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Generalized Ohm's Laws for Relativistically Streaming Plasmas

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Abstract

A theoretical framework is given for treating energy dissipation and the associated diffusion of magnetic field lines resulting from 'frictional' interactions between different species in plasmas in which the species have relativistic streaming velocities. The plasmas are not necessarily neutral. One result is a form for generalized Ohm's laws. Details are given for binary plasmas, with particular attention paid to the analysis of inertial effects. Some remarks are made on the relevance of the results to pulsar magnetospheres.

1. Introduction

Burman *et al.* (1976) introduced a technique for treating energy dissipation and the associated diffusion of magnetic field lines resulting from 'frictional' interactions between different species in nonrelativisitic plasmas. They made no restriction on the number or kinds of species present, and they left the nature of the 'frictional' interactions unspecified: no particular choice of expression was made for the force on each species which arises from its 'frictional' interactions with the other species. These forces could represent collisions or less-well-understood phenomena giving rise to anomalous resistivity. The foundations of the technique are the expansions of the relative momentum densities and the 'frictional' forces as linear combinations of three nonorthogonal vector fields. The basic idea is to choose those vector fields (so far as is possible) to be generalized currents relating naturally to different types of diffusion processes that occur in the plasma, such as electric current flow and ambipolar diffusion. One is likely to be more interested in such quantities than in the momentum densities of individual species. The technique is capable of extension to general systems of vector fluxes and forces in the thermodynamics of irreversible processes (Burman 1977).

The formalism introduced by Burman *et al.* (1976) has two purposes. One is to elucidate the equations underlying the theory of dissipation. In particular, Burman *et al.* obtained a form for generalized Ohm's laws for nonrelativistic plasmas, setting out the mathematical conditions needed to reduce it to the familiar form. From this, an equation describing the dissipation or generation of magnetic flux follows. The other purpose is to obtain specific results for physical quantities, such as electrical conductivities of plasmas with several species—the formalism provides a systematic scheme for such calculations. In particular, for 'frictional' forces representing collisions described by the standard relaxation model, Byrne (1977) has used the formalism to obtain specific results for fully ionized ternary plasmas that are not necessarily neutral and for quaternary plasmas with two neutral species.

In many situations, plasma species have relativistic streaming velocities, so that it is of interest to have a similar formalism for describing dissipative effects in these circumstances. Such a formalism will serve the same two basic purposes mentioned above in the nonrelativistic case.

In Section 2 of the present paper, the formalism of Burman *et al.* (1976) is extended to apply to plasmas in which the species can have relativistic streaming velocities. The thermal speeds are restricted to be nonrelativistic, which is not a significant restriction for most purposes, since only for temperatures exceeding 10^{10} K are the electron thermal speeds highly relativistic. As with the nonrelativistic formalism, inertial terms are included throughout. These terms are likely to be particularly important for relativistically streaming plasmas; e.g. they will be essential in applications of the results to pulsar magnetospheres (Mestel 1973; Mestel *et al.* 1976). Relativistic plasmas like those constituting pulsar magnetospheres may well be nonneutral, and so the neutrality condition is *not* imposed in this paper.

In Section 3 below, the type of generalized Ohm's law adumbrated in Section 2 is obtained explicitly for the case of binary plasmas, and inertial effects are analysed in some detail. In Section 4, some remarks are made on the relevance of the results to pulsar magnetospheres.

2. Method

Consider a plasma consisting of N species of any kind. The rth component species has mass density ρ_r, proper mass density ρ_{0r} and fluid or streaming 3-velocity $\boldsymbol{v}_r$, with corresponding Lorentz factor γ_r. Also, κ_r denotes the ratio of charge to rest mass for particles of this species. The fluid velocities are unrestricted, but the thermal speeds are nonrelativistic, implying that $\rho_r = \gamma_r^2 \rho_{0r}$. Pressure gradients are neglected. The force density that acts on the rth species through its 'frictional' interactions with the other species is denoted by $\rho_r \boldsymbol{F}_r$. As usual, $\boldsymbol{E}$, $\boldsymbol{H}$, c and t denote the electric and magnetic fields, the free-space speed of light and the time coordinate. The equation of motion of the rth species is now given by

$$\rho_r \boldsymbol{A}_r = \kappa_r \rho_{0r} \gamma_r(\boldsymbol{E} + c^{-1} \boldsymbol{v}_r \times \boldsymbol{H}) + \rho_r \boldsymbol{F}_r, \tag{1}$$

where $\boldsymbol{A}_r$ is defined by (see e.g. Rindler 1966, Sections 57 and 58)

$$\rho_r \boldsymbol{A}_r \equiv \partial(\rho_r \boldsymbol{v}_r)/\partial t + \nabla . (\rho_r \boldsymbol{v}_r \boldsymbol{v}_r) \tag{2a}$$

$$= \rho_{0r} \gamma_r (\partial/\partial t + \boldsymbol{v}_r . \nabla)(\gamma_r \boldsymbol{v}_r). \tag{2b}$$

Equation (2b) follows from (2a) on using the continuity and energy equations of the rth species, neglecting internal energy generation by $\boldsymbol{F}_r$.

