Aust. J. Phys., 1978, **31**, 151–61

On the Statistical Theory of Nuclear Reactions

W. K. Bertram

AAEC Research Establishment, Private Mail Bag, Sutherland, N.S.W. 2232.

Abstract

The statistical theory of energy-averaged reaction cross sections is examined using the pole expansion of the S-matrix. Exact expressions for the average cross sections in terms of the parameters of the S-matrix are derived for the case when there are two open channels. It is shown that when the number of channels exceeds two, the average cross sections can be evaluated provided the poles of the S-matrix are evenly spaced.

1. Introduction

The behaviour of energy-averaged reaction cross sections in terms of transmission coefficients was originally formalized by Hauser and Feshbach (1952; hereinafter abbreviated HF) using Bohr's compound nucleus assumption. Subsequent investigations, by means of the R-matrix formalism for example, showed that in the context of resonance theory (Thomas 1955; Lane and Lynn 1957) the HF theory, modified to include fluctuation corrections, could only be expected to apply when the ratio $\bar{\Gamma}/D$ of the averaged level width to the average level spacing was sufficiently small.

Attempts to improve upon the HF theory, using various representations of the S-matrix, have met with little success (Moldauer 1964; Ullah 1969; Kawai *et al.* 1973). The most notable attempt was Moldauer's (1964) theory. From the unitarity and analyticity of the S-matrix, he derived expressions for the average cross section in terms of the parameters of the statistical S-matrix. However, these results did not provide a practical alternative to HF theory owing to the fact that the cross sections contained terms which were found to be extremely difficult to evaluate. In order to overcome this difficulty, Moldauer (1975) introduced the 'M-cancellation principle', where, as a result of cancellation of certain terms, the average cross section is given by the HF theory in which the fluctuation correction is obtained by associating with each channel c a variable number of degrees of freedom ν_c. This procedure, originally proposed by Lane and Lynn (1957), is also formally identical with that put forward by Tepel *et al.* (1974) and Hofmann *et al.* (1975), whose analysis is based on Bohr's compound nucleus assumption.

It has been found (Moldauer 1975; Hofmann *et al.* 1975) that, with suitable parameterization of the quantities ν_c, the HF formula yields results which are much more accurate than those obtained from the HF theory with all $\nu_c = 1$. On the other hand, the basic HF expression for the cross section has only been derived from resonance theories in the case of weak absorption ($\bar{\Gamma}/D \ll 1$) and in the case of many channels with strong absorption (Agassi *et al.* 1975).

In the present paper we investigate energy-averaged cross sections using the pole expansion of the S-matrix. When there are only two open channels, the averages can be evaluated exactly using only the unitarity and analyticity of the S-matrix. When more than two channels are open, the problem becomes much more difficult; however, in the special case when the poles of the S-matrix are evenly spaced, an exact solution is possible even when the number of open channels exceeds two.

2. General Theory

The energy-averaged cross section $\langle\sigma_{cc'}\rangle$ for a reaction $c \to c'$ can be expressed in terms of the energy averages of the elements $U_{cc'}$ of the collision matrix and of their absolute squares $|U_{cc'}|^2$ as

$$\langle\sigma_{cc'}\rangle = \pi\lambda\!\!\!^{-2}_c\langle|\delta_{cc'}-U_{cc'}|^2\rangle \tag{1}$$

and the average total cross section as

$$\langle\sigma_{c\mathrm{T}}\rangle = 2\pi\lambda\!\!\!^{-2}_c(1-\mathrm{Re}\langle U_{cc}\rangle). \tag{2}$$

It is customary to separate $\langle\sigma_{cc'}\rangle$ into a direct part $\sigma^{\mathrm{d}}_{cc'}$ and a compound nucleus part $\sigma^{\mathrm{f}l}_{cc'}$ (Feshbach *et al.* 1953),

$$\sigma^{\mathrm{d}}_{cc'} = \pi\lambda\!\!\!^{-2}_c|\delta_{cc'}-\langle U_{cc'}\rangle|^2, \qquad \sigma^{\mathrm{f}l}_{cc'} = \pi\lambda\!\!\!^{-2}_c(\langle|U_{cc'}|^2\rangle-|\langle U_{cc'}\rangle|^2). \tag{3}$$

With these definitions, the transmission coefficient of channel c is given by the familiar relation

$$T_c = 1-|\langle U_{cc}\rangle|^2. \tag{4}$$

In order to evaluate the necessary energy averages, we assume a collision matrix of the form

$$U = \Omega S \Omega, \tag{5}$$

where Ω is the matrix of hard-sphere scattering phase shifts ϕ_c,

$$\Omega_{cc'} = \exp(-\mathrm{i}\phi_c)\,\delta_{cc'}, \tag{6}$$

and S is a statistical S-matrix as defined by Moldauer (1964) with elements

$$S_{cc'} = S^0_{cc'}+\mathrm{i}\sum_\lambda g_{\lambda c}g_{\lambda c'}/(\mathscr{E}_\lambda-E), \tag{7}$$

where

$$\mathscr{E}_\lambda = E_\lambda-\tfrac{1}{2}\mathrm{i}\mathscr{G}_\lambda \tag{8}$$

and S^0 is assumed to be energy independent. The energy average of S, denoted by $\bar{S} \equiv \langle S\rangle$, is obtained from the relation (Moldauer 1967*a*)

$$\bar{S}_{cc'}-(\bar{S}^{*-1})_{cc'} = -2\pi\langle g_{\lambda c}g_{\lambda c'}\rangle/D, \tag{9}$$

where D is the average level spacing, as before.

