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Basic Properties of the Exceptional Lie Groups

B. G. Wybourne and M. J. Bowick

Department of Physics, University of Canterbury, Christchurch 1, New Zealand.

Abstract

The exceptional Lie groups play a significant role in elementary particle models involving octonionic coloured quark fields. Simple methods for calculating the basic properties of these groups are outlined here. Among the properties computed are the dimensions and Dynkin index eigenvalues of irreps and branching rules for the most important group–subgroup structures. Kronecker products and symmetrized Kronecker powers of irreps of the exceptional groups are resolved. The concepts of elementary multiplets and Schur functions (S-functions) are used to greatly simplify the calculations, making possible manual calculations that are well beyond the capabilities of modern computer algorithms based on the enumeration of weights.

1. Introduction

The classification of the complex semisimple Lie algebras was undertaken by Elie Cartan in his thesis of 1894 (Cartan 1894). Four great classes of Lie algebras were identified and designated by Cartan as A_n, B_n, C_n and D_n. These Lie algebras, often referred to as the classical Lie algebras, may be associated with infinitesimal forms of the semisimple Lie groups SU_{n+1}, SO_{2n+1}, Sp_{2n} and SO_{2n} respectively. These Lie algebras and their associated Lie groups exist for every positive integer n. In addition to the four classical Lie algebras, Cartan identified five exceptional Lie algebras which he designated as G_2, F_4, E_6, E_7 and E_8, where the subscripted integers are the ranks of the respective algebras. An alternative classification of the Lie algebras in terms of their root systems was given by van der Waerden (1933), while a more elegant formulation in terms of simple roots was developed by Dynkin (1952*a*, 1952*b*). A modern description of the exceptional Lie algebras has been given by Jacobson (1962, 1971) and many others (e.g. Bourbaki 1968; Freudenthal and de Vries 1968; Hausner and Schwartz 1968; Wan 1975).

The exceptional group G_2 was introduced to physics by Racah (1949) in his classification of the states of electrons in the atomic f-shell and later by Flowers (1952) in his analogous treatment of the nuclear f-shell. In these cases G_2 was used simply as a mathematical device to simplify otherwise complex calculations—no physical significance was attached to the use of G_2 (Wybourne 1965).

The introduction of SU_3 into particle physics via a quark model has stimulated the study of higher groups that contain SU_3 as a subgroup. It has been suggested at various times that octonionic Hilbert spaces (Pais 1961; Günaydin and Gürsey 1973) could play an important role in the description of quarks and their associated colour gauge bosons. It is well known that it is possible to generate the exceptional groups starting with octonions (Schafer 1966; Jacobson 1971). Gürsey and his associates (Günaydin and Gürsey 1973; Gürsey *et al.* 1975, 1976; Gürsey and Sikivie 1976) have introduced the concept of octonionic quark fields and have attempted to develop unified theories of strong, electromagnetic and weak interactions based on maximal subgroups of the five exceptional groups, i.e.

$$\begin{array}{ll} G_2 & SU_3^{\mathrm{C}}, \\ F_4 & SU_3 \times SU_3^{\mathrm{C}}, \\ E_6 & SU_3 \times SU_3 \times SU_3^{\mathrm{C}}, \\ E_7 & SU_6 \times SU_3^{\mathrm{C}}, \\ E_8 & E_6 \times SU_3^{\mathrm{C}}, \end{array}$$

where SU_3^{C} is interpreted as the unbroken SU_3 of colour.

These new applications of the exceptional groups have stimulated attempts to establish properties of the groups and their relevant subgroups (Patera and Sankoff 1973; McKay *et al.* 1976*a*, 1976*b*; Patera *et al.* 1976). The exceptional groups, apart from G_2, are characterized by a predominance of representations of very high dimensions. Thus in F_4 there are only four nontrivial irreducible representations (irreps) of dimension $<10^3$, while for E_8 there is only one, and even that has a dimension of 248. There are already applications in particle physics requiring a detailed knowledge of the properties of high dimension irreps of the exceptional groups (Ramond 1976).

The current algorithms used to compute branching rules and to resolve Kronecker products for the irreps of the exceptional groups and their subgroups make use of projection onto the one-dimensional weight subspaces of the irreps (Navon and Patera 1967; Beck 1972; Beck and Kolman 1972; Kolman and Beck 1973). These methods have the great advantage of universality: the same algorithm may be used for all group structures once the root systems and embeddings are specified. Given the high dimension of many irreps it has been essential to resort to large sophisticated computer programs, and even then calculations have been limited to those few irreps of dimension $<10^4$.

It is well known (Littlewood 1950) that the characters of the classical Lie groups can be expressed as series of Schur functions (S-functions) and that these S-functions can be multiplied via the standard Young tableaux or the Littlewood–Richardson rule. An exposition of these methods for handling the properties of the classical Lie groups together with many relevant tabulations has been given elsewhere (Wybourne 1970).

The techniques outlined in this paper basically involve projection onto the irreps of the largest maximal classical Lie subgroup followed by the use of S-functions to systematically compute Kronecker products and branching rules for all the exceptional groups and various relevant subgroups. The approach used is amenable to simple calculation and can readily yield results well beyond the capabilities of the computer algorithms alluded to above. For example, it has been possible to decompose a 31 702 671 dimensional irrep of F_4 into irreps of SO_9 by hand in a matter of minutes with the result being checked on a hand calculator using a list of computed dimensions.

Our aim in this paper is to outline the basic methods for computing branching rules and resolving Kronecker products for all the exceptional groups and to collect together these properties in tabular form for later applications. These results are essential for the calculation of the $3jm$ and $6j$ symbols involving the exceptional groups (Butler 1975; Butler and Wybourne 1976). These symbols play a vital role, via the Wigner–Eckart theorem (Wybourne 1974), in making quantitative calculations of physical properties of systems involving various group structures.

We start by briefly describing the irreps of the special unitary groups and then develop a systematic notation for labelling the irreps of the exceptional groups. The classification of the irreps of the exceptional groups is reviewed and the dimensions and Dynkin index eigenvalues of the irreps are discussed and tabulated. The concept of elementary multiplets is developed and applied especially to the exceptional groups G_2 and F_4. Methods of computing branching rules and resolving Kronecker products are outlined and applied to the exceptional groups and their relevant subgroup structures. Attention is also given to the use of S-function plethysm to resolve the Kronecker powers of irreps.

2. Irreps of SU_n

The special unitary groups SU_n occur as important subgroups or covering groups of the exceptional groups. The irreps of SU_n will here be labelled by ordered partitions of integers. The appropriate partitions are enclosed in braces $\{...\}$. The irreps of SU_n involving n nonzero parts are equivalent to irreps involving $<n$ nonzero parts via

$$\{\lambda_1, \lambda_2, ..., \lambda_n\} \equiv \{\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, ..., \lambda_{n-1} - \lambda_n, 0\}\,. \tag{1}$$

Thus in practice we need only consider partitions into just $n-1$ parts. The familiar quark and antiquark irreps of SU_3 are in our notation designated as $\{1\}$ and $\{1^2\}$ respectively.

The partitions

$$\{\lambda_1, \lambda_2, ..., \lambda_n\} \qquad \text{and} \qquad \{\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, ...\} \tag{2}$$

label irreps of SU_n that are said to be *contragredient* to one another. Partitions where

$$\{\lambda_1, \lambda_2, ..., \lambda_n\} \equiv \{\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, ...\} \tag{3}$$

are said to label irreps of SU_n that are *self-contragredient* and will be distinguished here by attaching an asterisk as a right superscript, e.g. $\{\lambda\}^*$. Numerous useful theorems concerning contragredient irreps have been given by Mal'cev (1944), and we shall exploit some of these in Section 4 below.

The irreps of all the groups discussed here involve sets of integers or half integers, and distinctive brackets are used to enclose these sets: braces $\{...\}$ for special unitary group (SU_n) irreps; square brackets [...] for special orthogonal group (SO_n) irreps; angular brackets $\langle ... \rangle$ for symplectic group (Sp_n) irreps; and parentheses (...) for exceptional group irreps. Sets involving half-integers will be designated by enclosing just their numerators in the appropriate brackets and attaching a prime as a right superscript, e.g. $(3111)' \equiv (\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})$. We shall often use numerical superscripts to indicate the number of times a given part is repeated, e.g. $(21^6) \equiv (2111111)$.

The dimensions and Kronecker products for many of the relevant irreps of the classical Lie groups, and of G_2, have been given elsewhere (Wybourne 1970) and will find frequent use here.

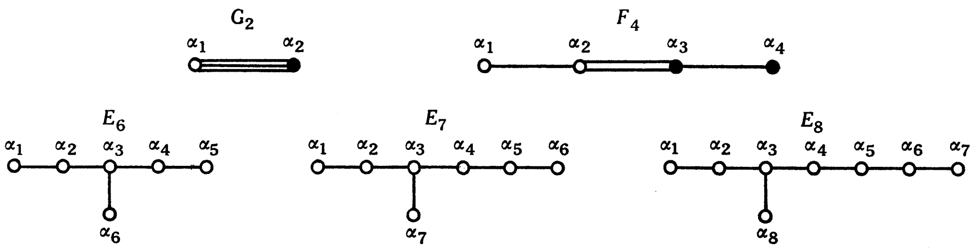

Fig. 1. Dynkin diagrams associated with the exceptional groups.

3. Labelling of Exceptional Group Irreps

Dynkin (1952*a*, 1952*b*) has shown that the irreps of a compact semisimple Lie group of rank l may be uniquely labelled by exploiting the properties of the l simple roots α_i ($i = 1, 2, \ldots, l$) of its associated Lie algebra. The properties of the simple roots are displayed in the form of simple diagrams. The Dynkin diagrams appropriate to the five exceptional groups are shown in Fig. 1. The irreps of a semisimple Lie group of rank l are then uniquely labelled by associating l non-negative integers $(a_1\, a_2 \ldots a_l)$ with the circles of its Dynkin diagram (Wybourne 1974). The integers $\{f_1 \ldots f_{n-1}\}$ we have introduced for SU_n are related to the Dynkin integers $(a_1\, a_2 \ldots a_{n-1})$ by

$$f_i = \sum_{j=i}^{n-1} a_j \,. \tag{4}$$

The irreps of a compact semisimple Lie group of rank l may also be uniquely labelled in terms of the highest weights $M^{(i)}$ of the l basic irreps after the manner of Cartan (1913). Unfortunately Cartan's prescription often leads to cumbersome fractional weights which we wish to avoid. The weights $M^{(i)}$ may be written in terms of the simple roots to give, for the exceptional groups (apart from E_6 which we will treat separately), the following results

$$G_2\colon\quad M^{(1)} = \alpha_1 + 2\alpha_2 , \qquad M^{(2)} = 2\alpha_1 + 3\alpha_2 \,; \tag{5}$$

$$F_4\colon\quad M^{(1)} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 , \tag{6a}$$

$$M^{(2)} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 , \tag{6b}$$

$$M^{(3)} = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4 , \tag{6c}$$

$$M^{(4)} = 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4 \,; \tag{6d}$$

$$E_7\colon\quad M^{(1)} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7 , \tag{7a}$$

$$M^{(2)} = 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6 + 4\alpha_7 , \tag{7b}$$

$$M^{(3)} = 4\alpha_1 + 8\alpha_2 + 12\alpha_3 + 9\alpha_4 + 6\alpha_5 + 3\alpha_6 + 6\alpha_7 , \tag{7c}$$

$$M^{(4)} = \tfrac{1}{2}(6\alpha_1 + 12\alpha_2 + 18\alpha_3 + 15\alpha_4 + 10\alpha_5 + 5\alpha_6 + 9\alpha_7) , \tag{7d}$$

$$M^{(5)} = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 2\alpha_6 + 3\alpha_7 , \tag{7e}$$

$$M^{(6)} = \tfrac{1}{2}(2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 3\alpha_7) , \tag{7f}$$

$$M^{(7)} = \tfrac{1}{2}(4\alpha_1 + 8\alpha_2 + 12\alpha_3 + 9\alpha_4 + 6\alpha_5 + 3\alpha_6 + 7\alpha_7) \,; \tag{7g}$$

$$
\begin{aligned}
E_8:\quad M^{(1)} &= 2\alpha_1+3\alpha_2+4\alpha_3+5\alpha_4+6\alpha_5+4\alpha_6+2\alpha_7+3\alpha_8\,, && \text{(8a)}\\
M^{(2)} &= 3\alpha_1+6\alpha_2+8\alpha_3+10\alpha_4+12\alpha_5+8\alpha_6+4\alpha_7+6\alpha_8\,, && \text{(8b)}\\
M^{(3)} &= 4\alpha_1+8\alpha_2+12\alpha_3+15\alpha_4+18\alpha_5+12\alpha_6+6\alpha_7+9\alpha_8\,, && \text{(8c)}\\
M^{(4)} &= 5\alpha_1+10\alpha_2+15\alpha_3+20\alpha_4+24\alpha_5+16\alpha_6+8\alpha_7+12\alpha_8\,, && \text{(8d)}\\
M^{(5)} &= 6\alpha_1+12\alpha_2+18\alpha_3+24\alpha_4+30\alpha_5+20\alpha_6+10\alpha_7+15\alpha_8\,, && \text{(8e)}\\
M^{(6)} &= 4\alpha_1+8\alpha_2+12\alpha_3+16\alpha_4+20\alpha_5+14\alpha_6+7\alpha_7+10\alpha_8\,, && \text{(8f)}\\
M^{(7)} &= 2\alpha_1+4\alpha_2+6\alpha_3+8\alpha_4+10\alpha_5+7\alpha_6+4\alpha_7+5\alpha_8\,, && \text{(8g)}\\
M^{(8)} &= 3\alpha_1+6\alpha_2+9\alpha_3+12\alpha_4+15\alpha_5+10\alpha_6+5\alpha_7+8\alpha_8\,. && \text{(8h)}
\end{aligned}
$$