Summations will be over r from 1 to N unless otherwise indicated, and we may now define

$$\rho \equiv \sum \rho_r, \qquad \rho \boldsymbol{v} \equiv \sum \rho_r \boldsymbol{v}_r, \qquad \hat{p} \equiv \sum \rho_r (\boldsymbol{v}_r - \boldsymbol{v})(\boldsymbol{v}_r - \boldsymbol{v}), \tag{3a–3c}$$

$$\nu \equiv \sum \kappa_r \rho_{0r} \gamma_r, \qquad \boldsymbol{j} \equiv \sum \kappa_r \rho_{0r} \gamma_r \boldsymbol{v}_r. \tag{3d, e}$$

These relations define the mass density, barycentric velocity and pressure tensor of the plasma as a whole (equations 3a–3c), together with the net charge density and the electric current density (equations 3d and 3e).

Summing equation (2a) over all species, using (3a–3c) and noting that we have

$$\sum \rho_r \boldsymbol{v}_r \boldsymbol{v}_r = \rho \boldsymbol{v}\boldsymbol{v} + \hat{p},$$

gives

$$\sum \rho_r \boldsymbol{A}_r = \partial(\rho \boldsymbol{v})/\partial t + \nabla . (\rho \boldsymbol{v}\boldsymbol{v}) + \nabla . \hat{p}.$$

Hence, summing equation (1) over all species, using (3d) and (3e) and taking

$$\sum \rho_r \boldsymbol{F}_r = \boldsymbol{0},$$

gives the equation of motion of the plasma as a whole:

$$\partial(\rho \boldsymbol{v})/\partial t + \nabla . (\rho \boldsymbol{v}\boldsymbol{v}) + \nabla . \hat{p} = \nu \boldsymbol{E} + c^{-1} \boldsymbol{j} \times \boldsymbol{H}. \tag{4}$$

Taking the scalar product of $\rho_r \boldsymbol{A}_r$ (equation 1) with $\boldsymbol{v}_r$, summing the result over all species, using the definition (3e) of $\boldsymbol{j}$, defining $\boldsymbol{J}_r$ as $\rho_r(\boldsymbol{v}_r - \boldsymbol{v})$ and hence substituting for $\boldsymbol{v}_r$, and then using (4) leads to

$$\boldsymbol{j}.\boldsymbol{E} = \boldsymbol{v}.\{\partial(\rho \boldsymbol{v})/\partial t + \nabla . (\rho \boldsymbol{v}\boldsymbol{v})\} + \boldsymbol{v}.(\nabla . \hat{p}) + \sum \boldsymbol{J}_r.(\boldsymbol{A}_r - \boldsymbol{F}_r). \tag{5}$$

The last term, which represents processes associated with the relative velocities of the species, will be analysed here.

Since in ordinary space no more than three vectors can be linearly independent, the N relative momentum densities and the N 'frictional' forces can be expressed in terms of three basic nonorthogonal vector fields $\boldsymbol{m}_i$:

$$\boldsymbol{J}_r = \sum_{i=1}^{3} S_{ri} \boldsymbol{m}_i \quad \text{and} \quad \boldsymbol{F}_r = \sum_{i=1}^{3} A_{ri} \boldsymbol{m}_i. \tag{6a, b}$$

The idea is eventually to choose (so far as possible) the $\boldsymbol{m}_i$ as generalized currents relating naturally to different types of diffusion processes, such as electric current flow and ambipolar diffusion, that occur in the medium. By the definitions (3a) and (3b) of ρ and $\boldsymbol{v}$, the $\boldsymbol{J}_r$ sum to zero, so that

$$\sum S_{ri} = 0. \tag{7}$$

Using the equations (6), (5) can be written

$$\boldsymbol{j}.\boldsymbol{E} + \sum_{i=1}^{3} \boldsymbol{m}_i.\boldsymbol{X}_i = \boldsymbol{v}.\{\partial(\rho \boldsymbol{v})/\partial \mathrm{t} + \nabla . (\rho \boldsymbol{v}\boldsymbol{v})\} + \boldsymbol{v}.(\nabla . \hat{p}) + Q. \tag{8}$$