The Engelbrecht–Weidenmüller (1973) transformation allows us to consider, without loss of generality, only those S-matrices for which

$$\langle g_{\lambda c}g_{\lambda c'}\rangle = \langle g^2_{\lambda c}\rangle\delta_{cc'} \qquad \text{and} \qquad \mathrm{Im}\langle g^2_{\lambda c}\rangle = 0. \tag{10}$$

Equation (9) can then be solved to yield

$$\langle S_{cc'}\rangle = (\cosh\alpha_c - \sinh\alpha_c)\delta_{cc'}, \tag{11}$$

where α_c is defined by the relation

$$\pi\langle g_{\lambda c}^2\rangle/D = \sinh\alpha_c. \tag{12}$$

The constant background matrix S^0 then has elements

$$S_{cc'}^0 = \cosh\alpha_c\,\delta_{cc'}, \tag{13}$$

and from equation (4) the transmission coefficients become

$$T_c = 1-\exp(-2\alpha_c). \tag{14}$$

The energy average of the quantity $|S_{cc'}|^2$ can be obtained by considering the integration around a rectangular contour which is based on the real energy axis and has vertical sides of length W with $W \gg \langle\mathscr{Y}_\lambda\rangle/D$. The integral of $|S_{cc'}|^2$ over an energy interval Δ on E is then equal to the value of $\Delta\,|S_{cc'}(z)|^2$ as $z \to \mathrm{i}W$, plus $2\pi\mathrm{i}$ times the sum of the residues within the contour. Using equation (7) we obtain

$$\langle|S_{cc'}|^2\rangle = \delta_{cc'} - Z_{cc'}, \tag{15}$$

where

$$Z_{cc'} = 2\pi\langle g_{\lambda c}\,g_{\lambda c'}\,S_{cc'}^*(\mathscr{E}_\lambda^*)\rangle/D. \tag{16}$$

The principal difficulty in the derivation of a statistical theory is the evaluation of $Z_{cc'}$ in terms of known quantities, such as the transmission coefficients.

3. One and Two Open Channels

In the case where there is only one open channel, S is a scalar function and can be represented as an infinite product (Ning Hu 1948; van Kampen 1953; Simonius 1974*a*):

$$S(z) = \prod_\lambda (\mathscr{E}_\lambda^* - z)/(\mathscr{E}_\lambda - z). \tag{17}$$

From this it can be shown (Moldauer 1969; Simonius 1974*b*) that the average of the residues of S is given by

$$\pi\langle g_\lambda^2\rangle/D = \sinh\alpha, \tag{18a}$$

and the average of S is

$$\langle S\rangle = S(\mathrm{i}W) = \cosh\alpha - \sinh\alpha, \tag{18b}$$

where

$$\alpha = \pi\langle\mathscr{Y}_\lambda\rangle/D. \tag{18c}$$

For the purpose of practical applications, the single-channel case is not a very useful one. However, the importance of the above results lies in the fact that the determinant of an N-channel S-matrix of the form (7) is just the single-channel S-matrix given by equation (17). This property allows us to solve the less trivial two-channel case. If we write the unitarity condition as

$$S^*(z^*) = \Delta^{-1}\begin{pmatrix} S_{22}(z) & -S_{12}(z) \\ -S_{12}(z) & S_{11}(z) \end{pmatrix}, \tag{19}$$

the determinant Δ must be of the form

$$\Delta = \Delta^0 + \mathrm{i}\sum_\lambda h_\lambda^2/(\mathscr{E}_\lambda - z), \tag{20a}$$

where

$$\pi\langle h_\lambda^2\rangle/D = \sinh\alpha, \qquad \Delta^0 = \cosh\alpha. \tag{20b, c}$$