The simple roots α_i may be realized in terms of a set of orthogonal unit vectors e_k $(k = 1,2,\ldots,l)$. We make the following choices

$$G_2:\quad \alpha_1 = (1\,-2),\qquad \alpha_2 = (0\,1)\,; \tag{9}$$

$$F_4:\quad \alpha_1 = (01-10),\quad \alpha_2 = (001-1),\quad \alpha_3 = (0001),\quad \alpha_4 = (\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{2})\,; \tag{10}$$

$$E_7:\quad \alpha_i = e_i - e_{i+1}\qquad (i = 1,2,\ldots,6)\,,\quad \alpha_7 = e_4+e_6+e_7\,; \tag{11}$$

$$E_8:\quad \alpha_i = e_i - e_{i+1}\qquad (i = 1,2,\ldots,7)\,,\quad \alpha_8 = e_6+e_7+e_8\,. \tag{12}$$

Use of the realizations in equations (5)–(9) gives the highest weights of the basic irreps as

$$G_2:\quad M^{(1)} = (10)\,,\qquad M^{(2)} = (2-1)\,; \tag{13}$$

$$F_4:\quad M^{(1)} = (1000)\,,\ M^{(2)} = (1100)\,,\ M^{(3)} = (3111)'\,,\ M^{(4)} = (2110)\,; \tag{14}$$

$$
\begin{aligned}
E_7:\quad M^{(1)} &= (21^6)\,,\quad M^{(2)} = (3^2 2^5)\,,\ M^{(3)} = (4^3 3^4)\,,\ M^{(4)} = (3^4 2^3)\,, && \text{(15a)}\\
M^{(5)} &= (2^5 1^2)\,,\quad M^{(6)} = (1^6)\,,\quad M^{(7)} = (2^7)\,; && \text{(15b)}
\end{aligned}
$$

$$
\begin{aligned}
E_8:\quad M^{(1)} &= (21^7)\,,\quad M^{(2)} = (3^2 2^6)\,,\ M^{(3)} = (4^3 3^5)\,,\ M^{(4)} = (5^4 4^4)\,, && \text{(16a)}\\
M^{(5)} &= (6^5 5^3)\,,\ M^{(6)} = (4^6 3^2)\,,\ M^{(7)} = (2^7 1)\,,\quad M^{(8)} = (3^8)\,. && \text{(16b)}
\end{aligned}
$$

The appearance of negative integers for G_2 may be avoided by writing

$$(u_1\,u_2) \equiv (m_1+m_2,\,-m_2)\,, \tag{17}$$

thus reproducing the standard labelling adopted by Racah (1949).

An arbitrary irrep of a rank l semisimple Lie group may be characterized by its highest weight Λ and written as a linear combination of the highest weights $M^{(i)}$ of the l basic irreps:

$$\Lambda = \sum_{i=1}^{l} a_i M^{(i)}\,, \tag{18}$$

where the a_i are non-negative integers and have been chosen here to be exactly equivalent to Dynkin integers. The components Λ_i of Λ satisfy the general condition

$$\Lambda_i \geqslant \Lambda_{i+1} \geqslant 0 \qquad (i = 1,2,\ldots,l-1)\,. \tag{19}$$

In addition there are the specific conditions

$$F_4: \quad \Lambda_1 \geqslant \Lambda_2 + \Lambda_3 + \Lambda_4 \,; \tag{20}$$

$$E_7: \quad \Lambda_4 + \Lambda_5 + \Lambda_6 + \Lambda_7 \geqslant \Lambda_1 + \Lambda_2 + \Lambda_3 \,; \tag{21}$$

$$E_8: \quad 2(\Lambda_6 + \Lambda_7 + \Lambda_8) - (\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4 + \Lambda_5) = 0 \,(\mathrm{mod}\, 3)\,. \tag{22}$$

The conditions (19)–(22) permit every irrep of the exceptional groups G_2, F_4, E_7 and E_8 to be uniquely labelled by appropriate partitions. These partitions may be related to the usual Dynkin integers by inversion of equation (18).

We adopt a somewhat different approach for labelling the irreps of E_6. The Lie algebra E_6 has a maximal subalgebra $A_5 + A_1$. The weights of the elementary irrep of A_5 are designated as $\lambda_1, \lambda_2, \ldots, \lambda_6$ and those of A_1 as $\pm\lambda$. The simple roots may then be expressed in terms of these weights by writing

$$\alpha_i = \lambda_i - \lambda_{i+1} \quad (i = 1, 2, \ldots, 5)\,, \quad \alpha_6 = \lambda + \lambda_4 + \lambda_5 + \lambda_6\,. \tag{23}$$

An arbitrary irrep Λ of E_6 may then be represented as a linear combination of the weights of $A_5 + A_1$ by writing

$$\Lambda = \sum_{i=1}^{6} l_i \lambda_i + l\lambda\,, \qquad \text{where} \qquad \sum_{i=1}^{6} l_i = 0\,, \tag{24}$$

leading to (Wybourne 1974)

$$l_i = l_6 + \sum_{k=i} a_k\,, \tag{25a}$$

$$l_6 = -\sum_{i=1}^{5} \tfrac{1}{6} i a_i\,, \tag{25b}$$

$$l = a_1 + 2a_2 + 3a_3 + 2a_4 + a_5 + 2a_6\,. \tag{25c}$$

While it is quite feasible to label the irreps of E_6 with the seven numbers (l_i, l), the l_i will normally involve fractions of integers. To obtain a system of labels involving just integers we need only recall how the irreps of SU_n may be related to ordered partitions. To this end we introduce five integers m_i, where

$$m_i = \sum_{k=i}^{4} a_{5-k}\,, \tag{26}$$

and a sixth integer $m = l$. It is readily seen that

$$m_i \geqslant m_{i+1} \geqslant 0 \qquad (i = 1, \ldots, 4) \tag{27}$$

and

$$m - m_1 - m_2 - m_3 + m_4 + m_5 = 0 \,(\mathrm{mod}\, 2)\,. \tag{28}$$

The set of six integers $(m_1 m_2 m_3 m_4 m_5 : m)$ uniquely label the irreps of E_6.

The method we have developed for labelling the irreps of the exceptional groups has several advantages over the usual Dynkin labels, and these will become apparent later in this paper.

4. Classification of Exceptional Group Irreps

The classification of the irreps of a group as real (orthogonal), pseudo-real (symplectic) or complex plays an important role in the determination of the phases associated with the calculation of the *nj* and *njm* symbols of the group (Butler and King 1974; Butler and Wybourne 1976). It also plays an essential part in determining branching rules and in resolving Kronecker products. If the representation is equivalent to a real representation we shall term it *orthogonal* while if the character is real but the representation is not equivalent to a real representation we shall term it *symplectic*. However, if the character is complex then the representation will be termed *complex*.

The classification of the representations (λ) of a group proceeds by first noting that the given irrep (λ) will be orthogonal, symplectic or complex respectively according as

$$(\lambda) \otimes \{2\} \supset (0), \quad (\lambda) \otimes \{1^2\} \supset (0) \quad \text{or} \quad (\lambda)^2 \not\supset (0) \tag{29}$$

(Butler and King 1974). The Kronecker product of two symplectic or two orthogonal irreps is necessarily orthogonal while the Kronecker product of an orthogonal irrep with a symplectic irrep is necessarily symplectic (Mal'cev 1944). The representations of a group may be fully classified once the classification is found for the basic irreps of the group (Mehta 1966; Mehta and Srivastava 1966).

The irreps of the exceptional groups G_2, F_4 and E_8 are all orthogonal. The irreps of E_6 are self-contragredient if

$$(m_1 m_2 m_3 m_4 m_5 : m) \equiv (m_1, m_1 - m_5, m_1 - m_4, m_1 - m_3, m_1 - m_2 : m). \tag{30}$$

If the equivalence does not hold then the two irreps are contragredient and complex. The self-contragredient irreps of E_6 are all real and orthogonal (Mal'cev 1944). The irreps of E_7 are orthogonal or symplectic according as

$$\phi = (-1)^{a_4 + a_6 + a_7} = (-1)^{\frac{1}{2}(3\Lambda_4 - \Lambda_5 - \Lambda_6 - \Lambda_1 - \Lambda_2 - \Lambda_3 - \Lambda_7)} \tag{31}$$

is positive or negative.

We note that the irreps $[\Lambda_1 \ldots]$ of SO_{2n+1} are orthogonal or symplectic according as

$$\phi = (-1)^{\frac{1}{2}n(n+1)\Lambda_1} \tag{32}$$

is positive or negative. Thus the irreps of SO_9, which is a maximal subgroup of F_4, are all orthogonal.

The self-contragredient irreps of SU_n will be orthogonal if n is *even*. If $n = 1(\text{mod}\,4)$ the self-contragredient irreps with Λ_1 *even* are orthogonal while those with Λ_1 *odd* are symplectic. All other irreps of SU_n are complex (Mal'cev 1944).

The classification of the irreps has important consequences in determining branching rules. For example, we have just stated that the irreps of E_7 are all orthogonal or symplectic and the self-contragredient irreps of SU_8 are all orthogonal. Thus we can be assured that if an irrep of E_7 is symplectic then the decomposition $E_7 \to SU_8$ will yield no self-contragredient irreps of SU_8. Likewise we may also conclude that SU_8 irreps will always occur as contragredient pairs under $E_7 \to SU_8$.

5. Dimensions of Irreps

A knowledge of the dimensions of irreps of the compact semisimple Lie groups plays an important part in our methods and as a procedural check. The dimensions are calculated using the standard Weyl (1925) formula. An extensive tabulation for many of the groups has been given by P. H. Butler (Wybourne 1970). Additional tables for all the exceptional groups and for the spin irreps of SO_9 were generated using Butler's interactive S-function program.

The tables of dimensions give a simple check on branching rules and Kronecker products. Thus if under $G \to H$ we have the decomposition of the (λ) irrep of G into

$$(\lambda) \to \sum k(\omega)(\omega), \tag{33}$$

where $k(\omega)$ is the multiplicity index for the (ω) irrep of H, then we have the dimensional check

$$N(\lambda) = \sum k(\omega) N(\omega). \tag{34}$$

Likewise, for the Kronecker product

$$(\lambda)(\lambda') = \sum k(\lambda'')(\lambda'') \tag{35}$$

we have

$$N(\lambda)N(\lambda') = \sum k(\lambda'')N(\lambda''). \tag{36}$$

It is important to note that dimensional checks do not provide a complete verification since often different irreps of a group may be of the same dimensions. This is certainly the case for contragredient irreps but the equality of irrep dimensions also occurs for other cases, e.g. in SO_9 the irreps [2211], [2222] and [4100] are all of dimension 2772. In other cases several irreps may sum to give the dimension of another irrep of the group. For precisely for these reasons we consider that the prevalent practice of specifying irreps by their dimensions is an unfortunate and ambiguous notation.

6. Dynkin Indexes

The dimensional problem, just alluded to, can in many cases be overcome by use of the Dynkin index (Dynkin 1952*a*, 1952*b*; Patera *et al.* 1976)

$$j(\lambda) = N(\lambda)\{\boldsymbol{K}^2(\lambda) - \boldsymbol{R}^2\}/r, \tag{37}$$

where r is the order of the group, $\boldsymbol{R}$ is half the sum of the positive weights of the adjoint representation and

$$\boldsymbol{K}(\lambda) = \boldsymbol{M}(\lambda) + \boldsymbol{R}. \tag{38}$$

The quantity $\boldsymbol{K}^2(\lambda) - \boldsymbol{R}^2$ is essentially the second-order Casimir invariant. The calculation of the eigenvalues of the second-order Casimir invariants has been outlined elsewhere (Wybourne 1974). A more practical index having smaller integer eigenvalues is obtained by defining

$$B(\lambda) = j(\lambda)/j(1), \tag{39}$$

Table 1. Dimensions and Dynkin index eigenvalues for irreps of exceptional groups