Here

$$\boldsymbol{X}_i \equiv -\sum S_{ri} \boldsymbol{A}_r \quad \text{and} \quad Q \equiv \sum_{i=1}^{3} \sum_{j=1}^{3} K_{(ij)} \boldsymbol{m}_i.\boldsymbol{m}_j, \tag{9a, b}$$

with

$$K_{ij} \equiv -\sum S_{ri} A_{rj}, \tag{9c}$$

the symmetric and skewsymmetric parts being denoted by $K_{(ij)}$ and $K_{[ij]}$ respectively.

Certain electrical properties of plasmas are often expressed by generalized Ohm's laws, which relate the conduction current density $\boldsymbol{i}$ to quantities of the form

$E + c^{-1} \boldsymbol{u}_H \times \boldsymbol{H}$, where $\boldsymbol{u}_H$ is some velocity field. Multiplying equation (1) through by S_{ri}/ρ_r, summing over all species and using the condition (7), results in

$$-\sum S_{ri} \boldsymbol{F}_r = \alpha_i \boldsymbol{E} + c^{-1} \left(\sum \kappa_r S_{ri} \boldsymbol{v}_r/\gamma_r\right) \times \boldsymbol{H} + \boldsymbol{X}_i, \tag{10}$$

where

$$\alpha_i \equiv \sum \kappa_r S_{ri}/\gamma_r. \tag{11}$$

Suppose that at least one of the α_i, say α_1, does not vanish, so that equation (10) for $i = 1$ can be written

$$-\sum S_{ri} \boldsymbol{F}_r/\alpha_1 = \boldsymbol{E} + c^{-1} \boldsymbol{u}_H \times \boldsymbol{H} + \boldsymbol{E}', \tag{12}$$

where

$$\boldsymbol{u}_H \equiv \sum \frac{\kappa_r S_{r1}}{\gamma_r \alpha_1} \boldsymbol{v}_r = \boldsymbol{v} + \sum_{i=1}^{3} \sum \frac{\kappa_r}{\gamma_r \rho_r} \frac{S_{r1}}{\alpha_1} S_{ri} \boldsymbol{m}_i \tag{13a, b}$$

and

$$\boldsymbol{E}' \equiv \boldsymbol{X}_1/\alpha_1. \tag{13c}$$

Equation (12) can be thought of as a generalized Ohm's law, with $\boldsymbol{E}'$ an equivalent electric field arising from the inertial or acceleration term in the equation of motion (1): the right-hand side of (12) has the usual form for a generalized Ohm's law, while the left-hand side is a linear combination of 'frictional' forces generalizing the usual expression $\boldsymbol{i}/\sigma$, where σ is the conductivity. The quantities

$$\boldsymbol{E}'.\boldsymbol{j} \quad \text{and} \quad \boldsymbol{i}.\left(-\sum S_{ri} \boldsymbol{F}_r/\alpha_1\right)$$

represent the rate at which $\boldsymbol{E}'$ does work in driving $\boldsymbol{j}$ and the rate of Joule heating respectively.

Taking the curl of equation (12) and using Faraday's law to eliminate $\boldsymbol{E}$ results in

$$\partial \boldsymbol{H}/\partial t - \nabla \times (\boldsymbol{u}_H \times \boldsymbol{H}) = c \nabla \times \left(\boldsymbol{E}' + \sum S_{r1} \boldsymbol{F}_r/\alpha_1\right). \tag{14}$$

If the right-hand side of equation (14) happens to be negligible, then magnetic flux through any surface moving everywhere with velocity $\boldsymbol{u}_H$ is conserved, so that $\boldsymbol{u}_H$ can conveniently be called the velocity of the magnetic field lines. In general, the right-hand side of equation (14) will not be negligible, but will describe the dissipation or generation of magnetic flux.

The three nonorthogonal vector fields $\boldsymbol{m}_i$ are arbitrary, apart from the requirement of linear independence, but their choice is now restricted in such a way that the 'frictional work' term Q in equation (8) takes the form of a sum of squares, i.e. the $\mathbf{m}_i$ are chosen so that $K_{(ij)}$ is diagonal:

$$K_{(ij)} = \theta_i^{-1} \delta_{ij}. \tag{15}$$

The $\boldsymbol{m}_i$ represent nine quantities, of which three are mere normalization functions and three are required in order that equation (15) be satisfied. Thus, three quantities remain free, and an abitrary choice can be made for $\boldsymbol{m}_1$, say, together with the angle between $\boldsymbol{m}_2$ and $\boldsymbol{m}_1$.

