10) and (11) give

$$\Gamma/\Gamma_0 = \{1 + (\beta/3)(y/R)\}^{-2}. \tag{13}$$

Thus a stable vortex must be followed over a large distance before the Reynolds number approaches Γ_t. For example, equation (13) implies that a vortex travels about $22\,R$ before Γ falls to half its initial value.

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