Irrep (λ)	Dynkin label	Dimen-sions	Dynkin index eigenvalues	Irrep (λ)	Dinkyn label	Dimen-sions	Dynkin index eigenvalues
			(a) G_2 irreps				
(00)	(00)	1	0	(30)	(03)	77	44
(10)	(01)	7	1	(31)	(12)	189	144
(11)	(10)	14	4	(32)	(21)	286	286
(20)	(02)	27	9	(33)	(30)	273	351
(21)	(11)	64	32	(40)	(04)	182	156
(22)	(20)	77	55				
			(b) F_4 irreps				
(0000)	(0000)	1	0	(3100)	(1002)	10 829	1 666
(1000)	(0001)	26	1	(3110)	(0101)	19 278	3 213
(1100)	(1000)	52	3	(3111)	(0020)	19 448	3 366
(3111)′	(0010)	273	21	(3200)	(2001)	17 901	3 213
(2000)	(0002)	324	27	(3210)	(1100)	29 172	5 610
(2100)	(1001)	1 053	108	(3300)	(3000)	12 376	2 618
(2110)	(0100)	1 274	147	(7111)′	(0012)	34 749	6 237
(2200)	(2000)	1 053	135	(7311)′	(1011)	106 496	21 504
(5111)′	(0011)	4 096	512	(7331)′	(0110)	107 406	23 409
(5311)′	(1010)	8 424	1 242	(7511)′	(2010)	119 119	27 489
(3000)	(0003)	2 652	357	(4000)	(0004)	16 302	3 135
			(c) E_6 irreps				
(0:0)*	(000000)	1	0	$(1^2:4)$	(000101)	17 550	2 300
(1:1)	(000010)	27	1	(2:4)	(000021)	19 305	2 695
(0:2)*	(000001)	78	4	$(21^2:4)$	(001010)	51 975	7 700
$(1^2:2)$	(000100)	351	25	$(21^4:4)$*	(100011)	34 749	4 752
(2:2)	(000020)	351	28	$(2^2:4)$	(000200)	34 398	5 390
$(21^4:2)$*	(100010)	650	50	$(2^21^2:4)$*	(010100)	70 070	10 780
(1:3)	(000011)	1 728	160	(31:4)	(000120)	54 054	8 932
$(1^3:3)$*	(001000)	2 925	300	$(31^3:4)$	(010020)	78 975	12 825
(21:3)	(000110)	5 824	672	$(321^3:4)$	(100110)	112 320	18 080
$(21^3:3)$	(010010)	7 371	840	(4:4)	(000040)	19 305	3 520
(3:3)	(000030)	3 003	385	$(41^4:4)$	(100030)	61 425	10 675
$(31^4:3)$	(100020)	7 722	946	$(42^4:4)$*	(200020)	85 293	14 580
(0:4)*	(000002)	2 430	270				
			(d) E_7 irreps				
(0)	(0000000)	1	0	(43^42^2)	(1000100)	152 152	9 152
(1^6)	(0000010)	56	1	(43^51)	(1000020)	150 822	9 450
(21^6)	(1000000)	133	3	(43^6)	(1000001)	86 184	4 995
(2^51^2)	(0000100)	1 539	54	(4^23^42)	(0100010)	362 880	23 760
(2^6)	(0000020)	1 463	55	(4^33^4)	(0010000)	365 750	24 750
(2^7)	(0000001)	912	30	(4^43^22)	(0001010)	980 343	71 253
(32^51)	(1000010)	6 480	270	(4^52^2)	(0000200)	617 253	46 410
(3^22^5)	(0100000)	8 645	390	(4^531)	(0000120)	915 705	71 200
(3^42^3)	(0001000)	27 664	1 430	(4^53^2)	(0000101)	861 840	61 830
(3^521)	(0000110)	51 072	2 832	(4^6)	(0000040)	293 930	24 255
(3^6)	(0000030)	24 320	1 440	(4^62)	(0000011)	885 248	65 728
(3^62)	(0000011)	40 755	2 145	(4^7)	(0000002)	253 935	17 820
(42^6)	(2000000)	7 371	351				
			(e) E_8 irreps				
(21^7)	(10000000)	248	1	(54^7)	(10000001)	26 411 008	372 736
(2^71)	(00000010)	3 875	25	(5^44^4)	(00010000)	146 325 270	2 360 085
(3^22^6)	(01000000)	30 380	245	(5^74)	(00000011)	301 694 976	5 068 800
(3^8)	(00000001)	147 250	1 425	(63^7)	(30000000)	1 763 125	22 750
(42^7)	(20000000)	27 000	225	(65^24^5)	(10100000)	344 452 500	5 740 875
(43^62)	(10000010)	779 247	8 379	(65^54^2)	(10000100)	1 094 951 000	19 426 550
(4^33^5)	(00100000)	2 450 240	29 640	(6^25^6)	(01000001)	2 275 896 000	42 214 200
(4^63^2)	(00000100)	6 696 000	88 200	(6^35^44)	(00100010)	4 825 673 125	93 400 125
(4^72)	(00000020)	4 881 384	65 610	(6^55^3)	(00001000)	6 899 079 264	139 094 340
(543^6)	(11000000)	4 096 000	51 200	(6^654)	(00000110)	8 634 368 000	177 561 600
(5^24^51)	(01000010)	76 272 625	1 148 175				

* Self-contragredient.

where (1) is the vector irrep of the group. The analogue of equation (34) then becomes

$$B(\lambda) = \sum k(\omega)\, B(\omega) \tag{40}$$

and of equation (36)

$$B(\lambda)\, N(\lambda') + B(\lambda')\, N(\lambda) = \sum k(\omega)\, B(\omega)\,. \tag{41}$$

The eigenvalues of the modified Dynkin index are integers and are computed from products of the dimension $N(\lambda)$ and the eigenvalues of the second-order Casimir invariant. Patera *et al.* (1976) have also introduced higher order indexes but the index of equation (39) suffices for our purposes. It is interesting to note that the modified Dynkin index distinguishes the irreps of the same dimension in SO_9 and thus equations (40) and (41) tend to provide more powerful checks than the customary dimensional checks. The Dynkin index eigenvalues will be the same for pairs of contragredient irreps; however, this will rarely cause difficulty if the classification of the irreps into orthogonal, symplectic or complex is known, as indicated in Section 4.

The dimensions and Dynkin index eigenvalues for irreps of each of the exceptional groups are given in Tables 1.

7. Elementary Multiplets

In the method of elementary multiplets (Bargmann and Moshinsky 1961; Devi and Venkatarayudu 1968; Sharp and Lam 1969; Sharp 1970) one seeks to define a minimal set of elementary multiplets for a given group–subgroup combination such that all other multiplets of the combination can be represented as stretched products of members of the set. The *stretched product* gives the multiplet containing the maximal weight in the Kronecker product. Normally, this multiplet is found by simply adding the weights of the multiplets making up the product.

The elementary multiplets of a given group–subgroup combination are found by first determining the decomposition of the vector representation of the group into the irreps of the subgroup. This decomposition will be defined by the manner in which the subgroup is embedded in the group. If the group has spinorial irreps, the decomposition of the basic spinor irreps must also be determined. These multiplets are necessarily members of the set of elementary multiplets but usually will not suffice to yield the complete set. Stretched products of the elementary multiplets just found are formed with the powers of the products chosen to yield multiplets associated with the decomposition of the irrep of the group having the next highest weight. The dimension of this irrep of the group is compared with the sum of the dimensions of the irreps of the subgroup obtained in the stretching process. If the dimensions agree, no new multiplets need be added to the set while, if there is a deficiency in the subgroup, additional elementary multiplets must be added to the set; alternatively, if there is an excess in the subgroup, some of the stretched products must be regarded as redundant and be so specified. In most cases it is obvious from simple dimensional considerations which multiplets must be added to the set or which products must be declared redundant. If there is doubt the modified Dynkin index provides a resolution of the problem.

8. Connection with State-labelling Problem

The problem of determining the set of elementary multiplets of a group–subgroup combination, say $G \supset H$, is closely associated with the problem of finding the corresponding integrity basis (Weyl 1946; Judd *et al.* 1974; McLellan 1974; Bickerstaff and Wybourne 1976), which in turn is related to the state-labelling problem. The exponents of the elementary multiplets can be regarded as supplying a complete set of labels (Sharp 1976). Some combinations of the exponents will be ruled out by subsidiary conditions required to remove redundant states. In the group–subgroup labelling problem one wishes to construct a set of basis functions that are common eigenstates of a complete set of Hermitian opertators. Racah (1965) has shown that, besides the Casimir operators of the G and H groups and the appropriate internal subgroup operators, a number

$$p = \tfrac{1}{2}(l_G - r_G - l_H - r_H) \tag{42}$$

of additional operators must be found, where r_G, l_G, and r_H, l_H are the rank and order of the group and subgroup. The number of missing labels will be a minimum if H is the largest possible subgroup of G. For $G_2 \supset SU_3$ we have $p = 1$ while for $F_4 \supset SO_9$ we have $p = 4$. This result would lead us to expect that the $G_2 \supset SU_3$ elementary multiplets will be fewer, and the conditions on their exponents simpler, than those for $F_4 \supset SO_9$, as is indeed the case.

9. Elementary Multiplets for $G_2 \supset SU_3$ and $F_4 \supset SO_9$

The elementary multiplets for the maximal subgroup SU_3 of G_2 may be readily found using the methods outlined above to give

$$(10)\{0\}, \quad (10)\{1\}, \quad (10)\{1^2\}, \quad (11)\{1\}, \quad (11)\{1^2\}, \quad (11)\{21\}, \tag{43}$$

which apart from a notational difference is equivalent to those found by Sharp and Lam (1969). Stretched states associated with a given irrep $(u_1 u_2)$ of G_2 and an SU_3 irrep $\{\lambda\mu\}$ may be formed from the product of the elementary multiplets, as

$$(u_1 u_2)\{\lambda\mu\} \sim [(10)\{0\}]^a [(10)\{1\}]^b [(10)\{1^2\}]^c [(11)\{1\}]^d [(11)\{1^2\}]^e [(11)\{21\}]^f, \tag{44}$$

where

$$u_1 = a+b+c+d+e+f, \qquad u_2 = d+e+f, \tag{45a}$$

with $u_1 \geqslant u_2 \geqslant 0$, and

$$\lambda = b+c+d+e+2f, \qquad \mu = c+e+f. \tag{45b}$$

In applying the result (44) certain states are excluded by the subsidiary condition

$$af = 0. \tag{46}$$

The elementary multiplets (43) taken with the subsidiary conditions (45) give a simple method of determining the branching rules for $G_2 \to SU_3$; a convenient tabulation of these rules is given in Table 2. It is a trivial matter to extend the table as required. The highest weight irrep of SU_3 arising from an irrep $(u_1 u_2)$ of G_2 is readily seen to be $\{u_1 + u_2, u_1\}$.

We note for later use that the elementary multiplets for obtaining the branching rules for $SO_7 \to G_2$ have been given elsewhere (Wybourne 1972). In that case, seven

Table 2. $G_2 \to SU_3$ branching rules

$(u_1 u_2)$	Branching to SU_3
(00)	$\{0\}$
(10)	$\{1^2\}+\{1\}+\{0\}$
(11)	$\{21\}+\{1^2\}+\{1\}$
(20)	$\{2^2\}+\{2\}+\{21\}+\{1^2\}+\{1\}+\{0\}$
(21)	$\{32\}+\{31\}+\{2^2\}+\{20\}+2\{21\}+\{1^2\}+\{1\}$
(22)	$\{42\}+\{32\}+\{31\}+\{2^2\}+\{2\}+\{21\}$
(30)	$\{3^2\}+\{3\}+\{32\}+\{31\}+\{2^2\}+\{2\}+\{21\}+\{1^2\}+\{1\}+\{0\}$
(31)	$\{43\}+\{41\}+\{42\}+2\{32\}+2\{31\}+\{3^2\}+\{3\}+\{2^2\}+\{2\}+2\{21\}+\{1^2\}+\{1\}$
(32)	$\{53\}+\{52\}+\{43\}+\{41\}+2\{42\}+\{3^2\}+\{3\}+2\{32\}+2\{31\}+\{2^2\}+\{2\}+\{21\}$
(33)	$\{63\}+\{52\}+\{53\}+\{43\}+\{41\}+\{42\}+\{3^2\}+\{3\}+\{32\}+\{31\}$
(40)	$\{4^2\}+\{4\}+\{43\}+\{41\}+\{42\}+\{3^2\}+\{3\}+\{32\}+\{31\}+\{2^2\}$ $+\{2\}+\{21\}+\{1^2\}+\{1\}+\{0\}$

Table 3. $F_4 \supset SO_9$ elementary multiplets

m_i	Multiplet	Subsidiary conditions
m_1	(1000) [0000]	$m_1 m_{10} = m_1 m_{18} = m_3 m_{18}$
m_2	[1000]	
m_3	[1111]′	$= m_3 m_{20} = m_4 m_{19}$
m_4	(1100) [1100]	
m_5	[1111]′	$= m_6 m_{19} = m_7 m_{15}$
m_6	(3111)′[1000]	
m_7	[1100]	$= m_7 m_{20} = m_8 m_{20}$
m_8	[1110]	
m_9	[1111]′	$= m_9 m_{11} = m_9 m_{19}$
m_{10}	[3111]′	
m_{11}	(2100) [1110]	$= m_9 m_{20} = m_{10} m_{13}$
m_{12}	(2110) [1100]	
m_{13}	[1110]	$= m_{10} m_{17} = m_{10} m_{19}$
m_{14}	[2110]	
m_{15}	[3111]′	$= m_{10} m_{20} = m_{15} m_{20}$
m_{16}	[3311]′	
m_{17}	(5311)′[1110]	$= m_{16} m_{17} = m_{16} m_{18}$
m_{18}	[2110]	
m_{19}	(3110) [2210]	$= m_{19} m_{20} = m_{15} m_{16} m_{11} = 0$
m_{20}	(3210) [2110]	

elementary multiplets were required and there was one subsidiary condition. Here we again have $p = 1$ and the situation is simple.

The largest subgroup of F_4 is SO_9. A few branching rules for low-dimension irreps of F_4 under $F_4 \to SO_9$ have been given by Wadzinski (1969). Vastly more can be easily found using the method of elementary multiplets. Since $p = 4$ we would anticipate a larger set of elementary multiplets and subsidiary conditions than for the $p = 1$ cases just treated. The required elementary multiplets are given in Table 3. The entries in this table were used to calculate the $F_4 \to SO_9$ branching rules for all irreps of F_4 up to (9711)′ inclusive. It was found to be a comparatively easy task to obtain the correct decomposition of the 31 702 671-dimensional irrep (6320) of F_4 by a hand calculation.

The list of subsidiary conditions is only established up to the limits just indicated; undoubtedly additional conditions will arise for higher irreps of F_4 but there would appear to be no difficulty in obtaining these as, or if, they are required. A list of $F_4 \to SO_9$ branching rules for all irreps of F_4 up to (4000) is given in Table 4.