A case in which $K_{[ij]}$ is known to vanish (Byrne 1977) is that in which the $\boldsymbol{F}_r$ are linear sums of the velocity differences between species, with the reciprocity relations (Delcroix 1965, p. 258) being valid. If $K_{[ij]}$ vanishes then equations (6b), (9c) and

(15) show that $-\sum S_{ri} \boldsymbol{F}_r$ reduces to $\boldsymbol{m}_i/\theta_i$, and so the right-hand side of equation (10) becomes an expression for $\boldsymbol{m}_i/\theta_i$:

$$\boldsymbol{m}_i/\theta_i = \alpha_i \boldsymbol{E} + c^{-1}\left(\sum \kappa_r S_{ri} \boldsymbol{v}_r/\gamma_r\right) \times \boldsymbol{H} + \boldsymbol{X}_i . \tag{16}$$

The definitions of $\boldsymbol{j}$, $\boldsymbol{J}_r$ and ν together with the expansion (6a) for $\boldsymbol{J}_r$ and the definition (11) of α_i show that

$$\boldsymbol{j} - \nu \boldsymbol{v} = \sum_{i=1}^{3} \alpha_i \boldsymbol{m}_i . \tag{17}$$

If we consider the particular case of neutral plasmas, equation (17) shows that, because of the linear independence of the $\boldsymbol{m}_i$, choice of $\boldsymbol{j}/\alpha_1$ for $\boldsymbol{m}_1$ implies that α_2 and α_3 must vanish. Furthermore, if we have $K_{[1j]} = 0$ then the left-hand side of the generalized Ohm's law (12), the rate of Joule heating, and the right-hand side of the flux theorem (14) reduce to familiar forms. The first two quantities become $\boldsymbol{j}/\sigma$ and $\boldsymbol{j}^2/\sigma$, with $\sigma \equiv \alpha_1^2 \theta_1$. Hence σ can be interpreted as the conductivity relevant to Joule heating. The right-hand side of equation (14) becomes $c\nabla \times (\boldsymbol{E}' - \boldsymbol{j}/\sigma)$ which, if $\boldsymbol{E}'$ and the displacement current happen to be negligible, reduces to $-\nabla \times \{(c^2/4\pi\sigma)\nabla \times \boldsymbol{H}\}$. Hence σ is the conductivity relevant to the decay of magnetic flux through any surface that moves at each of its points with velocity $\boldsymbol{u}_H$.

For nonneutral plasmas, the total current density $\boldsymbol{j}$ can be separated into the sum of convection and conduction current densities $\boldsymbol{c}$ and $\boldsymbol{i}$, but this separation is not unique. For example, the separation may be chosen so that Ohm's law can be written in an explicitly covariant form (see e.g. Costa de Beauregard 1966, Section 3.7; Schwartz 1968, Section 7.3)—the convection and conduction 4-current densities are then time-like and space-like respectively. In this case we have $\boldsymbol{c} = \gamma \nu^* \boldsymbol{v}$, where ν^* is the net charge density in the local rest frame of the medium and γ is the Lorentz factor corresponding to $\boldsymbol{v}$. Such a choice has the disadvantage that $\boldsymbol{c}$ and ν do not vanish together (Schwartz 1968, Section 7.3). A more natural choice is to take $\nu\boldsymbol{v}$ for $\boldsymbol{c}$, corresponding to $\boldsymbol{j} = \boldsymbol{i} + \nu\boldsymbol{v}$ and this choice is made here.

3. Binary Plasmas

In the context of pulsar magnetosphere theory, Ardavan (1976) obtained a generalized Ohm's law for relativistic nonneutral binary plasmas, with inertial effects included. He restricted the plasmas to those in which the particles are nonrelativistic in the proper frame of the plasma. Thus, as well as having nonrelativistic thermal speeds, the relative streaming speed of the two species also has to be nonrelativistic. The latter requirement is a serious limitation on the range of applicability of Ardavan's results. In particular, it is not clear that this requirement will be satisfied in pulsar magnetospheres, although perhaps 'frictional' mechanisms may limit the relative streaming speed. On the other hand, the form for generalized Ohm's laws obtained as a result of the formalism developed in Section 2 (above) is not restricted to binary plasmas, nor are the relative streaming velocities of the species restricted to be nonrelativistic.

