Aust. J. Phys., 1978, **31**, 131–5

Calculation of 6*j* Symbols for the Exceptional Group E_7

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Abstract

The 6*j* symbols of the exceptional group E_7 are studied and evaluated explicitly for a number of important cases involving the fundamental and adjoint representations. These 6*j* symbols suffice to calculate all the 3*jm* factors (or isoscalar factors) involving the fundamental or adjoint representations of E_7 at least twice, except in the latter case those involving the power-4 irreps (42^6) and $(3^2 2^5)$.

Introduction

The exceptional group E_7 has recently become of interest to particle physicists. Attempts have been made to develop unified theories of strong, electromagnetic and weak interactions using the group structure $E_7 \supset SU_6^{\mathrm{fl}} \times SU_3^{\mathrm{c}}$, where SU_6^{fl} is the group of quark flavours and SU_3^{c} is the unbroken group of colour (Gürsey 1975; Gürsey *et al.* 1976; Ramond 1976). In these theories the basic fermions (quarks, leptons and their antiparticles) are associated with the 56-dimensional fundamental irreducible representation (irrep) of E_7, and the gauge vector bosons that mediate the interactions are associated with the 133-dimensional adjoint irrep.

Quantitative calculations require a knowledge of the 3*jm* factors (or isoscalar factors) for $E_7 \supset SU_6 \times SU_3$ and of the 6*j* symbols of E_7. In this paper we calculate some 6*j* symbols that involve the fundamental and adjoint irreps of E_7. These 6*j* symbols suffice to calculate all the 3*jm* factors involving the fundamental or adjoint irreps of E_7 at least twice, except in the latter case those involving the power-4 irreps (42^6) or $(3^2 2^5)$. Such 3*jm* factors arise in the evaluation of the matrix elements of the generators of E_7 in the $E_7 \supset SU_6 \times SU_3$ basis, a subject we shall report on later.

A detailed discussion of the basic properties of the exceptional groups has been given by Wybourne and Bowick (1977) while the general properties, and evaluation, of the 6*j* symbols and 3*jm* factors for compact groups have been considered by Butler (1975, 1978) and by Butler and Wybourne (1976). We refer to these papers for much of the basic theory, notation and definitions.

Irreps of E_7

A number of properties of the irreps of E_7 must first be enumerated. Wybourne and Bowick (1977) have shown that the irreps of E_7 may be uniquely labelled by partitions (λ) of even integers l into six or seven integral parts λ_i such that

$$\lambda_i \geqslant \lambda_{i+1} \geqslant 0 \qquad (i = 1, 2, \ldots, 6)$$

and

$$\lambda_4+\lambda_5+\lambda_6+\lambda_7 \geqslant \lambda_1+\lambda_2+\lambda_3 .$$

In this notation the fundamental irrep is designated as (1^6) and the adjoint irrep as (21^6). The dimension $|\lambda|$ of each irrep (λ) may be readily evaluated (Wybourne 1974). The power p_λ of an irrep (λ) is defined as the smallest integer p_λ for which the p_λth Kronecker power of the fundamental irrep contains (λ). In this paper we shall restrict our attention to irreps with $p_\lambda \leqslant 3$. For subsequent brevity it is convenient to associate a serial number λ with each irrep (λ).

The irreps of E_7 are all real and are orthogonal or symplectic as

$$\phi_\lambda = (-1)^{l/2} \tag{1}$$

is positive or negative. The phase ϕ_λ is often referred to as the $2j$ symbol (Butler and Wybourne 1976). It follows that all the $6j$ symbols of E_7 may be taken as real. The above-mentioned properties of the E_7 irreps with $p_\lambda \leqslant 3$ are listed in Table 1. We note that $\lambda = 0$ corresponds here to the identity irrep (0) of E_7.

Table 1. Some E_7 irreps and their associated properties

Irrep (λ)	Dimension $\lvert\lambda\rvert$	Serial No. λ	Power p_λ	Phase ϕ_λ
(0)	1	0	0	1
(1^6)	56	1	1	−1
(21^6)	133	2	2	1
(2^6)	1 463	3	2	1
$(2^5 1^2)$	1 539	4	2	1
(2^7)	912	5	3	−1
$(32^5 1)$	6 480	6	3	−1
$(3^4 2^3)$	27 664	7	3	−1
$(3^5 21)$	51 072	8	3	−1
(3^6)	24 320	9	3	−1

Triads and $3j$ Symbols for E_7 Irreps

The $6j$ symbol

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4} \tag{2}$$

will be null unless the triple Kronecker product for each of the four triads $(\lambda_1\, \mu_2^*\, \mu_3)$, $(\mu_1\, \lambda_2\, \mu_3^*)$, $(\mu_1^*\, \mu_2\, \lambda_3)$ and $(\lambda_1\, \lambda_2\, \lambda_3)$ contains the identity irrep $\lambda = 0$. The four indices r_i attached to the $6j$ symbol are associated with the product multiplicities that may arise in the four triads. The triple Kronecker products associated with each triad follow trivially from the tables of E_7 Kronecker products given by Wybourne and Bowick (1977).