Table 4. $F_4 \to SO_9$ branching rules

$D_{(u)}$	$(u_1 u_2 u_3 u_4)$	Branching to SO_9
1	(0000)	[0000]
26	(1000)	[0000]+[1000]+[1111]′
52	(1100)	[1100]+[1111]′
273	(3111)′	[1000]+[1100]+[1110]+[1111]′+[3111]′
324	(2000)	[0000]+[1000]+[1111]+[2000]+[1111]′+[3111]′
1 053	(2100)	[1100]+[1110]+[1111]+[2100]+[1111]′+[3111]′+[3311]′
1 274	(2110)	[1100]+[1110]+[2110]+[3111]′+[3311]′
1 053	(2200)	[1111]+[2200]+[3311]′
4 096	(5111)′	[1000]+[1100]+[1110]+[1111]+[2000]+[2100]+[2110]+[2111]+[1111]′ +2[3111]′+[3311]′+[3331]′+[5111]′
8 424	(5311)′	[1110]+[1111]+[2100]+[2110]+[2111]+[2200]+[2210]+[3111]′+2[3311]′ +[3331]′+[5311]′
2 652	(3000)	[0000]+[1000]+[1111]+[2000]+[2110]+[3000]+[1111]′+[3111]′+[3333]′ +[5111]′
10 829	(3100)	[1100]+[1110]+[1111]+[2100]+[2110]+[2111]+[2211]+[3100]+[1111]′ +[3111]′+[3311]′+[3331]′+[3333]′+[5111]′+[5311]′
19 278	(3110)	[1100]+[1110]+[2100]+2[2110]+[2111]+[2210]+[2211]+[3110]+[3111]′ +2[3311]′+[3331]′+[5111]′+[5311]′+[5331]′
19 448	(3111)	[1111]+[2000]+[2100]+[2110]+[2111]+[2200]+[2210]+[2220]+[3111] +[3111]′+[3311]′+[3331]′+[5111]′+[5311]′+[5331]′
17 901	(3200)	[1111]+[2111]+[2200]+[2210]+[2211]+[3200]+[3311]′+[3331]′+[3333]′ +[5311]′+[5511]′
29 172	(3210)	[2110]+[2111]+[2200]+[2210]+[2211]+[3210]+[3311]′+[3331]′+[5311]′ +[5331]′+[5511]′
12 376	(3300)	[2211]+[3300]+[3333]′+[5511]′
34 749	(7111)′	[1000]+[1100]+[1110]+[1111]+[2000]+[2100]+[2110]+2[2111]+[2211] +[2221]+[3000]+[3100]+[3110]+[1111]′+2[3111]′+[3311]′ +[3331]′+[3333]′+2[5111]′+[5311]′+[5331]′+[5333]′+[7111]′
106 496	(7311)′	[1110]+[1111]+[2100]+2[2110]+2[2111]+[2200]+2[2210]+2[2211] +[2220]+[2221]+[3100]+[3110]+[3111]+[3200]+[3210]+[3211] +[3111]′+2[3311]′+2[3331]′+[3333]′+[5111]′+3[5311]′+2[5331]′ +[5333]′+[5511]′+[5531]′+[7311]′
107 406	(7331)′	[2100]+[2110]+[2111]+[2200]+2[2210]+[2211]+[2220]+[3110]+[3111] +[3210]+[3211]+[3220]+[3311]′+[3331]′+[5111]′+2[5311]′ +2[5331]′+[5511]′+[5531]′+[7331]′
119 119	(7511)′	[2111]+[2210]+2[2211]+[2221]+[3200]+[3210]+[3211]+[3300]+[3310] +[3331]′+[3333]′+[5311]′+[5331]′+[5333]′+2[5511]′+[5531]′ +[7511]′
16 302	(4000)	[0000]+[1000]+[1111]+[2000]+[2111]+[2222]+[3111]+[4000]+[1111]′ +[3111]′+[3333]′+[5111]′+[5333]′+[7111]′

10. Elementary Multiplets for $SO_{26} \supset F_4$

It is well known (Dynkin 1952*a*, 1952*b*; Wadzinski 1969) that F_4 can be embedded in the group SO_{26}, a situation analogous to the $SO_7 \supset G_2$ embedding (Racah 1949; Judd 1962). A few branching rules for $SO_{26} \to F_4$ have been given by Wadzinski (1969). In the case of $SO_{26} \supset F_4$ we have from equation (42) that $p = 128$ so that the missing-label problem assumes colossal proportions; however, some progress is possible. The elementary multiplets associated with one-part irreps of SO_{26} may be readily and completely determined. The number of elementary multiplets for partitions into two or more parts grows dramatically. The branching rules for the true irreps of SO_{26} involving partitions of weight four or less were determined by use of elementary multiplets and a knowledge of some F_4 Kronecker products. The elementary multiplets and the resulting $SO_{26} \to F_4$ branching rules are given in Table 5. There are no inherent difficulties in extending these listings as required. An alternative approach to the determination of $SO_{26} \to F_4$ branching rules is given in Section 15.

Table 5. $SO_{26} \supset F_4$ elementary multiplets and $SO_{26} \to F_4$ branching rules

(a) Elementary multiplets

[1] (1000)
[2] (1000)
[3] (0000)(3111)′
[11] (1100)(3111)′
[21] (1000)(1100)(2000)(3111)′
[31] (1000)(1100)(2110)2(3111)′
[22] (0000)(1000)2(2000)(2100)(3000)(3111)′(5111)′
[111] (2100)(2110)(3111)′
[211] (1000)(1100)(2000)2(2100)2(2110)2(3111)′(5111)′(5311)′
[1111] (2000)(2100)(2200)(5111)′(5311)′

(b) $SO_{26} \to F_4$ branching rules

$D_{[\lambda]}$	$[\lambda]$	Branching to F_4
1	[0]	(0000)
26	[1]	(1000)
325	[11]	(1100)+(3111)′
350	[2]	(1000)+(2000)
3 250	[3]	(0000)+(2000)+(3000)+(3111)′
5 824	[21]	(1000)+(1100)+(2000)+(2100)+(3111)′+(5111)′
2 600	[111]	(2100)+(2110)+(3111)′
23 400	[4]	(1000)+(2000)+(3000)+(4000)+(5111)′
60 750	[31]	(1000)+(1100)+(2000)+2(2100)+(2110)+(3000)+(3100)+2(3111)′+2(5111)′ +(7111)′
37 674	[22]	(0000)+(1000)+2(2000)+(2100)+(2200)+(3000)+(3111)+(3111)′+(5111)′ +(5311)′
52 325	[211]	(1000)+(1100)+(2000)+2(2100)+2(2110)+(3100)+(3110)+2(3111)′+2(5111)′ +(5311)′
14 950	[1111]	(2000)+(2100)+(2200)+(5111)′+(5311)′

11. Highest Weight Irreps and Branching Rules

The irreps of a compact semisimple Lie group are uniquely labelled, to within an equivalence, by their highest weight (Wybourne 1974). Our choice of labelling for the irreps of F_4, E_6, E_7 and E_8 is such that we necessarily have under

$$F_4 \to SO_9: \qquad (\lambda_1\lambda_2\lambda_3\lambda_4) \supset [\lambda_1\lambda_2\lambda_3\lambda_4], \tag{47}$$

$$E_6 \to SU_6 \times SU_2: \qquad (\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5:\lambda) \supset \{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\}\{\lambda\}, \tag{48}$$

$$E_7 \to SU_8: \qquad (\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6\lambda_7) \supset \{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6\lambda_7\}, \tag{49}$$

$$E_8 \to SU_9: \qquad (\lambda_1\,\lambda_2\,\lambda_3\,\lambda_4\,\lambda_5\,\lambda_6\,\lambda_7\,\lambda_8) \supset \{\lambda_1\,\lambda_2\,\lambda_3\,\lambda_4\,\lambda_5\,\lambda_6\,\lambda_7\,\lambda_8\}, \tag{50}$$

where in each case the subgroup irrep given is the highest weight irrep of the subgroup contained in the decomposition. The situation is slightly different for $G_2 \supset SU_3$ due to our adoption of Racah's (1949) scheme via equation (17). The results (47)–(50) play a key role in our subsequent determination of branching rules and Kronecker product resolutions.

12. Kronecker Products for F_4

The evaluation of the Kronecker products of the irreps of G_2 may be readily made (Smith and Wybourne 1967) by expressing the characters of G_2 as linear combinations of those of SO_7, thus forming the Kronecker products for the SO_7 irreps and then decomposing the SO_7 irreps into those of G_2. An extensive tabulation has been given elsewhere (Wybourne 1970). Judd and Wadzinski (1967) have given the resolution of the Kronecker squares of some G_2 irreps into their symmetric and antisymmetric terms.

The Kronecker products of the irreps of F_4 may be determined by exploiting our knowledge of the $F_4 \to SO_9$ decompositions. The two F_4 irreps appearing in the Kronecker product are each expanded into SO_9 irreps using the results of Table 4, and the Kronecker products of the SO_9 irreps are then resolved using standard S-function theory (Wybourne 1970). The procedure is best illustrated by a simple example.

To evaluate $(1000)\times(1100)$ we have from Table 4

$$(1000) \to [0000]+[1000]+[1111]'$$

and

$$(1100) \to [1100]+[1111]',$$

and hence

$$\begin{aligned}(1000)\times(1100) \to {} & [[0000]+[1000]+[1111]']\times[[1100]+[1111]'] \\ = {} & [0000]+2[1000]+2[1100]+2[1110]+[1111]+[2100] \\ & +3[1111]'+2[3111]'+[3311]'. \end{aligned} \tag{51}$$

Since under $F_4 \to SO_9$ we necessarily have $(\lambda_1\,\lambda_2\,\lambda_3\,\lambda_4) \supset [\lambda_1\,\lambda_2\,\lambda_3\,\lambda_4]$ we pick out of equation (51) the highest weight irrep of SO_9, in this case [2100]. Thus the Kronecker product $(1000)\times(1100) \supset (2100)$. Now remove from the right-hand side of (51) all SO_9 irreps contained in (2100) to give

$$\begin{aligned}(1000)\times(1100) \supset {} & (2100)+[0000]+2[1000]+[1100]+[1110] \\ & +2[1111]'+[3111]'. \end{aligned}$$

The highest weight irrep of SO_9 in the residue is $[3111]'$ and hence

$$(1000)\times(1100) \supset (2100)+(3111)'+[0000]+[1000]+[1111]'.$$

Inspection of the new SO_9 residue leads immediately to the conclusion that

$$(1000)\times(1100) = (1000)+(2100)+(3111)'.$$

The only difficult part in the calculation is the evaluation of the SO_9 Kronecker products. Tables for the true irreps exist (Wybourne 1970). The products involving spin irreps may be evaluated using the methods outlined elsewhere. In our case we were able to get the necessary results from P. H. Butler's interactive program that readily evaluates all the S-function procedures. A sample list of Kronecker products of F_4 irreps is given in Table 6. In the case of Kronecker squares the irreps contained in the symmetric part are enclosed in braces {...} and those of the antisymmetric part in square brackets [...].

Table 6. F_4 Kronecker products

Product	Evaluation
(0000) × (0000)	(0000)
(1000) × (0000)	(1000)
(1000) × (1000)	{(0000) + (1000) + (2000)} + [(1100) + (3111)′]
(1100) × (1000)	(1000) + (2100) + (3111)′
(1100) × (1100)	{(0000) + (2000) + (2200)} + [(1100) + (2110)]
(3111)′ × (1000)	(1000) + (1100) + (2000) + (2100) + (2110) + (3111)′ + (5111)′
(3111)′ × (1100)	(1000) + (2000) + (2100) + (3111)′ + (5111)′ + (5311)′
(3111)′ × (3111)′	{(0000) + (1000) + 2(2000) + (2100) + (2200) + (2200) + (3000) + (3111)′ + (5111)′ + (5311)′} + [(1100) + (2100) + (2110) + (3100) + (3110) + 2(3111)′ + (5111)′]
(2000) × (1000)	(1000) + (2000) + (2100) + (3000) + (3111)′ + (5111)′
(2000) × (1100)	(1100) + (2000) + (2110) + (3100) + (3111)′ + (5111)′
(2000) × (3111)′	(1000) + (1100) + (2000) + 2(2100) + (2110) + (3000) + (3100) + (3110) + 2(3111)′ + 2(5111)′ + (5311)′ + (7111)′
(2000) × (2000)	{(0000) + (1000) + 2(2000) + (2200) + (3000) + (3111) + (4000) + (5111)′ + (5311)′} + [(1100) + (2100) + (2110) + (3100) + (3111)′ + (5111)′ + (7111)′]

13. Kronecker Products and Branching Rules

The method just used for the resolution of the Kronecker products of F_4 irreps can be readily extended to the remaining exceptional groups E_6, E_7 and E_8 and used as the basis for a building-up principle that simultaneously yields branching rules for the maximal subgroup embeddings $E_6 \supset SU_6 \times SU_2$, $E_7 \supset SU_8$ and $E_8 \supset SU_9$. In this method we exploit the fact that we can resolve products and symmetrized powers of SU_n irreps by the standard S-function calculus (Wybourne 1970).