Evaluation of the ratios $S_{cc'}/\Delta$ in the limit $z \to \mathscr{E}_\lambda$ leads to the result

$$S_{cc'}^*(\mathscr{E}_\lambda^*) = (g_\lambda^2/h_\lambda^2)\delta_{cc'} - g_{\lambda c}g_{\lambda c'}/h_\lambda^2, \tag{21}$$

where we have defined

$$g_\lambda^2 = g_{\lambda 1}^2 + g_{\lambda 2}^2.$$

The term $Z_{cc'}$ defined by equation (16) becomes

$$Z_{cc'} = \frac{2\pi}{D}\left\langle\frac{g_{\lambda c}^2 g_\lambda^2}{h_\lambda^2}\right\rangle\delta_{cc'} - \frac{2\pi}{D}\left\langle\frac{g_{\lambda c}^2 g_{\lambda c'}^2}{h_\lambda^2}\right\rangle, \tag{22}$$

which, by introducing the quantities $\Phi_{\lambda c}$ defined as

$$\Phi_{\lambda c} = 2\pi D^{-1} g_{\lambda c}^2 g_\lambda^2/h_\lambda^2, \tag{23}$$

can be written

$$Z_{cc'} = \langle\Phi_{\lambda c}\rangle\delta_{cc'} - \langle\Phi_{\lambda c}\Phi_{\lambda c'}/\Phi_\lambda\rangle, \tag{24}$$

where

$$\Phi_\lambda = \Phi_{\lambda 1} + \Phi_{\lambda 2}.$$

The compound nucleus part of the cross section is then of the form

$$\sigma_{cc'}^{fl} = \pi\bar{\lambda}_c^2\langle\Phi_{\lambda c}\Phi_{\lambda c'}/\Phi_\lambda\rangle + \pi\bar{\lambda}_c^2(T_c - \langle\Phi_{\lambda c}\rangle)\delta_{cc'}. \tag{25}$$

This expression for the cross section is, of course, exact in the two-channel case.

Finally, in order to establish a connection between α_c as defined by equation (12) and the quantity α (equation 18c), we note that in the N-channel case $S(z)$ becomes a diagonal matrix as $z \to \mathrm{i}W$, that is,

$$S_{cc}(\mathrm{i}W) = \exp(-\alpha_c). \tag{26}$$

In this limit, the determinant of S is just the product of these diagonal elements:

$$\Delta(\mathrm{i}W) = \prod_c S_{cc}(\mathrm{i}W) = \exp\Big(-\sum_c \alpha_c\Big). \tag{27}$$

On the other hand, from equation (18b) we have $\Delta(\mathrm{i}W) = \exp(-\alpha)$. Therefore it follows that

$$\alpha = \sum_c \alpha_c. \tag{28}$$

If there are no fluctuations in the widths and level spacings, the cross section σ_{12} is obtained by replacing $g_{\lambda c}^2$ and h_λ^2 by the values of their averages, and thus

$$\sigma_{12}^{fl} = 2\pi\lambda\!\!\!^{-}{}_1^2(\sinh\alpha_1 \sinh\alpha_2)/\sinh\alpha\,. \tag{29}$$

Using equation (14) this can be expressed in terms of the transmission coefficients as

$$\sigma_{12}^{fl} = \pi\lambda\!\!\!^{-}{}_1^2\, T_1\, T_2/(T_1+T_2-T_1\, T_2)\,. \tag{30}$$

This result is identical with that obtained from the R-matrix picket fence model (Moldauer 1967*b*).

4. General N-channel Case

The method used in Section 3 to derive the average cross section when only two reaction channels are open does not lend itself to generalization to the multichannel case. Instead, we will use a method which makes more use of the analytic properties of the S-matrix.

Let us write the S-matrix of equation (7) in the form

$$S = B + \mathrm{i}G_\lambda/(\mathscr{E}_\lambda - z)\,, \tag{31}$$

where

$$B_{cc'} = \cosh\alpha_c\,\delta_{cc'} + \mathrm{i}\sum_{\mu\neq\lambda} g_{\mu c}\, g_{\mu c'}/(\mathscr{E}_\mu - z)\,, \qquad G_{\lambda cc'} = g_{\lambda c}\, g_{\lambda c'}\,.$$

G_λ is a rank one matrix and has the property $G_\lambda^2 = G_\lambda \operatorname{trace} G_\lambda$. Therefore, since $G_\lambda B^{-1}$ is also a rank one matrix, we can evaluate the inverse of S as

$$S^{-1} = S^* = B^{-1} - \mathrm{i}B^{-1}\, G_\lambda\, B^{-1}/(\mathscr{E}_\lambda - z + \mathrm{i}\operatorname{trace} G_\lambda\, B^{-1})\,. \tag{32}$$

The determinant Δ of S can be expressed as

$$\Delta = \Delta_B\{1 + \mathrm{i}(\operatorname{trace} G_\lambda\, B^{-1})/(\mathscr{E}_\lambda - z)\}\,, \tag{33}$$

where Δ_B is the determinant of B.