Ardavan (1976) used his generalized Ohm's law to investigate the important problem of the range of applicability of the magnetohydrodynamic approximation in relativistically streaming plasmas. It is sometimes thought that the approximation requires the inertial forces to be negligible in comparison with the electromagnetic forces, but Ardavan found otherwise. When the Lorentz factor γ corresponding to $\boldsymbol{v}$ is much greater than unity, he found that, regardless of the relative magnitude of the inertial and electromagnetic forces acting on the bulk plasma,

$$\boldsymbol{E} + c^{-1}\boldsymbol{v} \times \boldsymbol{H} \approx \mathbf{0} \qquad \text{for} \qquad (c/\omega_e^* L)^2 \ll 1\,,$$

where ω_e^* is the electron angular plasma frequency in the local proper frame of the plasma and L is the scale length over which the macroscopic plasma properties vary appreciably. We now proceed to remove Ardavan's restriction, that the two species have a relative streaming velocity that is nonrelativistic, by obtaining a generalized Ohm's law for binary plasmas using the formalism of the last section and then performing an analysis of inertial effects similar to Ardavan's.

Note that, if n_r and n_{0r} denote the number density and proper number density of particles of the rth species, Ardavan's (1976) symbol ρ denotes (in the present notation) $\Sigma\, m_r n_r$, that is, $\Sigma\, \rho_r/\gamma_r$, reducing to ρ/γ on putting $\gamma_r \approx \gamma$.

With but two species present, only the first of the $\boldsymbol{m}_i$ is needed in equation (6a), and we choose the vector $\boldsymbol{i}/\alpha_1$ for $\boldsymbol{m}_1$. Since we have $\boldsymbol{J}_1 + \boldsymbol{J}_2 = \mathbf{0}$, it follows that $S_{11} + S_{21} = 0$. Also, from equation (11) we have

$$\alpha_1 \equiv \kappa_1 S_{11}/\gamma_1 + \kappa_2 S_{21}/\gamma_2\,,$$

and hence

$$S_{11}/\alpha_1 = (\kappa_1/\gamma_1 - \kappa_2/\gamma_2)^{-1} = -S_{21}/\alpha_1\,. \tag{18a, b}$$

With the equations (18), the generalized Ohm's law (12) becomes

$$\frac{\boldsymbol{F}_2 - \boldsymbol{F}_1}{\kappa_1/\gamma_1 - \kappa_2/\gamma_2} = \boldsymbol{E} + c^{-1}\boldsymbol{u}_H \times \boldsymbol{H} + \boldsymbol{E}'\,, \tag{19}$$

with

$$\boldsymbol{u}_H = \frac{\kappa_1 \boldsymbol{v}_1/\gamma_1 - \kappa_2 \boldsymbol{v}_2/\gamma_2}{\kappa_1/\gamma_1 - \kappa_2/\gamma_2} = \boldsymbol{v} + \frac{\kappa_1/\gamma_1^3 \rho_{01} + \kappa_2/\gamma_2^3 \rho_{02}}{(\kappa_1/\gamma_1 - \kappa_2/\gamma_2)^2}\,\boldsymbol{i} \tag{20a, b}$$

and

$$\boldsymbol{E}' = \frac{\boldsymbol{A}_2 - \boldsymbol{A}_1}{\kappa_1/\gamma_1 - \kappa_2/\gamma_2}\,. \tag{20c}$$

The unspecified parameter α_1 has cancelled from the generalized Ohm's law and from the expressions for $\boldsymbol{u}_H$ and $\boldsymbol{E}'$. No restriction has been placed on the $\boldsymbol{F}_r$. If, for example, $\boldsymbol{F}_2 - \boldsymbol{F}_1$ happens to be parallel to $\boldsymbol{i}$ then the condition

$$\rho_1 \boldsymbol{F}_1 + \rho_2 \boldsymbol{F}_2 = \mathbf{0}$$

implies that both equations (6) are satisfied with only the one $\boldsymbol{m}_i$, namely m_1.