The method is best demonstrated by the example of E_7. We first make the $E_7 \to SU_8$ decomposition for the elementary irrep (1^6). It follows from the result (49) that

$$(1^6) \supset \{1^6\} \tag{52}$$

and, since (1^6) is a symplectic irrep, the irreps of SU_8 can only occur as contragredient partners and hence $\{1^2\}$ must be added to (52). A dimensional check then assures us that we have the complete branching rule as

$$(1^6) \to \{1^6\} + \{1^2\}. \tag{53}$$

We now resolve the Kronecker square of (1^6) into its symmetric terms. Consider the symmetric part. From the rule (53)

$$\begin{aligned}(1^6) \otimes \{2\} &= (\{1^6\} + \{1^2\}) \otimes \{2\} \\ &= \{1^6\} \otimes \{2\} + \{1^2\} \otimes \{2\} + \{1^6\}\{1^2\}.\end{aligned} \tag{54}$$

Noting that the plethysms $\{1^6\} \otimes \{2\}$ and $\{1^2\} \otimes \{2\}$ will yield irreps of SU_8 that are contragredient to one another, we obtain

$$(1^6) \otimes \{2\} = (\{2^6\}+\{1^4\})+(\{2^2\}+\{1^4\})+(\{2^2 1^4\}+\{21^6\}+\{0\}). \qquad (55)$$

(The relevant plethysms may be evaluated directly (Wybourne 1970) or read off existing tables (Butler and Wybourne 1971).) Remembering the result (49), we deduce that (2^6) necessarily occurs in $(1^6) \otimes \{2\}$. Inspection of the SU_8 irreps in equation (55) shows that the only other irreps of E_7 that could arise would be (21^6) and possibly (0). Dimensional considerations, or the fact that (1^6) is symplectic, rule out the possibility of (0) and hence

$$(1^6) \otimes \{2\} = (2^6)+(21^6). \qquad (56)$$

It follows from stretching the SU_8 irreps of rule (53) that

$$(2^6) \supset \{2^6\}+\{2^2\}+\{2^2 1^4\}$$

and necessarily

$$(21^6) \supset \{21^6\}.$$

Simple dimensional arguments and comparison of equation (55) with (56) lead immediately to the results

$$(2^6) \rightarrow \{2^6\}+\{2^2\}+\{2^2 1^4\}+\{1^4\}+\{0\},$$
$$(21^6) \rightarrow \{21^6\}+\{1^4\}.$$

Exactly the same analysis can be made for

$$(1^6) \otimes \{1^2\} = (2^5 1^2)+(0)$$

and then trivially

$$(2^5 1^2) \rightarrow \{2^5 1^2\}+\{21^2\}+\{2^2 1^4\}+\{21^6\}.$$

Thus at the same time as resolving Kronecker products we have been able to establish new branching rules.

Having obtained the first few branching rules it then becomes possible to resolve further Kronecker products and hence to establish further branching rules. At each step the calculations are checked using dimensions and the Dynkin index eigenvalues. The resulting branching rules for $E_6 \rightarrow SU_6 \times SU_2$, $E_7 \rightarrow SU_8$ and $E_8 \rightarrow SU_9$ are given in Tables 7*a*, 7*b* and 7*c* respectively. Contragredient irreps are grouped together in pairs and self-contragredient irreps are indicated by an asterisk.

The Kronecker products evaluated for the groups E_6, E_7 and E_8 are given in Tables 8. In the case of E_6 (Table 8*a*) the list is shortened by noting that if the Kronecker product of two irreps (m_1) and (m_2) is given as

$$(m_1) \times (m_2) = \sum (m_{12}), \qquad (57)$$

then for the contragredient irreps $(m_1)^c$ and $(m_2)^c$ we have

$$(m_1)^c \times (m_2)^c = \sum (m_{12})^c. \qquad (58)$$

Table 7. E_n group branching rules

Asterisks indicate self-contragredient irreps throughout

(λ)	Branching
	(a) $E_6 \to SU_6 \times SU_2$ branching rules
$(0:0)$	$\{0\}\{0\}$
$(1:1)$	$\{1\}\{1\}+\{1^4\}\{0\}$
$(0:2)^*$	$\{0\}^*\{2\}+\{1^3\}^*\{1\}+\{21^4\}^*\{0\}$
$(1^2:2)$	$\{1^2\}\{2\}+(\{21^3\}+\{1^5\})\{1\}+(\{2^31^2\}+\{2\})\{0\}$
$(2:2)$	$\{2\}\{2\}+\{21^3\}\{1\}+(\{2^4\}+\{1^2\})\{0\}$
$(21^4:2)^*$	$\{21^4\}^*\{2\}+(\{2^41\}+\{21\}+\{1^3\}^*)\{1\}+(\{21^4\}^*+\{2^21^2\}^*+\{0\}^*)\{0\}$
$(1:3)$	$\{1\}\{3\}+(\{1^4\}+\{21^2\})\{2\}+(\{31^4\}+\{2^31\}+\{2^21^3\}+\{1\})\{1\}$ $+(\{32^31\}+\{21^2\}+\{1^4\})\{0\}$
$(1^3:3)^*$	$\{1^3\}^*\{3\}+(\{2^21^2\}^*+\{21^4\}^*+\{0\}^*)\{2\}+(\{32^21^2\}^*+\{2^41\}+\{21\}+\{1^3\}^*)\{1\}$ $+(\{3^22^3\}+\{31^3\}+\{2^3\}^*+\{21^4\}^*)\{0\}$
$(21:3)$	$\{21\}\{3\}+(\{31^3\}+\{2^21^2\}^*+\{21^4\}^*)\{2\}$ $+(\{32^21^2\}^*+\{32^3\}+\{3\}+\{2^41\}+\{21\}+\{1^3\})\{1\}$ $+(\{3^321\}+\{31^3\}+\{2^21^2\}^*+\{21^4\}^*)\{0\}$
$(21^3:3)$	$\{21^3\}\{3\}+(\{321^3\}+\{2^4\}+\{2\}+\{2^31^2\}+\{1^2\})\{2\}$ $+(\{3^22^21\}+\{32^4\}+\{31^2\}+2\{21^3\}+\{2^21\}+\{1^5\})\{1\}$ $+(\{3^42\}+\{32^21\}+\{321^3\}+\{2^31^2\}+\{2\}+\{1^2\})\{0\}$
$(3:3)$	$\{3\}\{3\}+\{31^3\}\{2\}+(\{32^3\}+\{21\})\{1\}+(\{3^4\}+\{2^21^2\}^*)\{0\}$
$(31^4:3)$	$\{31^4\}\{3\}+(\{32^31\}+\{31\}+\{21^2\})\{2\}$ $+(\{3^41\}+\{321^2\}+\{31^4\}+\{2^31\}+\{2^21^3\}+\{1\})\{1\}$ $+(\{3^22^2\}+\{32^31\}+\{2^2\}+\{21^2\}+\{1^4\})\{0\}$
$(0:4)^*$	$\{0\}^*\{4\}+\{1^3\}^*\{3\}+(\{2^3\}^*+\{21^4\}^*)\{2\}$ $+(\{32^21^2\}^*+\{1^3\}^*)\{1\}+(\{42^4\}^*+\{2^21^2\}^*+\{0\}^*)\{0\}$
$(1^2:4)$	$\{1^2\}\{4\}+(\{2^21\}+\{21^3\}+\{1^5\})\{3\}+(\{32^21\}+\{321^3\}+2\{2^31^2\}+\{2\}+\{1^2\})\{2\}$ $+(\{42^31\}+\{3^31^2\}+\{3^22^21\}+\{32^4\}+\{31^2\}+\{2^21\}+2\{21^3\}+\{1^5\})\{1\}$ $+(\{43^22^2\}+\{41^4\}+\{32^21\}+\{321^3\}+\{2^4\}+\{2^31^2\}+\{1^2\})\{0\}$
$(2:4)$	$\{2\}\{4\}+(\{31^2\}+\{21^3\})\{3\}+(\{41^4\}+\{32^21\}+\{321^3\}+\{2^4\}+\{2\}+\{1^2\})\{2\}$ $+(\{42^31\}+\{3^32\}+\{3^22^21\}+\{31^2\}+\{2^21\}+2\{21^3\})\{1\}$ $+(\{43^31\}+\{32^21\}+\{321^3\}+\{2^4\}+\{2^31^2\}+\{2\})\{0\}$
$(21^2:4)$	$\{21^2\}\{4\}+(\{321^2\}+\{31^4\}+\{2^31\}+\{2^21^3\}+\{1\})\{3\}$ $+(\{42^21^2\}+\{3^22^2\}+\{3^221^2\}+2\{32^31\}+\{31\}+\{2^2\}+2\{21^2\}+2\{1^4\})\{2\}$ $+(\{43^221\}+\{432^3\}+\{41^3\}+\{3^41\}+\{3^32^2\}+\{32^2\}+2\{321^2\}+2\{31^4\}$ $+2\{2^31\}+2\{2^21^3\}+\{1\})\{1\}$ $+(\{4^23^22\}+\{42^3\}+\{42^21^2\}+\{3^21\}+\{3^221^2\}+2\{32^31\}+\{31\}+\{2^5\}$ $+2\{21^2\})\{0\}$
$(21^4:4)^*$	$\{21^4\}^*\{4\}+(\{32^21^2\}^*+\{2^41\}+\{21\}+\{1^3\}^*)\{3\}+(\{42^4\}^*+\{3^321\}+\{321\}+\{3^22^3\}$ $+\{31^3\}+\{2^3\}^*+2\{2^21^2\}^*+2\{21^4\}^*+\{0\})\{2\}+(\{43^32\}+\{421^3\}+\{3^221\}^*$ $+\{3^21^3\}+\{32^3\}+3\{32^21^2\}^*+2\{2^41\}+2\{21\}+2\{1^3\}^*)\{1\}$ $+(\{432^21\}^*+\{42^4\}^*+\{3^321\}+\{321\}+\{3^22^3\}+\{31^3\}+\{2^3\}^*+2\{2^21^2\}^*$ $+\{21^4\}^*)\{0\}$
$(2^2:4)$	$\{2^2\}\{4\}+(\{321^2\}+\{2^21^3\})\{3\}+(\{42^3\}+\{3^221^2\}+\{32^31\}+\{31\}+\{2^5\}+\{21^2\})\{2\}$ $+(\{43^221\}+\{41^3\}+\{3^32^2\}+\{321^2\}+\{31^4\}+\{2^31\}+\{2^21^3\})\{1\}$ $+(\{4^32^2\}+\{42^21^2\}+\{4\}+\{3^22^2\}+\{32^31\}+\{2^2\}+\{1^4\})\{0\}$
$(2^21^2:4)^*$	$\{2^21^2\}^*\{4\}+(\{3^21^3\}+\{32^3\}+\{32^21^2\}^*+\{2^41\}+\{21\}+\{1^3\}^*)\{3\}$ $+(\{432^21\}^*+\{3^321\}+\{321\}+2\{3^22^3\}+2\{31^3\}+\{2^3\}^*+2\{2^21^2\}^*$ $+2\{21^4\}^*)\{2\}+(\{4^232^2\}+\{42^21\}+\{43^32\}+\{421^3\}+\{3^5\}+\{3\}+\{3^221\}^*$ $+\{3^21^3\}+\{32^3\}+3\{32^21^2\}^*+\{2^41\}+\{21\}+\{1^3\}^*)\{1\}$ $+(\{4^33^2\}+\{41^2\}+\{43^21^2\}^*+\{432^21\}^*+\{42^4\}^*+\{3^321\}+\{321\}$ $+\{3^22^3\}+\{31^3\}+2\{2^21^2\}^*+\{21^4\}^*+\{0\}^*)\{0\}$
$(31:4)$	$\{31\}\{4\}+(\{41^3\}+\{321^2\}+\{31^4\})\{3\}$ $+(\{42^3\}+\{42^21^2\}+\{4\}+\{3^22^2\}+\{32^31\}+\{31\}+\{2^2\}+\{21^2\})\{2\}$ $+(\{43^3\}+\{41^3\}+\{3^41\}+2\{321^2\}+\{31^4\}+\{2^31\}+\{2^21^3\})\{1\}$ $+\{4^331\}+\{42^3\}+\{3^22^2\}+\{3^221^2\}+\{32^31\}+\{31\}+\{21^2\})\{0\}$

Table 7 (*Continued*)