In the limit when $z \to \mathscr{E}_\lambda$, equation (32) yields

$$S^*(\mathscr{E}_\lambda^*) = (B_\lambda^{-1} \operatorname{trace} G_\lambda\, B_\lambda^{-1} - B_\lambda^{-1}\, G_\lambda\, B_\lambda^{-1})/(\operatorname{trace} G_\lambda\, B_\lambda^{-1})\,, \tag{34}$$

where B_λ is the value of B at $z = \mathscr{E}_\lambda$. From equation (33) it follows that

$$h_\lambda^2 = \Delta_B(\mathscr{E}_\lambda) \operatorname{trace} G_\lambda\, B_\lambda^{-1}\,, \tag{35}$$

so that equation (34) can be written

$$S^*(\mathscr{E}_\lambda^*) = \{\Delta_B(\mathscr{E}_\lambda)/h_\lambda^2\}(B_\lambda^{-1} \operatorname{trace} G_\lambda\, B_\lambda^{-1} - B_\lambda^{-1}\, G_\lambda\, B_\lambda^{-1})\,. \tag{36}$$

Furthermore, if we denote the matrix of cofactors of B_λ by V_λ,

$$V_\lambda = \Delta_B(\mathscr{E}_\lambda)\, B_\lambda^{-1}\,, \tag{37}$$

we find from equation (36)

$$\langle g_{\lambda c} g_{\lambda c'} S^*_{cc'}(\mathscr{E}^*_\lambda)\rangle = \left\langle \frac{g_{\lambda c} g_{\lambda c'}}{h^2_\lambda} \sum_{ef} g_{\lambda e} g_{\lambda f} \left(\frac{V_{\lambda cc'} V_{\lambda ef} - V_{\lambda ce} V_{\lambda c'f}}{\Delta_B(\mathscr{E}_\lambda)} \right) \right\rangle$$

$$= \left\langle \frac{g_{\lambda c} g_{\lambda c'}}{h^2_\lambda} \sum_{ef} g_{\lambda e} g_{\lambda f} U_{cc'ef} \right\rangle . \tag{38}$$

The quantity $U_{cc'dd'}$, defined as

$$U_{cc'dd'} = (V_{\lambda cc'} V_{\lambda dd'} - V_{\lambda cd} V_{\lambda c'd'})/\Delta_B(\mathscr{E}_\lambda) , \tag{39}$$

is the determinant of the matrix which results when we delete from B_λ the two rows and two columns which intersect at the elements $B_{\lambda cc'}$ and $B_{\lambda dd'}$.

In the case when there are two open channels the summation on the right-hand side of equation (38) contains only one nonzero term, namely $U_{cc'cc'} = -1$ ($c \neq c'$), so that we obtain the result derived in Section 3, for $c \neq c'$:

$$\langle g_{\lambda c} g_{\lambda c'} S^*_{cc'}(\mathscr{E}^*_\lambda)\rangle = -\langle g^2_{\lambda c} g^2_{\lambda c'}/h^2_\lambda\rangle . \tag{40}$$

When more than two channels are open, the evaluation of equation (38) is difficult to carry out in general. However, it can be done exactly for a certain class of statistical S-matrices which have their poles evenly spaced along the line $z = E - \frac{1}{2}\mathrm{i}\overline{\mathscr{Y}}$.

5. Evenly Spaced Poles

If the positions of the poles of a statistical S-matrix are given by

$$\mathscr{E}_\lambda = \lambda\pi D - \tfrac{1}{2}\mathrm{i}\overline{\mathscr{Y}} , \tag{41}$$

its determinant can be written in the form

$$\Delta = \cosh\alpha + \mathrm{i} \sum_\lambda \frac{h^2_\lambda/D}{\lambda\pi - \frac{1}{2}\mathrm{i}\mathscr{Y} - z'} = \cosh\alpha - \mathrm{i}\sinh\alpha\cot(z + \tfrac{1}{2}\mathrm{i}\mathscr{Y}) , \tag{42}$$

where

$$\mathscr{Y} = \overline{\mathscr{Y}}/D , \qquad z' = E/D , \qquad (\pi/D)h^2_\lambda = \sinh\alpha .$$

In such an S-matrix the values of $g_{\lambda c}$ are restricted by the unitarity condition and by the condition that all $\mathscr{Y}_\lambda$ be equal. The second condition is expected to become more restrictive when the number of channels is small; whereas when the number of open channels is large the condition $\mathscr{Y}_\lambda \approx \overline{\mathscr{Y}}$ is usually automatically satisfied.

Let us define the function F as

$$F(z') = -\mathrm{i}\cot(z' + \tfrac{1}{2}\mathrm{i}\mathscr{Y}) = H(z') + \mathrm{i}(\mathscr{E}_\lambda/D - z')^{-1} \tag{43}$$

and denote its nth derivative by $F^{(n)}$,

$$F^{(n)}(z') = H^{(n)}(z') + \mathrm{i}n!(\mathscr{E}_\lambda/D - z')^{-n-1} . \tag{44}$$

The values of H and its derivatives at $z = \mathscr{E}_\lambda/D$ can then be evaluated as

$$H_\lambda = H(\mathscr{E}_\lambda/D) = 0$$

and

$$H'_\lambda = \tfrac{1}{3}\mathrm{i}, \quad H''_\lambda = 0, \quad H'''_\lambda = \tfrac{2}{15}\mathrm{i}, \quad \ldots,$$

where, in general,

$$H_\lambda^{(2n-1)} = (-1)^{n-1}/\mathrm{i}(2^{2n-1}/n)\mathscr{B}_{2n}, \qquad H_\lambda^{(2n)} = 0, \tag{45}$$

the $\mathscr{B}_n$ being the Bernoulli numbers.