The acceleration or inertial term $\boldsymbol{E}'$ in the generalized Ohm's law (19) can be expressed in terms of the variables $\boldsymbol{v}$ and $\boldsymbol{i}$ relating to the plasma as a whole by eliminating $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. For binary plasmas, the definitions of $\boldsymbol{v}$ and $\boldsymbol{j}$ can be inverted to give

$$\boldsymbol{v}_1 = \boldsymbol{v} + \frac{\boldsymbol{i}/\rho_1}{\kappa_1/\gamma_1 - \kappa_2/\gamma_2} \qquad \text{and} \qquad \boldsymbol{v}_2 = \boldsymbol{v} - \frac{\boldsymbol{i}/\rho_2}{\kappa_1/\gamma_1 - \kappa_2/\gamma_2}\,. \tag{21a, b}$$

From the definition (2a) of A_r, it follows that

$$A_r = \frac{\partial v_r}{\partial t} + \nabla.(v_r v_r) + \frac{v_r}{\rho_r}\left(\frac{\partial}{\partial t} + v_r.\nabla\right)\rho_r. \tag{22}$$

Writing the equations (21) as $v_r = v + \beta_r i$ and substituting this into (22) shows that, for binary plasmas,

$$A_r = \frac{\partial v}{\partial t} + \nabla.(vv) + v\left(\frac{\mathrm{d}}{\mathrm{d}t} + \beta_r i.\nabla\right)\ln\rho_r + \beta_r i\left(\frac{\mathrm{d}}{\mathrm{d}t} + \beta_r i.\nabla\right)\ln\rho_r$$
$$+ \frac{\partial(\beta_r i)}{\partial t} + \nabla.\{\beta_r(vi+iv) + \beta_r^2 ii\}, \tag{23}$$

where $\mathrm{d}/\mathrm{d}t \equiv \partial/\partial t + v.\nabla$. Using equation (23) in (20c) gives an expression for E' in terms of v and i.

If we now consider the special situation in which the relative streaming velocity of the two species is nonrelativistic, so that $\gamma_1 \approx \gamma_2 \approx \gamma$ holds (where γ denotes the Lorentz factor corresponding to v) then equation (20b) reduces to

$$u_H = v + \frac{\kappa_1/\rho_{01} + \kappa_2/\rho_{02}}{\gamma(\kappa_1 - \kappa_2)^2} i. \tag{24}$$

On using equation (2b) for A_r together with (21), equation (20c) leads to

$$(\kappa_1 - \kappa_2)^2 E' = -\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(\frac{1}{\rho_{01}} + \frac{1}{\rho_{02}}\right)\frac{i}{\gamma}\right\} - \left(\frac{1}{\rho_{01}} + \frac{1}{\rho_{02}}\right)\frac{i.\nabla(\gamma v)}{\gamma}$$
$$- \frac{1}{\kappa_1 - \kappa_2}\left\{\frac{i.\nabla}{\gamma\rho_{01}}\left(\frac{i/\gamma}{\rho_{01}}\right) - \frac{i.\nabla}{\gamma\rho_{02}}\left(\frac{i/\gamma}{\rho_{02}}\right)\right\}. \tag{25}$$

We now consider the two species to be electrons and protons, denoted by subscripts e and i respectively, and we let e represent the electronic charge. Provided $n_{0i}/n_{0e} \gg (m_e/m_i)^2$ holds, meaning that the protons are not too much less numerous than the electrons, then we have

$$u_H \approx v + (\gamma e n_{0e})^{-1} i. \tag{26}$$

From the formula for the behaviour of the total electric current density under general Lorentz transformations (e.g. Ardavan 1976), we have

$$i = j^* - (1-\gamma^{-1}) j^*.vv/|v|^2, \tag{27}$$

where j^* is the total electric current density in the local rest frame of the plasma as a whole. Since the species are nonrelativistic in this frame, it follows that

$$|j^*| \sim |(n_{0e} - n_{0i})e|\, v_d^*,$$

where v_d^* is an average diffusion speed (Ardavan 1976); also, from (27) it follows that $|i| \lesssim |j^*|$. Hence, for $|v|$ near c, the term in i in the approximation (26) will be of small magnitude compared with $|\boldsymbol{v}|$, implying that the term in $\boldsymbol{i}\times\boldsymbol{H}$ in the generalized Ohm's law can be neglected (cf. Ardavan).

Consider an electron–proton plasma with γ_e not necessarily close to γ_i. Provided we have

$$\frac{\gamma_e}{\gamma_i} \ll \frac{m_i}{m_e} \quad \text{and} \quad \frac{\gamma_i^3 m_i^2 n_{0i}}{\gamma_e^3 m_e^2 n_{0e}} \gg 1\,, \tag{28a, b}$$

meaning that the electrons are not too much more relativistic than the protons and the protons are not too greatly less numerous than the electrons, the form (20b) shows that

$$\boldsymbol{u}_H \approx \boldsymbol{v} + (\gamma_e e n_{0e})^{-1}\boldsymbol{i}. \tag{29}$$