(λ)	Branching
	(a) $E_6 \to SU_6 \times SU_2$ (*Continued*)
$(31^3:4)$	$\{31^3\}\{4\}+(\{421^3\}+\{32^3\}+\{32^21^2\}^*+\{3\}+\{21\})\{3\}$ $+(\{432^21\}^*+\{42^4\}^*+\{41^2\}+\{3^4\}+\{3^321\}+\{321\}+2\{31^3\}+2\{2^21^2\}^*$ $+\{21^4\}^*)\{2\}+(\{4^23^21\}+\{42^21\}+\{43^32\}+\{421^3\}+\{3^21^3\}+\{3^221\}$ $+2\{32^3\}+2\{32^21^2\}^*+\{3\}+\{2^41\}+2\{21\}+\{1^3\}^*)\{1\}+(\{4^42\}+\{43^22\}$ $+\{432^21\}^*+\{3^321\}+\{321\}+\{3^22^3\}+2\{31^3\}+\{2^3\}^*+\{2^21^2\}^*+\{21^4\}^*)\{0\}$
$(321^3:4)$	$\{321^3\}\{4\}+(\{42^31\}+\{3^22^21\}+\{32^4\}+\{32\}+\{31^2\}+\{2^21\}+\{21^3\})\{3\}$ $+(\{43^31\}+\{43^22^2\}+\{421^2\}+\{41^4\}+\{3^42\}+\{3^21^2\}+2\{32^21\}+3\{321^3\}$ $+\{2^4\}+2\{2^31^2\}+\{2\}+\{1^2\})\{2\}+(\{4^332\}+\{432^2\}+\{4321^2\}+2\{42^31\}$ $+\{41\}+\{3^31^2\}+\{3^32\}+3\{3^22^21\}+2\{31^2\}+\{32\}+2\{2^21\}+3\{21^3\}$ $+\{1^5\})\{1\}+(\{4^2321\}+\{43^31\}+\{43^22^2\}+\{421^2\}+\{41^4\}+\{3^42\}$ $+\{3^21^2\}+2\{32^21\}+2\{321^3\}+\{2^4\}+2\{2^31^2\}+\{2\}+\{1^2\})\{0\}$
$(4:4)$	$\{4\}\{4\}+\{41^3\}\{3\}+(\{42^3\}+\{31\})\{2\}+(\{43^3\}+\{321^2\})\{1\}$ $+(\{4^4\}+\{3^22^2\}+\{2^2\})\{0\}$
$(41^4:4)$	$\{41^4\}\{4\}+(\{42^31\}+\{41\}+\{31^2\})\{3\}$ $+(\{43^31\}+\{421^2\}+\{41^4\}+\{32^21\}+\{321^3\}+\{2\})\{2\}$ $+(\{4^41\}+\{432^2\}+\{42^31\}+\{3^32\}+\{3^22^21\}+\{32\}+\{31^2\}+\{2^21\}+\{21^3\})\{1\}$ $+(\{4^23^3\}+\{43^31\}+\{3^21^2\}+\{32^21\}+\{321^3\}+\{2^4\}+\{1^2\})\{0\}$
$(42^4:4)^*$	$\{42^4\}^*\{4\}+(\{43^32\}+\{421^3\}+\{32^21^2\}^*)\{3\}+(\{4^42\}+\{42\}+\{432^21\}^*$ $+\{42^4\}^*+\{3^321\}+\{321\}+\{3^22^3\}+\{31^3\}+\{2^3\}^*+\{21^4\}^*)\{2\}$ $+(\{4^23^21\}+\{431^2\}+\{43^32\}+\{421^3\}+\{3^221\}^*+\{3^21^3\}+\{32^3\}$ $+2\{32^21^2\}^*+\{2^41\}+\{21\}+\{1^3\}^*)\{1\}+(\{4^22^2\}^*+\{432^21\}^*+\{42^4\}^*$ $+\{3^4\}+\{3^2\}+\{3^321\}+\{321\}+2\{2^21^2\}^*+\{21^4\}^*+\{0\}^*)\{0\}$
	(b) $E_7 \to SU_8$ *branching rules*
(0)	$\{0\}^*$
(1^6)	$\{1^6\}+\{1^2\}$
(21^6)	$\{21^6\}^*+\{1^4\}^*$
(2^51^2)	$\{2^51^2\}+\{21^2\}+\{2^21^4\}^*+\{21^6\}^*$
(2^6)	$\{2^6\}+\{2^2\}+\{2^21^4\}^*+\{1^4\}^*+\{0\}^*$
(2^7)	$\{2^7\}+\{2\}+\{2^31^4\}+\{21^4\}$
(32^51)	$\{32^51\}+\{321^5\}+\{2^41^2\}+\{2^21^2\}+\{2^31^4\}+\{21^4\}+\{1^6\}+\{1^2\}$
(3^22^5)	$\{3^22^5\}+\{31^5\}+\{32^31^3\}^*+\{2^51^2\}+\{21^2\}+\{2^31^2\}^*+\{21^6\}^*$
(3^42^3)	$\{3^42^3\}+\{31^3\}+\{3^22^31^2\}+\{32^21^3\}+\{32^51\}+\{321^5\}+\{2^7\}+\{2\}$ $+\{2^5\}+\{2^3\}+\{2^31^4\}+\{21^4\}$
(3^521)	$\{3^521\}+\{321\}+\{3^22^4\}+\{3^21^4\}+\{3^22^31^2\}+\{32^21^3\}+\{32^51\}$ $+\{321^5\}+\{2^41^2\}+\{2^21^2\}+\{2^31^4\}+\{21^4\}+\{1^6\}+\{1^2\}$
(3^6)	$\{3^6\}+\{3^2\}+\{3^22^4\}+\{3^21^4\}+\{2^41^2\}+\{2^21^2\}+\{1^6\}+\{1^2\}$
(3^62)	$\{3^62\}+\{31\}+\{3^32^31\}+\{321^3\}+\{3^22^5\}+\{31^5\}+\{3^221^4\}+\{32^41\}$ $+\{32^31^3\}^*+\{2^51^2\}+\{21^2\}+\{2^31^2\}^*+\{2^21^4\}^*+\{21^6\}^*+\{1^4\}^*$
(42^6)	$\{42^6\}^*+\{32^31^3\}^*+\{2^4\}^*+\{2^21^4\}^*+\{1^4\}^*+\{0\}^*$
(43^42^2)	$\{43^42^2\}+\{42^21^4\}+\{432^41\}^*+\{42^6\}^*+\{3^421^2\}+\{32^21\}+\{3^32^31\}+\{321^3\}$ $+\{3^22^5\}+\{31^5\}+\{3^22^21^2\}^*+\{3^221^4\}+\{32^41\}+2\{32^31^3\}^*+\{2^6\}+\{2^2\}$ $+\{2^51^2\}+\{21^2\}+\{2^31^2\}^*+2\{2^21^4\}^*+\{21^6\}^*+\{1^4\}^*$
(43^51)	$\{43^51\}+\{431^5\}+\{432^41\}^*+\{3^42^2\}+\{3^21^2\}+\{3^32^31\}+\{321^3\}+\{3^22^21^2\}^*$ $+\{3^221^4\}+\{32^41\}+\{32^31^3\}^*+\{2^6\}+\{2^2\}+\{2^51^2\}+\{21^2\}$ $+\{2^4\}^*+\{2^31^2\}^*+2\{2^21^4\}^*+\{21^6\}^*+\{1^4\}^*$
(43^6)	$\{43^6\}+\{41^6\}+\{43^22^4\}+\{42^41^2\}+\{3^42^3\}+\{31^3\}+\{3^321^3\}+\{32^31\}$ $+\{3^22^31^2\}+\{32^21^3\}+\{32^51\}+\{321^5\}+\{2^41^2\}+\{2^21^2\}+\{2^31^4\}+\{21^4\}+\{1^6\}$ $+\{1^2\}$

Table 7 (*Continued*)

(λ)	Branching
	(*b*) $E_7 \to SU_8$ (*Continued*)
(4^23^42)	$\{4^23^42\}+\{421^4\}+\{4^22^5\}+\{42^5\}+\{43^32^21\}+\{432^21^3\}+\{43^22^4\}+\{42^41^2\}+\{3^521\}+\{321\}+\{3^42^3\}+\{31^3\}+\{3^32^21\}+\{3^221^2\}+\{3^321^3\}+\{32^31\}+2\{3^22^31\}+2\{32^21^3\}+2\{32^51\}+2\{321^5\}+\{2^7\}+\{2\}+\{2^5\}+\{2^3\}+\{2^41^2\}+\{2^21^2\}+2\{2^31^4\}+2\{21^4\}$
(4^33^4)	$\{4^33^4\}+\{41^4\}+\{4^23^22^3\}+\{42^31^2\}+\{43^42^2\}+\{42^21^4\}+\{43^22^21^2\}^*+\{432^41\}^*+\{42^6\}^*+\{3^62\}+\{31\}+\{3^421^2\}+\{321^2\}+\{3^32^31\}+\{321^3\}+\{3^31^3\}+\{3^22^3\}+\{3^22^5\}+\{31^5\}+\{3^221^4\}+\{32^41\}+2\{32^31^3\}^*+\{2^51^2\}+\{21^2\}+\{2^31^2\}^*+\{2^21^4\}^*+\{21^6\}^*$
(4^43^22)	$\{4^43^22\}+\{421^2\}+\{4^23^321\}+\{4321^3\}+\{4^23^22^3\}+\{42^31^2\}+\{4^22^31^2\}+\{43^22^3\}+\{43^51\}+\{431^5\}+\{43^42^2\}+\{42^21^4\}+\{43^22^21^2\}^*+2\{432^41\}^*+\{3^62\}+\{31\}+\{3^51\}+\{3^22\}+\{3^421^2\}+\{32^21\}+2\{3^32^31\}+2\{321^3\}+\{3^31^3\}+\{3^22^3\}+2\{3^22^5\}+2\{31^5\}+\{3^22^21^2\}^*+2\{3^221^4\}+2\{32^41\}+2\{32^31^3\}^*+2\{2^51^2\}+2\{21^2\}+2\{2^31^2\}^*+\{2^21^4\}^*+\{21^6\}^*$
(4^52^2)	$\{4^52^2\}+\{42^2\}+\{4^23^321\}+\{4321^3\}+\{4^22^4\}^*+\{43^42^2\}+\{42^21^4\}+\{43^22^21^2\}^*+\{432^41\}^*+\{3^42^2\}+\{3^21^2\}+\{3^32^31\}+\{321^3\}+\{3^22^21^2\}^*+\{3^221^4\}+\{32^41\}+\{32^31^3\}^*+2\{2^61\}+2\{21\}+\{2^4\}^*+2\{2^21^4\}^*+\{1^4\}^*+\{0\}^*$
(4^531)	$\{4^531\}+\{431\}+\{4^23^4\}+\{4^21^4\}+\{4^23^321\}+\{4321^3\}+\{4^22^4\}^*+\{43^51\}+\{431^5\}+\{432^41\}^*+\{3^42^2\}+\{3^21^2\}+\{3^421^2\}+\{32^21\}+\{3^32^31\}+\{321^3\}+\{3^22^41\}+\{321^4\}+2\{3^22^21^2\}^*+\{2^6\}+\{2^2\}+\{2^51^2\}+\{21^2\}+\{2^31^2\}^*+2\{2^21^4\}^*+\{21^6\}^*+\{1^4\}^*$
(4^53^2)	$\{4^53^2\}+\{41^2\}+\{4^33^22^2\}+\{42^21^2\}+\{4^23^42\}+\{421^4\}+\{4^232^31\}+\{432^31\}+\{43^6\}+\{41^6\}+\{43^41^2\}+\{43^21^4\}+\{43^32^21\}+\{432^21^3\}+2\{43^22^4\}+2\{42^41^2\}+\{43^21^4\}+\{43^41^2\}+\{3^521\}+\{321\}+\{3^42^3\}+\{31^3\}+\{3^32^21\}+\{3^221^2\}+\{3^321^3\}+\{32^31\}+\{3^22^4\}+\{3^21^4\}+2\{3^22^31^2\}+2\{32^21^3\}+2\{32^51\}+2\{321^5\}+2\{2^41^2\}+2\{2^21^2\}+\{2^31^4\}+\{21^4\}+\{1^6\}+\{1^2\}$
(4^6)	$\{4^6\}+\{4^2\}+\{4^23^4\}+\{4^21^4\}+\{4^22^4\}^*+\{3^42^2\}+\{3^21^2\}+\{3^22^21^2\}^*+\{2^6\}+\{2^2\}+\{2^4\}^*+\{2^21^4\}^*+\{1^4\}^*+\{0\}^*$
(4^62)	$\{4^62\}+\{42\}+\{4^33^31\}+\{431^3\}+\{4^23^42\}+\{421^4\}+\{4^232^31\}+\{432^31\}+\{4^22^5\}+\{42^5\}+\{4^221^4\}+\{43^42\}+\{43^32^21\}+\{432^21^3\}+\{3^521\}+\{321\}+\{3^42^3\}+\{31^3\}+\{3^32^21\}+\{3^221^2\}+\{3^321^3\}+\{32^31\}+2\{3^22^31^2\}+2\{32^21^3\}+\{3^21^4\}+\{32^4\}+\{32^51\}+\{321^5\}+\{2^7\}+\{2\}+\{2^5\}+\{2^3\}+\{2^41^2\}+\{2^21^2\}+2\{2^31^4\}+2\{21^4\}$
(4^7)	$\{4^7\}+\{4\}+\{4^33^4\}+\{41^4\}+\{4^32^4\}+\{42^4\}+\{43^42^2\}+\{42^21^4\}+\{43^22^21^2\}^*+\{42^6\}^*+\{3^32^31\}+\{321^3\}+\{3^22^21^2\}^*+\{32^31^3\}^*+\{2^6\}+\{2^2\}+\{2^4\}^*+\{2^21^4\}^*+\{1^4\}^*+\{0\}^*$
	(*c*) $E_8 \to SU_9$ *branching rules*
(21^7)	$\{21^7\}^*+\{1^6\}+\{1^3\}$
(2^71)	$\{2^71\}+\{21\}+\{2^21^5\}^*+\{2^41^4\}+\{21^4\}+\{21^7\}^*$
(3^22^6)	$\{3^22^6\}+\{31^6\}+\{32^51^2\}+\{32^21^5\}+\{2^71\}+\{21\}+\{2^51^2\}+\{2^21^2\}+\{2^21^5\}^*+\{2^41^4\}+\{21^4\}+\{2^31^3\}^*+\{21^7\}^*+\{1^6\}+\{1^3\}+\{0\}$
(42^7)	$\{42^7\}^*+\{32^51^2\}+\{32^21^5\}+\{2^31^3\}^*+\{2^6\}+\{2^3\}+\{2^41^4\}+\{21^4\}+\{2^21^5\}^*+\{21^7\}^*+\{1^6\}+\{1^3\}+\{0\}$