The $3j$ symbols $\{(\pi)\lambda_1\, \lambda_2\, \lambda_3\}_{rr'}$ give the permutational symmetries of the $3jm$ factors and consequently arise in the reordering symmetries of the $6j$ symbols (Butler 1975). For simple phase irreps the $3j$ symbol is no more than a phase factor (Butler and King 1974). It may be shown that the irreps λ of E_7 with $p_\lambda \leqslant 3$ are indeed simple phase. As a consequence, for each of these irreps of E_7 we may associate a j value such that

$$\phi_\lambda = (-1)^{2j_\lambda}, \tag{3}$$

where j_λ is an integer if λ is orthogonal and half-integer if λ is symplectic. The 3j symbol may in this case be chosen so that (Butler and Wybourne 1976)

$$\{(123)\lambda_1\lambda_2\lambda_3\}_{rr'} = \{(132)\lambda_1\lambda_2\lambda_3\}_{rr'} = \delta_{rr'}, \tag{4a}$$

$$\begin{aligned}\{(12)\lambda_1\lambda_2\lambda_3\}_{rr'} &= \{(23)\lambda_1\lambda_2\lambda_3\}_{rr'} = \{(13)\lambda_1\lambda_2\lambda_3\}_{rr'} \\ &= \{\lambda_1\lambda_2\lambda_3 r\}\delta_{rr'} = (-1)^{j_{\lambda_1}+j_{\lambda_2}+j_{\lambda_3}+r}\delta_{rr'}.\end{aligned} \tag{4b}$$

The j_λ value to be associated with a given irrep λ of E_7 follows directly from an analysis of the Kronecker squares of λ. We readily deduce that for $p_\lambda \leqslant 3$ we may choose

$$j_\lambda = \tfrac{1}{2}p_\lambda, \tag{5}$$

except for the $(2^5 1^2)$ irrep where we must choose $j_4 = 0$.

A knowledge of the 3j symbols allows a determination of the behaviour of the 6j symbols under a reordering symmetry. Noting that the 6j symbols for E_7 are real because the irreps are real, we have

$$\begin{aligned}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \mu_1 & \mu_2 & \mu_3\end{Bmatrix}_{r_1r_2r_3r_4} &= \begin{Bmatrix}\lambda_1 & \mu_2 & \mu_3\\ \mu_1 & \lambda_2 & \lambda_3\end{Bmatrix}_{r_4r_2r_3r_1} \\ &= \phi_{\mu_1}\phi_{\mu_2}\phi_{\mu_3}\{\lambda_1\mu_2\mu_3 r_1\}\{\mu_1\lambda_2\mu_3 r_2\}\{\mu_1\mu_2\lambda_3 r_3\} \\ &\quad\times\{\lambda_1\lambda_2\lambda_3 r_4\}\begin{Bmatrix}\lambda_{\pi(1)} & \lambda_{\pi(2)} & \lambda_{\pi(3)}\\ \mu_{\pi(1)} & \mu_{\pi(2)} & \mu_{\pi(3)}\end{Bmatrix}_{r_{\pi(1)}r_{\pi(2)}r_{\pi(3)}r_{\pi(4)}},\end{aligned} \tag{6}$$

where π is a transposition.

Calculation of 6j Symbols for E_7

The trivial 6j symbol is essentially a 3j symbol:

$$\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \lambda_2 & \lambda_1 & 0\end{Bmatrix}_{00rs} = |\lambda_1\lambda_2|^{-\frac{1}{2}}\{\lambda_1\lambda_2\lambda_3 r\}\delta_{rs}. \tag{7}$$

These 6j symbols may be immediately evaluated using equation (4b) and the information contained in Table 1.

The next simplest 6j symbols to evaluate are the so-called primitive 6j symbols that involve the fundamental irrep 1 at least once (Butler and Wybourne 1976). These 6j symbols may be evaluated by use of the orthogonality relation.

$$\sum_{\mu_3 r_1 r_2} |\lambda_3\mu_3| \begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \mu_1 & \mu_2 & \mu_3\end{Bmatrix}_{r_1r_2r_3r_4}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3'\\ \mu_1 & \mu_2 & \mu_3\end{Bmatrix}_{r_1r_2r_3'r_4'} = \delta_{\lambda_3\lambda_3'}\delta_{r_3r_3'}\delta_{r_4r_4'} \tag{8}$$

and the Racah backcoupling relation

$$\begin{aligned}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \mu_1 & \mu_2 & \mu_3\end{Bmatrix}_{r_1r_2r_3r_4} &= \sum_{\nu rr'} |\nu|\,\phi_{\mu_2}\{\mu_1\lambda_2\mu_3 r_2\}\{\lambda_1\lambda_2\lambda_3 r_4\}\{\lambda_1\mu_1\nu r\} \\ &\quad\times\begin{Bmatrix}\lambda_2 & \lambda_1 & \lambda_3\\ \mu_1 & \mu_2 & \nu\end{Bmatrix}_{r'rr_3r_4}\begin{Bmatrix}\lambda_1 & \mu_1 & \nu\\ \lambda_2 & \mu_2 & \mu_3\end{Bmatrix}_{r_1r_2r'r}.\end{aligned} \tag{9}$$

Starting with the known values of the trivial $6j$ symbols it is possible to use equations (8) and (9) to generate simple sets of simultaneous equations in the primitive $6j$ symbols which may be solved to yield hitherto unknown primitive $6j$ symbols which may be returned to equations (8) and (9) to produce further primitive $6j$ symbols. In some cases the equations are nonlinear and it is necessary to choose the roots of a quadratic equation. In these cases it is always found, as would be expected, that the wrong choice of root leads to a subsequent contradiction.