When $z' \to \pm\mathrm{i}W$ with $W \to \infty$ we find from equation (43)

$$F(\pm\mathrm{i}W) = \mp 1, \qquad F^{(n)}(\pm\mathrm{i}W) = 0.$$

Therefore, if we use the form (31) for the S-matrix to evaluate the product $FS_{cc'}$ and carry out the integration around a large rectangular contour with vertical sides extending from $\mathrm{Im}(z') = -\mathrm{i}W$ to $+\mathrm{i}W$, we obtain

$$\langle B_{\lambda cc'}\rangle = \cosh\alpha_c\,\delta_{cc'}. \tag{46}$$

Similarly, by considering the residues of the poles of the expression $F^{(n)}S_{cc'}$, we find

$$\langle B_{\lambda cc'}^{(2n-1)}\rangle = (-1)^{n-1}\mathrm{i}(2^{2n-1}/n)\mathscr{B}_{2n}\sinh\alpha_c\,\delta_{cc'}, \qquad \langle B_{\lambda cc'}^{(2n)}\rangle = 0. \tag{47}$$

On the other hand, the contour integration of $S_{cc'}^2$ yields

$$\langle B_{\lambda cc'}g_{\lambda c}g_{\lambda c'}\rangle = \sinh\alpha_c\cosh\alpha_c\,\delta_{cc'}. \tag{48}$$

Comparison between equations (46) and (48) shows that

$$\langle B_{\lambda cc'}g_{\lambda c}g_{\lambda c'}\rangle = \langle B_{\lambda cc'}\rangle\langle g_{\lambda c}g_{\lambda c'}\rangle. \tag{49}$$

Therefore the quantities $g_{\lambda c}$ and $B_{\lambda cc'}$ behave as though they are uncorrelated. This result is consistent with the assumption that there are no level–level correlations. In general, $B_{\lambda cc'}$ is of the form

$$B_{\lambda cc'} = \sum_{\mu\neq\lambda} g_{\mu c}g_{\mu c'}/\{(E_\mu - E_\lambda) - \tfrac{1}{2}\mathrm{i}(\mathscr{Y}_\mu - \mathscr{Y}_\lambda)\}, \tag{50a}$$

so that $g_{\lambda c}$ and $B_{\lambda cc'}$ are correlated through the presence of $\mathscr{Y}_\lambda$ in each term of the summation on the right-hand side of equation (50a). However, when all $\mathscr{Y}_\lambda$ are equal, which is the case being considered here, this correlation disappears. It is therefore a plausible assumption that $g_{\lambda c}$ and $B_{\lambda cc'}^{(n)}$ are also uncorrelated, since the nth derivative of $B_{cc'}$ at $z' = \mathscr{E}_\lambda/D$ is of the form

$$B_{\lambda cc'}^{(n)} = \sum_{\mu\neq\lambda} g_{\mu c}g_{\mu c'}/\{(E_\mu - E_\lambda) - \tfrac{1}{2}\mathrm{i}(\mathscr{Y}_\mu - \mathscr{Y}_\lambda)\}^{n+1}. \tag{50b}$$

Consequently, by examining the residues of the poles of the functions $FS_{cc'}S_{dd'}$ and $S_{cc'}S_{dd'}S_{ee'}$, we obtain values for $\langle B_{\lambda cc'}B_{\lambda dd'}\rangle$ and $\langle g_{\lambda e}g_{\lambda e'}B_{\lambda cc'}B_{\lambda dd'}\rangle$ as follows

$$\langle B_{\lambda cc'}B_{\lambda dd'}\rangle = \{\cosh(\alpha_c+\alpha_d) - \tfrac{2}{3}\sinh\alpha_c\sinh\alpha_d\}\delta_{cc'}\delta_{dd'} - \tfrac{1}{3}(\pi/D)^2\langle g_{\lambda c}g_{\lambda c'}g_{\lambda d}g_{\lambda d'}\rangle \tag{51a}$$

and

$$\langle g_{\lambda e}g_{\lambda e'}B_{\lambda cc'}B_{\lambda dd'}\rangle = \langle g_{\lambda e}g_{\lambda e'}\rangle\langle B_{\lambda cc'}B_{\lambda dd'}\rangle. \tag{51b}$$

Similarly, if we form the product $F^{(n)} S_{cc'} S_{dd'}$, we can obtain the averages of the derivatives $(\mathrm{d}/\mathrm{d}z)^n(B_{cc'} B_{dd'})$ at $z = \mathscr{E}_\lambda/D$. If we keep repeating the above procedure we can, in principle, evaluate the average of any expression of the form $(B_{\lambda cc'} B_{\lambda dd'} \dots B_{\lambda nn'})$.