The local proper frame of the bulk plasma is defined by the condition

$$\sum \rho_r^* \boldsymbol{v}_r^* = \boldsymbol{0},$$

with an asterisk denoting a quantity referred to that frame. Hence, taking $\gamma_e^*/\gamma_i^* \ll m_i/m_e$ implies $|\boldsymbol{j}^*| \approx |en_e^* \boldsymbol{v}_e^*|$. Since equation (27) shows that $|\boldsymbol{i}| \lesssim |\boldsymbol{j}^*|$, it follows that $|\boldsymbol{i}| \lesssim |e|n_e^* c$. For the special case in which $\gamma_e \approx \gamma_i \approx \gamma$ holds, implying $n_e^* \approx n_{0e}$, equation (26) shows, as seen above (Ardavan 1976), that $\boldsymbol{u}_H \approx \boldsymbol{v}$ obtains for $\gamma \gg 1$. But when γ_e is not close to γ_i, so that the electrons are relativistic in the proper frame of the bulk plasma, $n_e^* = \gamma_e^* n_{0e}$ holds, and equation (29) shows that

$$|(\boldsymbol{u}_H - \boldsymbol{v})/c| \lesssim \gamma_e^*/\gamma_e = \gamma^{-1}(1 + \boldsymbol{v}\,.\,\boldsymbol{v}_e^*/c^2)^{-1}$$

obtains. For $\gamma \gg 1$ the term in $\boldsymbol{i}\times\boldsymbol{H}$ in the generalized Ohm's law can be neglected if $\boldsymbol{v}_e^*$ is not both near to c in magnitude and antiparallel to $\boldsymbol{v}$.

From the equations (21) and the condition (28a), we have

$$\beta_e \approx (\gamma_e e n_{0e})^{-1} \quad \text{and} \quad \beta_i \approx -(m_e \gamma_e/m_i \gamma_i)(\gamma_i e n_{0i})^{-1}. \tag{30a, b}$$

If we suppose

$$\rho_i \gtrsim \rho_e, \tag{31}$$

which implies $|\beta_i| \lesssim |\beta_e|$, then the condition (28a) ensures that (28b) is satisfied. Let us suppose further that

$$n_i \lesssim n_e \tag{32}$$

obtains in the region under consideration. This means that protons do not provide the predominant contribution to the net charge density or, at least for $|\boldsymbol{v}_i| \lesssim |\boldsymbol{v}_e|$, to the electric current density. Use of equation (23) for A_r in (20c) gives an expression for $\boldsymbol{E}'$, and it is seen that for fluid speeds near c the terms in $\boldsymbol{E}'$ typically have orders of magnitude less than or about $c^2\gamma_e m_e/|e|L$. Provided the displacement current does not predominate over $\boldsymbol{j}$, it follows from the Maxwell equation for $\nabla\times\boldsymbol{H}$ that $|\boldsymbol{H}| \sim 4\pi|e|n_{0e}\gamma_e L$. Under these circumstances, the ratio of the orders of magnitude of the terms $\boldsymbol{E}'$ and $c^{-1}\boldsymbol{v}\times\boldsymbol{H}$ in the generalized Ohm's law is typically $(c/\omega_{e0}L)^2$, where ω_{e0} is the proper electron plasma frequency. The acceleration or inertial term $\boldsymbol{E}'$ is negligible if the plasma gradients are sufficiently slow that (cf. Ardavan 1976)

$$(c/\omega_{e0}L)^2 \ll 1. \tag{33}$$

Then, if 'frictional' effects are negligible, the generalized Ohm's law (19) becomes, using (29) for $\boldsymbol{u}_H$,

$$\boldsymbol{E} + c^{-1}\boldsymbol{v} \times \boldsymbol{H} + (c\gamma_e e n_{0e})^{-1}\boldsymbol{i} \times \boldsymbol{H} \approx \boldsymbol{0}, \tag{34}$$

which reduces to the magnetohydrodynamic condition $\boldsymbol{E} + c^{-1}\boldsymbol{v} \times \boldsymbol{H} \approx \boldsymbol{0}$ under the conditions discussed in the last paragraph.

4. Remarks on Pulsar Magnetospheres

In the last few years it has become clear that inertial effects are crucial in pulsar magnetospheres. The necessity for the fields to be nonsingular at the light cylinder leads to difficulties for a plasma with inertia neglected. For the 'cylindrical pulsar' model, Mestel (1973) showed that if the field everywhere outside the star satisfies the magnetohydrodynamic condition $\boldsymbol{E} + c^{-1}\boldsymbol{v} \times \boldsymbol{H} \approx \boldsymbol{0}$ then there is no flow of energy across the light cylinder; furthermore, the solutions between the light cylinder and infinity are standing waves and so require a reflector at infinity (Mestel *et al.* 1976).