Once a primitive set of $6j$ symbols has been found it is possible to generate non-primitive $6j$ symbols using a generalization of the Biedenharn–Elliott sum rule (Butler and Wybourne 1976). A comprehensive computer program developed by one of us (P.H.B.; details to be published elsewhere) was used in generating many of the basic equations and in evaluating the symmetries of the $6j$ symbols.

Table 2. Some nontrivial $6j$ symbols for E_7

Symbol	Value	Symbol	Value	Symbol	Value
$\begin{Bmatrix}1&1&2\\1&1&2\end{Bmatrix}$	$-\frac{1}{2^3.19}$	$\begin{Bmatrix}1&1&3\\1&1&2\end{Bmatrix}$	$\frac{1}{2^3.7.19}$	$\begin{Bmatrix}1&1&3\\1&1&3\end{Bmatrix}$	$\frac{3}{2^3.7.11.19}$
$\begin{Bmatrix}1&1&4\\1&1&2\end{Bmatrix}$	$\frac{1}{2^3.3.7.19}$	$\begin{Bmatrix}1&1&4\\1&1&3\end{Bmatrix}$	$\frac{1}{2^3.3.7.19}$	$\begin{Bmatrix}1&1&4\\1&1&4\end{Bmatrix}$	$\frac{29}{2^3.3^4.7.19}$
$\begin{Bmatrix}2&1&5\\2&1&1\end{Bmatrix}$	$\frac{1}{2.7.19}$	$\begin{Bmatrix}2&1&5\\2&1&5\end{Bmatrix}$	$\frac{1}{2^4.3.7.19}$	$\begin{Bmatrix}2&1&5\\2&1&6\end{Bmatrix}$	$-\frac{1}{2^2.3^2.7.19}$
$\begin{Bmatrix}2&1&6\\2&1&1\end{Bmatrix}$	$-\frac{1}{2^2.3.7.19}$	$\begin{Bmatrix}2&1&6\\2&1&6\end{Bmatrix}$	$-\frac{103}{2^4.3^4.5.7.19}$		
$\begin{Bmatrix}2&2&2\\1&1&1\end{Bmatrix}$	$\frac{\sqrt{3}}{2.7.19}$	$\begin{Bmatrix}2&2&2\\1&1&5\end{Bmatrix}$	$-\frac{\sqrt{3}}{2.3.7.19}$	$\begin{Bmatrix}2&2&2\\1&1&6\end{Bmatrix}$	$\frac{\sqrt{3}}{2^2.3^2.7.19}$
$\begin{Bmatrix}2&2&4\\1&1&1\end{Bmatrix}$	$\frac{\sqrt{5}}{2.3.7.19}$	$\begin{Bmatrix}2&2&4\\1&1&5\end{Bmatrix}$	$\frac{\sqrt{5}}{2^2.3^2.7.19}$	$\begin{Bmatrix}2&2&4\\1&1&6\end{Bmatrix}$	$-\frac{\sqrt{5}}{3^4.5.7.19}$
$\begin{Bmatrix}2&2&2\\2&2&2\end{Bmatrix}$	$\frac{1}{2.7.19}$	$\begin{Bmatrix}2&2&4\\2&2&2\end{Bmatrix}$	$-\frac{2}{3^2.7.19}$		

A list of nontrivial $6j$ symbols for E_7 is given in Table 2. It so happens that all the $6j$ symbols listed there are invariant under all the reordering symmetries. This need not always be the case for the $6j$ symbols of E_7. For example, the $6j$ symbol

$$\begin{Bmatrix}4&4&2\\4&3&4\end{Bmatrix}_{0100}$$

changes sign under an odd transposition of columns. The $6j$ symbols given in Table 2 suffice to calculate all the $3jm$ factors involving the fundamental or adjoint irreps of E_7 at least twice, except in the latter case those including (42^6) or (3^22^5) irreps which are of power 4. We note that the evaluation of the $3jm$ factors requires only the primitive $6j$ symbols of E_7 (Butler and Wybourne 1976).

Conclusions

We have shown that even in a group as complex as E_7 it is possible to systematically evaluate the $6j$ symbols. This has the important implication that the machinery already exists to fully exploit the Racah–Wigner calculus for E_7 and its subgroups. The problems of phase specification are fully understood and there is no difficulty in extending the methods used here to any other compact group.

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Manuscript received 5 December 1977