The property that parameters belonging to different resonances are uncorrelated also allows us to simplify the expression for $\langle g_{\lambda c} g_{\lambda c'} S^*_{cc'}(\mathscr{E}^*_\lambda)\rangle$. For $c \neq c'$ the only nonzero term in the summation on the right-hand side of equation (38) is that for which $e = c$ and $f = c'$. Hence we have

$$\begin{aligned}(\pi/D)\langle g_{\lambda c} g_{\lambda c'} S^*_{cc'}(\mathscr{E}^*_\lambda)\rangle &= (\pi/D)\langle g^2_{\lambda c} g^2_{\lambda c'}/h^2_\lambda\rangle \langle U_{cc'cc'}\rangle \qquad (c \neq c') \\ &= -(\pi/D)\langle g^2_{\lambda c} g^2_{\lambda c'}/h^2_\lambda\rangle \langle U_{ccc'c'}\rangle\,, \end{aligned} \tag{52a}$$

whereas for $c = c'$ we obtain

$$(\pi/D)\langle g^2_{\lambda c} S^*_{cc}(\mathscr{E}^*_\lambda)\rangle = (\pi/D)\sum_d \langle g^2_{\lambda c} g^2_{\lambda d}/h^2_\lambda\rangle \langle U_{ccdd}\rangle\,. \tag{52b}$$

As indicated above, the values of $\langle U_{cc'dd'}\rangle$ are known; for example, when there are two open channels ($N = 2$) we have $\langle U_{1122}\rangle = 1$. From equation (46), for $N = 3$,

$$\langle U_{1122}\rangle = \cosh\alpha_3\,, \tag{53}$$

while from equation (51a), for $N = 4$,

$$\langle U_{1122}\rangle = \cosh(\alpha_3+\alpha_4) - \tfrac{2}{3}\sinh\alpha_3\sinh\alpha_4\,. \tag{54}$$

When the number of channels is larger, this method for evaluating $\langle U_{ccc'c'}\rangle$ becomes rather complicated, but can be simplified by using recurrence relations.

6. Recurrence Relations

We denote by $\tilde{S}$ the $m \times m$ matrix which results when we eliminate from the $N \times N$ S-matrix the $N-m$ rows and columns which intersect at $N-m$ different elements in the diagonal of S. Then, writing

$$\tilde{S} = \{1 + \mathrm{i}\tilde{G}_\lambda \tilde{B}^{-1}/(\mathscr{E}_\lambda - z)\}\tilde{B}\,, \tag{55}$$

we can express the determinant $\tilde{\varDelta}$ of $\tilde{S}$ as

$$\tilde{\varDelta} = \tilde{\varDelta}_B\{1 + \mathrm{i}(\operatorname{trace}\tilde{G}_\lambda \tilde{B}^{-1})/(\mathscr{E}_\lambda - z)\} = \tilde{\varDelta}_B + \mathrm{i}(\operatorname{trace}\tilde{G}_\lambda \tilde{V})/(\mathscr{E}_\lambda - z)\,, \tag{56}$$

where $\tilde{\varDelta}_B$ is the determinant of $\tilde{B}$, and $\tilde{V}$ is the matrix of cofactors of $\tilde{B}$. For the sake of simplicity we shall set $D = 1$ in equation (41), so that we obtain from equations (43) and (56)

$$F\tilde{\varDelta} = H\tilde{\varDelta}_B + \mathrm{i}\{H(\operatorname{trace} G_\lambda \tilde{V}) + \tilde{\varDelta}_B\}/(\mathscr{E}_\lambda - z) - (\operatorname{trace}\tilde{G}_\lambda \tilde{V})/(\mathscr{E}_\lambda - z)^2\,. \tag{57}$$

Since we have $H(\mathscr{E}_\lambda) = 0$, the residues a_λ of the poles at $z = \mathscr{E}_\lambda$ are given by

$$a_\lambda = \mathrm{i}\tilde{\varDelta}_B(\mathscr{E}_\lambda) + \operatorname{trace}\tilde{G}_\lambda \tilde{V}'_\lambda\,, \tag{58}$$

where $\tilde{V}'$ is the derivative of $\tilde{V}$ at $z = \mathscr{E}_\lambda$. Integrating equation (57) around the rectangular contour used in Section 5, we obtain

$$\langle \tilde{\Delta}_B(\mathscr{E}_\lambda)\rangle = \cosh\tilde{\alpha} + \mathrm{i}(\pi/D)\sum_{cc'}\langle g_{\lambda c}g_{\lambda c'}\tilde{V}'_{\lambda cc}\rangle$$

$$= \cosh\tilde{\alpha} + \mathrm{i}\sum_c \langle \tilde{V}'_{\lambda cc}\rangle \sinh\alpha_c. \tag{59}$$

Here $\tilde{\Delta}_B(\mathscr{E}_\lambda)$ is the determinant of the $m\times m$ matrix resulting from the elimination of $N-m$ rows and columns, corresponding to $N-m$ diagonal elements from B_λ, whereas $\tilde{V}_{cc}$ is the determinant of the $(m-1)\times(m-1)$ matrix which results from elimination of one additional row and column which correspond to the element B_{cc}.

