Aust. J. Phys., 1978, **31**, 115–16

Remark on the Polytrope of Index 5

H. A. Buchdahl

Department of Theoretical Physics, Faculty of Science,
Australian National University, P.O. Box 4, Canberra, A.C.T. 2600.

Abstract

Contrary to statements in the current literature the gravitational potential energy of the polytrope of index 5 is finite. With ξ_n denoting the least positive zero of the Emden function of index n, the value of the limit as $n \to 5$ of $(5-n)\xi_n$ is determined and made the basis of a simple rational approximation to ξ_n *qua* function of n.

It is well known that the gravitational potential energy Ω of a polytrope S_n of index n, mass M and radius R is given by

$$-\Omega = 3GM^2/(5-n)R, \tag{1}$$

where G is Newton's constant. Chandrasekhar (1957) comments on this result as follows: 'We see that $-\Omega$ is "infinite" for $n = 5$; the reason for this is that the polytrope of index $n = 5$ of finite radius is infinitely "concentrated" toward the centre'. Cox and Giuli (1968) comment on equation (1) in similar terms. Now, in the first place it is difficult to understand the reference to a 'polytrope of index 5 of finite radius' since the radius of S_5 is, after all, infinite. In any event, the potential energy of S_5 is in fact finite. The mistaken conclusion referred to above is absent from the older literature (see e.g. Emden 1907). Nevertheless it may be worth while to obtain in this note the correct explicit result which amounts to finding the limiting value, as $n \to 5$, of the product $q_n = (5-n)\xi_n$, where ξ_n is the least positive zero of $\theta_n(\xi)$, the Emden function of index n. Since not only θ_5 but also θ_0 and θ_1 are elementary functions, knowledge of q_0, q_1 and q_5 allows one to construct a simple elementary rational approximation to ξ_n *qua* function of n without recourse to numerical solutions of Emden's equation.

Where not otherwise indicated the notation is that of Chandrasekhar (1957) and the equations of Chapter IV of this work will be distinguished by the prefix C. Then

$$-\Omega = 4\pi G \int_0^R r\rho M(r)\,\mathrm{d}r$$

and using the first of equations (C8) (with $\lambda = \rho_c$) as well as (C67), (C69) and (C10) this becomes

$$\Omega = GM^2\omega_n^{-2}\alpha^{-1}\int_0^{\xi_n} \xi^3\,\theta_n^{\,n}\,\dot{\theta}_n\,\mathrm{d}\xi\,, \tag{2}$$

where a dot indicates differentiation with respect to ξ and

$$\omega_n = [\xi^2 \dot{\theta}_n(\xi)]_{\xi=\xi_n}.$$

Since $R = \alpha\xi_n$, comparison of equation (2) with (1) immediately gives the result

$$q_n = -3\omega_n^2\left(\int_0^{\xi_n} \xi^3 \theta_n{}^n \dot{\theta}_n \,\mathrm{d}\xi\right)^{-1}. \tag{3}$$

Taking now $n = 5$,

$$\theta_5 = (1 + \tfrac{1}{3}\xi^2)^{-\frac{1}{2}},$$

so that $\omega_5 = -\sqrt{3}$ (cf. equation C71), whilst the integral on the right-hand side of equation (3) becomes (bearing in mind that $\xi_5 = \infty$)

$$-\tfrac{1}{3}\int_0^{\infty} \xi^4(1 + \tfrac{1}{3}\xi^2)^{-4}\,\mathrm{d}\xi.$$

Its value is $-3\pi\sqrt{3}/32$. It thus follows that q_5 (regarded as the limit of q_n as $n \to 5$) is given by

$$q_5 = 32\sqrt{3}/\pi.$$

Accordingly

$$\Omega = -(\pi\sqrt{3}/32)GM^2/\alpha.$$

In view of equations (C38) and (C45), $q_0 = 5\sqrt{6}$ and $q_1 = 4\pi$. Evidently q_n is a slowly increasing function of n over the whole range, and a rational approximation to ξ_n of the kind

$$\xi_n = a(1+bn)/(5-n)(1+cn) \qquad (a, b, c \text{ const.})$$

suggests itself. The values of the three constants are determined by the known values of q_0, q_1 and q_5. In a crude approximation of this kind it suffices to retain three significant figures. Then

$$a = 12\cdot3, \qquad b = -0\cdot128, \qquad c = -0\cdot150.$$

To this degree of accuracy the approximation is very satisfactory.

References

Chandrasekhar, S. (1957). 'An Introduction to the Theory of Stellar Structure' (Dover: New York).
Cox, J. P., and Giuli, R. T. (1968). 'Stellar Structure', Ch. 23 (Gordon & Breach: New York).
Emden, R. (1907). 'Gaskugeln', Ch. 8 (Teubner: Leipzig).

Manuscript received 29 September 1977