Table 8. *E* group Kronecker products

Product	Evaluation
	(a) Kronecker products for E_6
$(1:1)\times(1:1)$	$\{(2:2)+(1^5:1)\}+[(1^2:2)]$
$(1:1)\times(1^5:1)$	$(21^4:2)^*+(0:2)^*+(0:0)^*$
$(0:2)^*\times(1:1)$	$(1:3)+(1^4:2)+(1:1)$
$(0:2)^*\times(0:2)^*$	$\{(0:4)^*+(21^4:2)^*+(0:0)^*\}+[(1^3:3)^*+(0:2)^*]$
$(1^2:2)\times(1:1)$	$(21:3)+(1^3:3)^*+(21^4:2)^*+(0:2)^*$
$(1^2:2)\times(1^5:1)$	$(2^21^3:3)+(1:3)+(1^4:2)+(1:1)$
$(1^2:2)\times(0:2)^*$	$(1^2:4)+(21^3:3)+(1^5:3)+(2:2)+(1^2:2)+(1^5:1)$
$(1^2:2)\times(1^2:2)$	$\{(2^2:4)+(1^4:4)+(31^4:3)+(1:3)+(2^5:2)+(1:1)\}$
	$+[(21^2:4)+(2^21^3:3)+(1:3)+(1^4:2)]$
$(1^2:2)\times(1^4:2)$	$(2^21^2:4)^*+(21^4:4)^*+(0:4)^*+(2^41:3)+(21:3)+(1^3:3)^*$
	$+2(21^4:2)^*+(0:2)^*+(0:0)^*$
$(2:2)\times(1:1)$	$(3:3)+(21:3)+(21^4:2)^*$
$(2:2)\times(1^5:1)$	$(31^4:3)+(1:3)+(1^5:1)$
$(2:2)\times(0:2)^*$	$(2:4)+(21^3:3)+(2:2)+(1^2:2)$
$(2:2)\times(1^2:2)$	$(31:4)+(21^2:4)+(31^4:3)+(1:3)+(2^21^3:3)+(1^4:2)$
$(2:2)\times(1^4:2)$	$(31^3:4)+(21^4:4)^*+(21:3)+(1^3:3)^*+(21^4:2)^*+(0:2)^*$
$(2:2)\times(2:2)$	$\{(4:4)+(2^2:4)+(31^4:3)+(2^5:2)\}+[(31:4)+(2^21^3:3)]$
$(2:2)\times(2^5:2)$	$(42^4:4)^*+(21^4:4)^*+(0:4)^*+(21^4:2)^*+(0:2)^*+(0:0)^*$
$(21^4:2)^*\times(1:1)$	$(31^4:3)+(2^21^3:3)+(1:3)+(2^5:2)+(1^4:2)+(1:1)$
$(21^4:2)^*\times(21^4:2)^*$	$\{(42^4:4)^*+(2^21^2:4)^*+(21^4:4)^*+(0:4)^*+(3^5:3)+(3:3)+(2^41:3)$
	$+(21:3)+2(21^4:2)^*+(0:2)^*+(0:0)^*\}$
	$+[(3^22^3:4)+(31^3:4)+(21^4:4)^*+(2^41:3)+(21:3)+2(1^3:3)^*$
	$+(21^4:2)^*+(0:2)^*]$
$(1:3)\times(1:1)$	$(2:4)+(1^2:4)+(21^3:3)+(1^5:3)+(2:2)+(1^2:2)$
$(1:3)\times(1^5:1)$	$(21^4:4)^*+(0:4)^*+(21:3)+(1^3:3)^*+(21^4:2)^*+(0:2)^*$
$(1^3:3)^*\times(1:1)$	$(21^2:4)+(1^4:4)+(2^21^3:3)+(1:3)+(1^4:2)$
$(21:3)\times(1:1)$	$(31:4)+(2^2:4)+(21^2:4)+(31^4:3)+(2^21^3:3)+(1:3)$
$(21:3)\times(1^5:1)$	$(321^3:4)+(2:4)+(1^2:4)+(21^3:3)+(2:2)+(1^2:2)$
$(3:3)\times(1:1)$	$(4:4)+(31:4)+(31^4:3)$
$(3:3)\times(1^5:1)$	$(41^4:4)+(2:4)+(2:2)$
	(b) Kronecker products for E_7
$(1^6)\times(1^6)$	$\{(2^6)+(21^6)\}+[(2^51^2)+(0)]$
$(21^6)\times(1^6)$	$(32^51)+(2^7)+(1^6)$
$(21^6)\times(21^6)$	$\{(42^6)+(2^51^2)+(0)\}+[(3^22^5)+(21^6)]$
$(2^51^2)\times(1^6)$	$(3^521)+(3^42^3)+(32^51)+(2^7)+(1^6)$
$(2^51^2)\times(21^6)$	$(43^42^2)+(3^62)+(3^22^5)+(2^6)+(2^51^2)+(21^6)$
$(2^51^2)\times(2^51^2)$	$\{(4^52^2)+(4^33^4)+(43^51)+(42^6)+(3^62)+2(2^51^2)+(0)\}$
	$+[(4^43^22)+(43^42^2)+(3^62)+(3^22^5)+(2^6)+(21^6)]$
$(2^6)\times(1^6)$	$(3^6)+(3^521)+(32^51)+(1^6)$
$(2^6)\times(21^6)$	$(43^51)+(3^62)+(2^51^2)+(2^6)$
$(2^6)\times(2^51^2)$	$(4^531)+(4^43^22)+(43^51)+(43^42^2)+(3^62)+(3^22^5)+(2^6)+(2^51^2)+(21^6)$
$(2^6)\times(2^6)$	$\{(4^6)+(4^52^2)+(43^51)+(42^6)+(2^51^2)+(0)\}$
	$+[(4^531)+(43^42^2)+(2^6)+(21^6)]$
$(2^7)\times(1^6)$	$(3^62)+(3^22^5)+(2^51^2)+(21^6)$
$(2^7)\times(21^6)$	$(43^6)+(3^42^3)+(32^51)+(2^7)+(1^6)$
$(2^7)\times(2^51^2)$	$(4^53^2)+(4^23^42)+(43^6)+(3^521)+(3^42^3)+2(32^51)+(2^7)+(1^6)$
$(2^7)\times(2^6)$	$(4^62)+(4^23^42)+(3^521)+(3^42^3)+(32^51)+(2^7)$
$(2^7)\times(2^7)$	$\{(4^7)+(43^42^2)+(3^22^5)+(2^6)+(21^6)\}+[(4^33^4)+(42^6)+(3^62)+(2^51^2)+(0)]$
$(32^51)\times(1^6)$	$(43^51)+(43^42^2)+(42^6)+(3^62)+(3^22^5)+(2^6)+(2^51^2)+(21^6)$
$(3^22^5)\times(1^6)$	$(4^23^42)+(43^6)+(3^42^3)+(32^51)+(2^7)$
$(3^42^3)\times(1^6)$	$(4^43^22)+(4^33^4)+(43^42^2)+(3^62)+(3^22^5)+(2^51^2)$

Table 8 (*Continued*)

Product	Evaluation
$(3^5 21)\times(1^6)$	$(4^5 31)+(4^5 2^2)+(4^4 3^2 2)+(43^5 1)+(43^4 2^2)+(3^6 2)+(2^5 1^2)$
$(3^6)\times(1^6)$	$(4^6)+(4^5 31)+(43^5 1)+(2^6)$
$(3^6 2)\times(1^6)$	$(4^6 2)+(4^5 3^2)+(4^2 3^4 2)+(43^6)+(3^5 21)+(32^5 1)+(2^7)$
	(*c*) *Kronecker products for* E_8
$(21^7)\times(21^7)$	$\{(42^7)+(2^7 1)+(0)\}+[(3^2 2^6)+(21^7)]$
$(2^7 1)\times(21^7)$	$(43^6 2)+(3^8)+(3^2 2^6)+(2^7 1)+(21^7)$
$(2^7 1)\times(2^7 1)$	$\{(4^7 2)+(4^3 3^5)+(42^7)+(3^8)+(2^7 1)+(0)\}+[(4^6 3^2)+(43^6 2)+(3^2 2^6)+(21^7)]$
$(3^2 2^6)\times(21^7)$	$(543^6)+(4^3 3^5)+(43^6 2)+(42^7)+(3^8)+(3^2 2^6)+(2^7 1)+(21^7)$
$(42^7)\times(21^7)$	$(63^7)+(543^6)+(43^6 2)+(42^7)+(3^2 2^6)+(21^7)$

14. Symmetrized Kronecker Powers for E_6, E_7 and E_8

A knowledge of the resolution of the symmetrized Kronecker powers of the irreps of a group is essential to the correct determination of the phase properties of the various *nj* and *njm* symbols of the group and its subgroups (Butler 1975; Butler and Wybourne 1976). These resolutions can also play an important role in the analysis of the invariants of groups.

The plethysm of S-functions supplies a natural tool for resolving the Kronecker powers of an irrep of a group into their different symmetry terms. The terms arising in the nth Kronecker power of an irrep (λ) of a group G are just the terms arising in the plethysm (Wybourne 1970)

$$(\lambda)\otimes\{\mu\} = \sum k_{\lambda'}(\lambda'), \tag{59}$$

where $k_{\lambda'}$ is the multiplicity of the irrep (λ') occurring in the resolution and $\{\mu\}$ is a partition of n appropriate to the symmetry terms being considered. The evaluation of equation (59) proceeds by first expressing (λ) as a series of S-functions and then evaluating the S-function plethysms (Wybourne 1970; Butler and Wybourne 1971). The resulting S-functions are then re-expressed as the characters of irreps of G.

The plethysms arising in equation (59) are simplest for the cases where $\{\mu\}$ corresponds to the partitions $\{n\}$ or $\{1^n\}$. The other symmetry terms can usually be derived most simply from these by noting that

$$A\otimes(BC) = (A\otimes B)(A\otimes C). \tag{60}$$

Thus to evaluate $(1^6)\otimes\{21\}$ for E_7, we note that

$$\begin{aligned}(1^6)\otimes(\{2\}\{1\}) &= ((1^6)\otimes\{2\})(\{1^6\}\otimes\{1\})\\ &= [(2^6)+(21^6)](1^6)\\ &= (3^6)+(3^5 21)+2(32^5 1)+(2^7)+2(1^6).\end{aligned} \tag{61}$$

But

$$(1^6)\otimes(\{2\}\{1\}) = (1^6)\otimes\{3\}+(1^6)\otimes\{21\}. \tag{62}$$

Comparison of equation (62) with (61) and the result for $(1^6)\otimes\{3\}$ leads immediately to the result for $(1^6)\otimes\{21\}$. These results can be checked dimensionally by noting that the partitions $\{\mu\}$ to the right of the plethys symbol $\otimes$ equivalently label the irreps of U_{56}.

The symmetrized powers of the fundamental irreps for E_6 and E_7 were evaluated up to the fourth power and the results are given in Tables 9*a* and 9*b* respectively. In view of the special significance of E_7 we give the resolution of the second and third powers of the adjoint irreps (21^6) in Table 9*c*.

Table 9. Kronecker powers for E_6 and E_7

Plethysm	Evaluation
	(a) Symmetrized powers of fundamental irrep of E_6
$(1:1)\otimes\{2\}$	$(2:2)+(1^5:1)$
$(1:1)\otimes\{1^2\}$	$(1^2:2)$
$(1:1)\otimes\{3\}$	$(3:3)+(21^4:2)+(0:0)$
$(1:1)\otimes\{21\}$	$(21:3)+(21^4:2)+(0:2)$
$(1:1)\otimes\{1^3\}$	$(1^3:3)$
$(1:1)\otimes\{4\}$	$(4:4)+(31^4:3)+(2^5:2)+(1:1)$
$(1:1)\otimes\{31\}$	$(31:4)+(31^4:3)+(2^21^3:3)+(1:3)+(1^4:2)+(1:1)$
$(1:1)\otimes\{2^2\}$	$(2^2:4)+(31^4:3)+(1:3)+(2^5:2)+(1:1)$
$(1:1)\otimes\{21^2\}$	$(21^2:4)+(2^21^3:3)+(1:3)+(1^4:2)$
$(1:1)\otimes\{1^4\}$	$(1^4:4)$
	(b) Symmetrized powers of fundamental irrep of E_7
$(1^6)\otimes\{2\}$	$(2^6)+(21^6)$
$(1^6)\otimes\{1^2\}$	$(2^51^2)+(0)$
$(1^6)\otimes\{3\}$	$(3^6)+(32^51)+(1^6)$
$(1^6)\otimes\{21\}$	$(3^521)+(32^51)+(2^7)+(1^6)$
$(1^6)\otimes\{1^3\}$	$(3^42^3)+(1^6)$
$(1^6)\otimes\{4\}$	$(4^6)+(43^51)+(42^6)+(2^51^2)+(2^6)+(0)$
$(1^6)\otimes\{31\}$	$(4^531)+(43^51)+(43^42^2)+(3^62)+(3^22^5)+2(2^6)+(2^51^2)+2(21^6)$
$(1^6)\otimes\{2^2\}$	$(4^52^2)+(43^51)+(42^6)+(3^62)+2(2^51^2)+(0)$
$(1^6)\otimes\{21^2\}$	$(4^43^22)+(43^42^2)+(3^62)+(3^22^5)+(2^6)+(2^51^2)+(21^6)$
$(1^6)\otimes\{1^4\}$	$(4^33^4)+(2^51^2)+(0)$
	(c) Kronecker powers of adjoint irrep of E_7
$(21^6)\otimes\{2\}$	$(42^6)+(2^51^2)+(0)$
$(21^6)\otimes\{1^2\}$	$(3^22^5)+(21^6)$
$(21^6)\otimes\{3\}$	$(63^6)+(43^42^2)+(3^22^5)+(2^6)+(21^6)$
$(21^6)\otimes\{21\}$	$(543^5)+(43^42^2)+(4^62)+(3^62)+(3^22^5)+(2^51^2)+2(21^6)$
$(21^6)\otimes\{1^3\}$	$(4^33^4)+(42^6)+(3^22^5)+(2^51^2)+(0)$

15. A Branching Rule Theorem

So far our discussion of branching rules has been limited to the largest maximal subgroups of the exceptional groups and we have used elementary multiplets and the building-up principle to construct them. To proceed to other subgroup structures we use a theorem given elsewhere (Butler and Wybourne 1969; Wybourne 1970) which may be stated as: *If, under the restriction $G \to H$, the character of the vector irrep of G, say $\{1\}$, decomposes as*

$$\{1\} \to \{\alpha\}+\{\beta\}+\ldots+\{\omega\}$$

then the character $[\lambda]$ of G decomposes into the characters $\{\rho\}$ of H according to the *characters of H contained in the plethysm*

$$[\{\alpha\}+\{\beta\}+\ldots+\{\omega\}]\otimes\{\lambda\}. \tag{63}$$

Thus in the case of $SO_{26}\to F_4$ we have

$$[1]\to(1000),$$

while for $F_4\to SO_9$ we have

$$(1000)\to[0000]+[1000]+[1111]',$$

and hence under $SO_{26}\to SO_9$ we have

$$[1]\to[0000]+[1000]+[1111]'.$$

Thus by (63) the $[\lambda]$ irrep of SO_{26} decomposes into the irreps of SO_9 contained in the plethysm

$$([0000]+[1000]+[1111]')\otimes[\lambda].$$

These plethysms may be readily evaluated using the properties of S-functions and existing tabulations of S-function plethysms (Butler and Wybourne 1971). If the resulting series of SO_9 characters are expressed in terms of F_4 characters using the result (47) and Table 4, we obtain the $SO_{26}\to F_4$ branching rules given in Table 5*b*.