Similarly, from the expression for $F^{(n)}\tilde{\Delta}$ we obtain the result

$$\langle \tilde{\Delta}_B^{(n)}(\mathscr{E}_\lambda)\rangle = \sum_c \sinh\alpha_c\{(n+1)^{-1}\langle \tilde{V}_{\lambda cc}^{(n+1)}\rangle - (-1)^n H_\lambda^{(n)}\langle \tilde{V}_{\lambda cc}\rangle\}. \tag{60}$$

Equations (59) and (60) with (45) enable us to evaluate the average $\langle U_{\lambda ccc'c'}\rangle$ which occurs in equations (52), for any number of channels. Starting with the two-channel case, for which $\langle \tilde{V}_{\lambda cc}\rangle = 1$ and $\langle \tilde{V}_{\lambda cc}^{(n)}\rangle = 0$, we find from equations (59) and (60), for $N = 3$,

$$\langle U_{\lambda 1122}\rangle = \langle \tilde{\Delta}_B(\mathscr{E}_\lambda)\rangle = \cosh\alpha_3 \tag{61}$$

and

$$\langle \tilde{\Delta}'_B(\mathscr{E}_\lambda)\rangle = \tfrac{1}{3}\mathrm{i}\sinh\alpha_3. \tag{62}$$

Then, for $N = 4$, equation (59) yields

$$\langle U_{\lambda 1122}\rangle = \langle \tilde{\Delta}_B(\mathscr{E}_\lambda)\rangle = \cosh(\alpha_3+\alpha_4) + \mathrm{i}(\sinh\alpha_3\langle \tilde{V}'_{\lambda 33}\rangle + \sinh\alpha_4\langle \tilde{V}'_{\lambda 44}\rangle). \tag{63}$$

However, $\tilde{V}_{33}$ is the determinant of the matrix which results when the rows and columns corresponding to the elements B_{11}, B_{22} and B_{33} are eliminated from the matrix B. Hence, in this case we have from equation (62)

$$\langle \tilde{V}'_{\lambda 33}\rangle = \tfrac{1}{3}\mathrm{i}\sinh\alpha_4,$$

so that equation (63) becomes

$$\langle U_{\lambda 1122}\rangle = \cosh\alpha_3\cosh\alpha_4(1+\tfrac{1}{3}t_3t_4), \tag{64}$$

where

$$t_c = \tanh\alpha_c. \tag{65}$$

If we continue this procedure, we find that

$$\langle U_{\lambda 1122}\rangle = A_{12}\prod_d(\cosh\alpha_d)/(\cosh\alpha_1\cosh\alpha_2), \tag{66}$$

where

$$\begin{aligned}
A_{12} &= 1, && \text{for } N=3;\\
&= 1+\tfrac{1}{3}t_3t_4, && N=4;\\
&= 1+\tfrac{1}{3}(t_3t_4+t_3t_5+t_4t_5), && N=5;\\
&= 1+\tfrac{1}{3}(t_3t_4+t_3t_5+t_3t_6+\ldots)+\tfrac{1}{5}t_3t_4t_5t_6, && N=6;\\
&= 1+\tfrac{1}{3}(t_3t_4+t_3t_5+\ldots)+\tfrac{1}{5}(t_3t_4t_5t_6+t_3t_4t_5t_7+\ldots), && N=7;
\end{aligned}$$

etc.

These results can be summarized by writing for N channels

$$\langle U_{\lambda ccc'c'}\rangle = A_{cc'} \prod_{d} (\cosh\alpha_d)/(\cosh\alpha_c \cosh\alpha_{c'}). \tag{67}$$

If we define a function $\mathscr{T}_{cc'}(x)$ as

$$\mathscr{T}_{cc'}(x) = \prod_{d\neq c,c'} (\tanh\alpha_d + x) \tag{68}$$

and denote the nth derivative of $\mathscr{T}$ with respect to x by $\mathscr{T}^{(n)}$, we can represent $A_{cc'}$ in the form

$$A_{cc'} = 1 + \sum_{k=1}^{M} \frac{1}{(2k+1)(N-2k-2)!} \mathscr{T}_{cc'}^{(N-2k-2)}(0), \tag{69}$$

where the upper limit M of the summation is given by

$$\begin{aligned} M &= \tfrac{1}{2}(N-2) \quad \text{when} \quad N \text{ even;} \\ &= \tfrac{1}{2}(N-3) \qquad\qquad\quad N \text{ odd.} \end{aligned}$$

7. Discussion

We have seen that an exact expression for the average cross section in terms of S-matrix parameters can be derived when there are two reaction channels. When more than two channels are open, formulae for average cross sections can be obtained in the case when the poles of the S-matrix are evenly spaced. However, in order to express these cross sections in terms of transmission coefficients we still require the value of $g_{\lambda c}^2 g_{\lambda c'}^2$. Since the effects of unitarity of the S-matrix on the statistical distribution of $g_{\lambda c}$ are extremely complicated, it is very difficult to make assumptions about the statistical properties of $g_{\lambda c}$ which are in accordance with unitarity. Notwithstanding this difficulty, if we assume that special cases exist for which

$$\langle g_{\lambda c}^2 g_{\lambda c'}^2\rangle = \langle g_{\lambda c}^2\rangle\langle g_{\lambda c'}^2\rangle, \tag{70}$$

the average cross section can be obtained from the transmission coefficients using equations (52), (67) and (14).