The development and analysis by Ardavan (1976) of a generalized Ohm's law for relativistically streaming plasmas has shown that the condition $\boldsymbol{E} + c^{-1}\boldsymbol{v} \times \boldsymbol{H} \approx \boldsymbol{0}$ fails in a small neighbourhood of the light cylinder, wherever the length scale for variation of macroscopic plasma properties becomes very small so that inertial effects in the law become important. For regions closer to the star, Ardavan's work shows that the magnetohydrodynamic condition is satisfied where the Lorentz factor corresponding to $\boldsymbol{v}$ is much greater than unity, regardless of the relative magnitude of the inertial and magnetic forces acting on the bulk plasma. According to Ardavan, it follows that the extent of the force-free regions of axisymmetric magnetospheres is not determined by the condition usually taken, namely that the ratio $\boldsymbol{H}^2/8\pi\rho c^2$ of magnetic to matter energy densities must there be large: the inertial effects are significant where $\boldsymbol{H}^2/8\pi\rho c^2 \sim \gamma^2$ holds. Ardavan interpreted this result as follows. When $\boldsymbol{E} \times c^{-1}\boldsymbol{v} \times \boldsymbol{H} \approx \boldsymbol{0}$, the magnetic force on the *convection* current is balanced by the electric force. But it is the magnetic force on the *conduction* current that maintains the plasma in rotation, and the ratio of the magnitude of this force to that of the plasma's inertial force is of order $\gamma^{-2}(\boldsymbol{H}^2/8\pi\rho c^2)$. However, Ardavan restricted his calculations to the special case in which the relative streaming velocity of the two species in the plasma is nonrelativistic. When this condition is not imposed, the magnetohydrodynamic approximation is valid under conditions discussed in Section 3 above.

In steady, axisymmetric pulsar magnetospheres there is no displacement current, and so $\nabla \times \boldsymbol{H}^* = 4\pi\boldsymbol{j}^*/c$. Let us now consider regions where the magnetohydrodynamic condition is valid, so that the plasma is tied to the magnetic field lines. Since $\boldsymbol{H}^*$ is poloidal, $\boldsymbol{j}^*$ is toroidal; hence $\boldsymbol{j}^*$ is essentially parallel to $\boldsymbol{v}$, and so equation (27) shows that $\boldsymbol{i} = \boldsymbol{j}^*/\gamma$ (Ardavan 1976). Ardavan also pointed out that $L^* = L$, since $\boldsymbol{v}$ is transverse to the gradient of the magnetic field, and that the magnetohydrodynamic condition and the transformation properties of magnetic fields imply $|\boldsymbol{H}^*| = |\boldsymbol{H}|/\gamma$. Hence the ratio of the magnetic force on the *conduction* current to the plasma's inertial force is about γ^{-2} multiplied by the ratio $\boldsymbol{H}^2/4\pi\rho c^2$ of the orders of magnitude of the total magnetic force to the plasma's inertial force (Ardavan). The right-hand side of the equation of motion (4) of the plasma as a whole can be written

$$\nu(\boldsymbol{E} + c^{-1}\boldsymbol{v} \times \boldsymbol{H}) + c^{-1}\boldsymbol{i} \times \boldsymbol{H}.$$

When the magnetohydrodynamic condition holds, meaning that there is a balance between the electric force on the plasma and the magnetic force on the *convection* current, the right-hand side of equation (4) reduces to $c^{-1}\boldsymbol{i} \times \boldsymbol{H}$. Here the plasma's inertial force, represented by the left-hand side of equation (4), is balanced by the magnetic force on the *conduction* current which, for $\gamma \gg 1$, is small compared with the magnetic force on the *convection* current.

5. Conclusions

The theoretical framework developed here for treating dissipation in relativistically streaming plasmas should fill a gap in the available formalism of plasma physics. Since relativistic plasmas may well be nonneutral, the neutrality condition has not been imposed. Inertial effects are very likely to be important in relativistic plasmas, and so inertial terms have been fully incorporated. Details have been given above for the binary case, with particular attention paid to the analysis of inertial effects for electron–proton plasmas. This work follows an analysis, in the context of pulsar magnetospheres, by Ardavan (1976), but his restriction to plasmas in which the two species have a nonrelativistic relative streaming velocity has been removed here. In pulsar magnetospheres, it is possible that pair production could create electron–positron binary plasmas or electron–positron–proton ternary plasmas. The general formalism of Section 2 could be used to perform a detailed analysis of such plasmas, and this is currently under consideration.

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