Table 10. $F_4\to SU_3\times SU_3$ branching rules

$(\mu_1\,\mu_2\,\mu_3\,\mu_4)$	Branching to $SU_3\times SU_3$
(0000)	$\{0\}\{0\}$
(1000)	$\{1^2\}\{1^2\}+\{1\}\{1\}+\{21\}\{0\}$
(1100)	$\{0\}\{21\}+\{21\}\{0\}+\{2^2\}\{1\}+\{2\}\{1^2\}$
(3111)′	$\{0\}\{0\}+\{1^2\}\{1^2\}+\{1\}\{1\}+\{21\}\{0\}+\{21\}\{21\}+\{1^2\}\{2\}$ $+\{1\}\{2^2\}+\{2\}\{1^2\}+\{2^2\}\{1\}+\{32\}\{1^2\}+\{31\}\{1\}$ $+\{3^2\}\{0\}+\{3\}\{0\}$
(2000)	$\{0\}\{0\}+\{1^2\}\{1^2\}+\{1\}\{1\}+\{0\}\{21\}+\{21\}\{0\}+\{21\}\{21\}$ $+\{2^2\}\{1\}+\{2\}\{1^2\}+\{2^2\}\{2^2\}+\{2\}\{2\}+\{32\}\{1^2\}$ $+\{31\}\{1\}+\{42\}\{0\}$

16. Further Branching Rules

Once a table of Kronecker products is available there is little difficulty in building up branching rules of any group structure involving the exceptional groups. We start with the smallest nontrivial irrep and then build up to higher dimensional irreps. Here we have restricted our attention to group structures of significance in theories of elementary particles that involve the exceptional groups, though other group structures present no special difficulties.

Results for $F_4\to SU_3\times SU_3$ are given in Table 10 while those for $E_6\to F_4$, $E_6\to SU_3\times SU_3\times SU_3$, $E_7\to E_6$, $E_7\to SU_6\times SU_3$, $E_8\to SU_2\times E_7$, $E_8\to SU_3\times E_6$ appear in Tables 11*a*–11*f* respectively. There is no difficulty in extending these tables as required.

Table 11. Further E_n group branching rules

(λ)	Branching
	(a) $E_6 \rightarrow F_4$ branching rules
$(0:0)$	(0000)
$(1:1)$	$(1000)+(0000)$
$(0:2)$	$(1100)+(1000)$
$(1^2:2)$	$(3111)'+(1100)+(1000)$
$(2:2)$	$(2000)+(1000)+(0000)$
$(21^4:2)$	$(2000)+(3111)'+2(1000)+(0000)$
$(1:3)$	$(2100)+(2000)+(3111)'+(1100)+(1000)$
$(1^3:3)$	$(2110)+(2100)+2(3111)'+(1100)$
$(21:3)$	$(5111)'+(2100)+(2000)+(3111)'+(1100)+(1000)$
$(21^3:3)$	$(5111)'+(2110)+(2100)+(2000)+2(3111)'+(1100)+(1000)$
$(3:3)$	$(3000)+(2000)+(1000)+(0000)$
$(31^4:3)$	$(3000)+(5111)'+2(2000)+(3111)'+2(1000)+(0000)$
$(0:4)$	$(2200)+(2100)+(2000)$
	(b) $E_6 \rightarrow SU_3 \times SU_3 \times SU_3$ branching rules
$(0:0)$	$\{0\}\{0\}\{0\}$
$(1:1)$	$\{1^2\}\{0\}\{1\}+\{1\}\{1\}\{0\}+\{0\}\{1^2\}\{1^2\}$
$(0:2)^*$	$\{21\}\{0\}\{0\}+\{0\}\{21\}\{0\}+\{0\}\{0\}\{21\}+\{1^2\}\{1\}\{1^2\}+\{1\}\{1^2\}\{1\}$
$(1^2:2)$	$\{2^2\}\{0\}\{1^2\}+\{21\}\{1\}\{1\}+\{2\}\{1^2\}\{0\}+\{1^2\}\{1^2\}\{21\}+\{1^2\}\{2\}\{0\}+\{1^2\}\{1^2\}\{0\}$ $+\{1\}\{21\}\{1^2\}+\{1\}\{0\}\{2\}+\{1\}\{0\}\{1^2\}+\{0\}\{2^2\}\{1\}+\{0\}\{1\}\{2^2\}$ $+\{0\}\{1\}\{1\}$
$(2:2)$	$\{2^2\}\{0\}\{2\}+\{21\}\{1\}\{1\}+\{2\}\{2\}\{0\}+\{1^2\}\{1^2\}\{21\}+\{1^2\}\{1^2\}\{0\}+\{1\}\{21\}\{1^2\}$ $+\{1\}\{0\}\{1^2\}+\{0\}\{2^2\}\{2^2\}+\{0\}\{1\}\{1\}$
$(21^4:2)^*$	$\{2^2\}\{1^2\}\{1\}+\{2\}\{1\}\{1^2\}+\{21\}\{0\}\{21\}+\{21\}\{21\}\{0\}+\{0\}\{21\}\{21\}+\{21\}\{0\}\{0\}$ $+\{0\}\{21\}\{0\}+\{0\}\{0\}\{21\}+\{1^2\}\{2^2\}\{1^2\}+\{1\}\{2\}\{1\}+\{1^2\}\{1\}\{2\}$ $+\{1\}\{1^2\}\{2^2\}+2\{1^2\}\{1\}\{1^2\}+2\{1\}\{1^2\}\{1\}+2\{0\}\{0\}\{0\}$
$(1:3)$	$\{32\}\{0\}\{1\}+\{0\}\{32\}\{1^2\}+\{0\}\{1^2\}\{32\}+\{31\}\{1\}\{0\}+\{1\}\{31\}\{0\}+\{1^2\}\{0\}\{31\}$ $+\{2^2\}\{1\}\{0\}+\{1\}\{2^2\}\{0\}+\{1^2\}\{0\}\{2^2\}+\{2^2\}\{1\}\{21\}+\{1\}\{2^2\}\{21\}$ $+\{1^2\}\{21\}\{2^2\}+\{21\}\{1^2\}\{2\}+\{21\}\{2\}\{1^2\}+\{2\}\{21\}\{1\}+2\{21\}\{1^2\}\{1^2\}$ $+2\{1^2\}\{21\}\{1\}+2\{1\}\{1\}\{21\}+\{2\}\{0\}\{1\}+\{0\}\{2\}\{1^2\}+\{0\}\{1^2\}\{2\}$ $+2\{1^2\}\{0\}\{1\}+2\{0\}\{1^2\}\{1^2\}+2\{1\}\{1\}\{0\}$
$(1^3:3)^*$	$\{3^2\}\{0\}\{0\}+\{3\}\{0\}\{0\}+\{0\}\{3^2\}\{0\}+\{0\}\{3\}\{0\}+\{0\}\{0\}\{3^2\}+\{0\}\{0\}\{3\}$ $+\{32\}\{1\}\{1^2\}+\{31\}\{1^2\}\{1\}+\{1\}\{32\}\{1\}+\{1^2\}\{31\}\{1^2\}+\{1^2\}\{1\}\{32\}$ $+\{1\}\{1^2\}\{31\}+\{2^2\}\{1^2\}\{2^2\}+\{2\}\{1\}\{1\}+\{1^2\}\{2^2\}\{2\}+\{1\}\{2\}\{2^2\}$ $+\{2\}\{2^2\}\{1^2\}+\{2^2\}\{2\}\{1\}+\{1\}\{1^2\}\{2^2\}+\{1^2\}\{1\}\{2\}+\{1^2\}\{2^2\}\{1^2\}$ $+\{1\}\{2\}\{1\}+\{2^2\}\{1^2\}\{1\}+\{2\}\{1\}\{1^2\}+\{21\}\{21\}\{21\}+2\{21\}\{21\}\{0\}$ $+2\{21\}\{0\}\{21\}+2\{0\}\{21\}\{21\}+\{21\}\{0\}\{0\}+\{0\}\{21\}\{0\}+\{0\}\{0\}\{21\}$ $+3\{1^2\}\{1\}\{1^2\}+3\{1\}\{1^2\}\{1\}+\{0\}\{0\}\{0\}$
	(c) $E_7 \rightarrow E_6$ branching rules
(0)	$(0:0)$
(1^6)	$(1:1)+(1^5:1)+2(0:0)$
(21^6)	$(0:2)+(1:1)+(1^5:1)$
$(2^5 1^2)$	$(21^4:2)+(1^2:2)+(1:2)+(0:2)+2(1:1)+2(1^5:1)+(0:0)$
(2^6)	$(2^5:2)+(2:2)+(21^4:2)+2(1:1)+2(1^5:1)+3(0:0)$
(2^7)	$(1^2:2)+(1^4:2)+2(0:2)+(1:1)+(1^5:1)$

Table 11 (*Continued*)

(λ)	Branching
	(*d*) $E_7 \to SU_6 \times SU_3$ *branching rules*
(0)	$\{0\}\{0\}'^*$
(1^6)	$\{1\}\{1\}'+\{1^5\}\{1^2\}'+\{1^3\}\{0\}'^*$
(21^6)	$\{1^2\}\{1^2\}'+\{1^4\}\{1\}'+\{0\}\{21\}'^*+\{21^4\}\{0\}'^*$
(2^51^2)	$\{0\}\{0\}^*+\{1^2\}\{1^2\}'+\{1^4\}\{1\}'+\{1^2\}\{2\}'+\{1^4\}\{2^2\}'+\{2\}\{1^2\}'+\{2^5\}\{1\}'$ $+\{0\}\{21\}'^*+\{21^2\}\{1\}'+\{2^31^2\}\{1^2\}'+\{21^4\}\{0\}'^*+\{21^4\}\{21\}'^*$ $+\{2^21^2\}\{0\}'^*$
(2^6)	$\{0\}\{0\}'^*+\{1^2\}\{1^2\}'+\{1^4\}\{1\}'+\{2\}\{2\}'+\{2^5\}\{2^2\}'+\{2^3\}\{0\}'^*+\{21^4\}\{0\}'^*$ $+\{21^4\}\{21\}'^*+\{21^2\}\{1\}'+\{2^31^2\}\{1^2\}'$
(2^7)	$\{1\}\{1\}'+\{1^5\}\{1^2\}'+\{1^5\}\{2\}'+\{1\}\{2^2\}'+\{21\}\{0\}'+\{2^41\}\{0\}'+\{1^3\}\{21\}'^*$ $+\{21^3\}\{1^2\}'+\{2^21^3\}\{1\}'$
	(*e*) $E_8 \to SU_2 \times E_7$ *branching rules*
(0)	$\{0\}(0)$
(21^7)	$\{2\}(0)+\{1\}(1^6)+\{0\}(21^6)$
(2^71)	$\{2\}(21^6)+\{1\}[(2^7)+(1^6)]+\{0\}[(2^51^2)+(0)]$
(3^22^6)	$\{3\}(1^6)+\{2\}[(2^51^2)+(21^6)+(0)]+\{1\}[(32^51)+(2^7)+(1^6)]$ $+\{0\}[(3^22^5)+(21^6)+(2^6)]$
(42^7)	$\{4\}(0)+\{3\}(1^6)+\{2\}[(2^6)+(21^6)]+\{1\}[(32^51)+(1^6)]$ $+\{0\}[(42^6)+(2^51^2)+(0)]$
	(*f*) $E_8 \to SU_3 \times E_6$ *branching rules*
(0)	$\{0\}(0:0)$
(21^7)	$\{21\}(0:0)+\{1^2\}(1:1)+\{1\}(1^5:1)+\{0\}(0:2)$
(2^71)	$\{2^2\}(1^5:1)+\{2\}(1:1)+\{21\}[(0:2)+(0:0)]+\{1^2\}[(1^4:2)+(1:1)]$ $+\{1\}[(1^2:2)+(1^5:1)]+\{0\}[(21^4:2)+(0:0)]$
(3^22^6)	$\{3^2\}(0:0)+\{3\}(0:0)+\{32\}(1:1)+\{31\}(1^5:1)+\{2^2\}[(1^2:2)+(1^5:1)]$ $+\{2\}[(1^4:2)+(1:1)]+\{21\}[(21^4:2)+2(0:2)+(0:0)]$ $+\{1^2\}[(1:3)+(2^5:2)+(1^4:2)+2(1:1)]+\{1\}[(1^5:3)+(2:2)+(1^2:2)+2(1^5:1)]$ $+\{0\}[(1^3:3)+(21^4:2)+(0:2)+(0:0)]$
(42^7)	$\{42\}(0:0)+\{32\}(1:1)+\{31\}(1^5:1)+\{2^2\}[(2:2)+(1^5:1)]$ $+\{2\}[(2^5:2)+(1:1)]+\{21\}[(21^4:2)+(0:2)+(0:0)]$ $+\{1^2\}[(1:3)+(1^4:2)+(1:1)]+\{1\}[(1^5:3)+(1^2:2)+(1^5:1)]$ $+\{0\}[(0:4)+(21^4:2)+(0:2)+(0:0)]$

17. Concluding Remarks

We have established simple labelling schemes for the irreps of the exceptional groups and then shown that it is possible to enumerate many properties using relatively simple techniques and a minimum of computation. The work reported here is a prerequisite for the calculation of the *nj* and *njm* symbols of the exceptional groups and their subgroup structures—a necessary step in making detailed applications. We note that there is basically little difficulty in extending the tabulations as required. Much of the material presented has been remarkably simple to obtain, which we believe demonstrates the superiority of methods based on the S-function calculus.

With the basic properties of the exceptional groups enumerated it should now be possible to start to examine the relevance of these groups in particle physics in a quantitative manner.

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