One example in which equation (70) holds is the three-channel picket fence model of Moldauer (1967*b*), when two channels are equivalent, e.g. $T_1 = T_3$. From equation (66) we find then

$$\sigma_{12}^{fl} = 2(\sinh\alpha_1 \sinh\alpha_2 \cosh\alpha_3)/\sinh\alpha.$$

This result is identical with that obtained by Moldauer (1967*b*) who used the notation $t_c = \tanh\frac{1}{2}\alpha_c$. The assumption of equation (70) also leads to a simple expression for $\langle g_{\lambda c}^2 S_{cc}^*(\mathscr{E}_\lambda^*)\rangle$ when $N > 3$. Using the unitarity relation

$$\langle g_{\lambda c}^2 S_{cc}^*(\mathscr{E}_\lambda^*)\rangle = \sum_{c'\neq c} \langle g_{\lambda c} g_{\lambda c'} S_{cc'}^*(\mathscr{E}_\lambda^*)\rangle, \tag{71}$$

we can verify from equations (52) and (67) that

$$(\pi/D)\langle g_{\lambda c}^2 S_{cc}^*(\mathscr{E}_\lambda^*)\rangle = \{\sinh\alpha_c \sinh(\alpha-\alpha_c)\}/\sinh\alpha. \tag{72}$$

On the other hand, from the pole expansions of Δ and S_{cc}^* we find

$$\Delta S_{cc}^* = \cosh(\alpha-\alpha_c) + \mathrm{i}\sum_\lambda h_\lambda S_{cc}^*(\mathscr{E}_\lambda^*)/(\mathscr{E}_\lambda - z). \tag{73}$$

Integration around a rectangular contour with vertical sides extending from $\mathrm{Im}(z) = +\mathrm{i}W$ to $-\mathrm{i}W (W \gg \bar{\mathscr{Y}})$ yields

$$(\pi/D)\langle h_\lambda^2 S_{cc}^*(\mathscr{E}_\lambda^*)\rangle = \sinh(\alpha-\alpha_c). \tag{74}$$

Since we are considering S-matrices in which h_λ^2 does not fluctuate, equations (72) and (74) imply

$$\langle g_{\lambda c}^2 S_{cc}^*(\mathscr{E}_\lambda^*)\rangle = \langle g_{\lambda c}^2\rangle\langle S_{cc}^*(\mathscr{E}_\lambda^*)\rangle. \tag{75}$$

In general, however, equation (70) does not hold and in order to evaluate $\langle g_{\lambda c}^2 g_{\lambda c'}^2\rangle$ one may have to resort to the R-matrix formalism, which has the advantage that the statistics of the parameters may be specified without being restricted by unitarity.

References

Agassi, D., Weidenmüller, H. A., and Mantzouranis, G. (1975). *Phys. Lett.* C **22**, 147.
Engelbrecht, C. A., and Weidenmüller, H. A. (1973). *Phys. Rev.* C **8**, 859.
Feshbach, H., Thomas, R. G., and Weisskopf, V. F. (1953). *Phys. Rev.* **90**, 166.
Hauser, W., and Feshbach, H. (1952). *Phys. Rev.* **87**, 366.
Hofmann, H. M., Richert, J., Tepel, J. W., and Weidenmüller, H. A. (1975). *Ann. Phys. (New York)* **90**, 403.
van Kampen, N. G. (1953). *Phys. Rev.* **91**, 1267.
Kawai, M., Kerman, A. K., and McVoy, K. W. (1973). *Ann. Phys. (New York)* **75**, 156.
Lane, A. M., and Lynn, J. E. (1957). *Proc. Phys. Soc. London* A **70**, 577.
Moldauer, P. A. (1964). *Phys. Rev.* B **135**, 642.
Moldauer, P. A. (1967*a*). *Phys. Rev. Lett.* **19**, 1047.
Moldauer, P. A. (1967*b*). *Phys. Rev.* **157**, 907.
Moldauer, P. A. (1969). *Phys. Rev. Lett.* **23**, 708.
Moldauer, P. A. (1975). *Phys. Rev.* C **11**, 426.
Ning Hu (1948). *Phys. Rev.* **74**, 131.
Simonius, M. (1974*a*). *Nucl. Phys.* A **218**, 53.
Simonius, M. (1974*b*). *Phys. Lett.* B **52**, 279.
Tepel, J. W., Hofmann, H. M., and Weidenmüller, H. A. (1974). *Phys. Lett.* B **49**, 1.
Thomas, R. G. (1955). *Phys. Rev.* **97**, 244.
Ullah, N. (1969). *Phys. Rev. Lett.* **22**, 648.

Manuscript received 4 August 1977