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Analytic Models of High-β Reversed Field Pinches

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Abstract

A class of simple analytic models for equilibria neighbouring Taylor's force-free states has been examined. It is shown that some members of this class are high-β equilibria which satisfy Suydam's criterion and Robinson's condition for magnetohydrodynamic stability.

1. Introduction

In order that a given amount of magnetic energy (supplied by externally driven currents) be most efficiently used for plasma confinement, the plasma β (ratio of plasma pressure to total magnetic pressure) should be as high as the gross stability of the system allows. On account of the high magnetic shear provided by the reversal of the axial component of the magnetic field in the outer regions of a plasma column, the reversed field pinch has good stability properties and it is one of the most promising devices for high-β ($\beta > 0\cdot 1$), stable, magnetic confinement of a hot plasma. Numerical models of stable high-β ($\beta \approx 0\cdot 3$) reversed field configurations have been found by Robinson and King (1968) and Robinson (1971) in the neighbourhood of the force-free paramagnetic model (Kadomtsev 1962). However, Taylor (1974*a*, 1974*b*) has shown that, in the completely relaxed state (accomplished by field line reconnections), the cylindrically symmetric state of minimum potential energy is the pressureless force-free state described by the Bessel function model: $B_\theta = B_0\,\mathrm{J}_1(\mu r)$ and $B_z = B_0\,\mathrm{J}_0(\mu r)$, where B_0 and μ are constants and the other symbols have their usual meanings. These configurations have been shown to be magnetohydrodynamically (MHD) stable (Voslamber and Callebaut 1962) provided $\mu b \leqslant 3\cdot 17$, where b is the radius of the conducting wall enclosing the plasma column; otherwise, the states of minimum energy are helically symmetric states (Taylor 1975). The importance of Bessel function configurations for magnetic plasma confinement arises from the fact that they can be deformed a finite amount by plasma pressure before MHD instabilities occur. Newton *et al.* (1975) have found stable high-β models by a process of numerical relaxation, starting from the Bessel function model. The resulting configurations, however, appear to depart markedly from the Bessel function model.

The purpose of this paper is to examine a class of deformations which have simple analytic representations and to obtain some properties of the resultant equilibria neighbouring the Taylor states. It will be shown that there exist high-β configurations which satisfy the Suydam (1958) criterion and other more stringent criteria (Robinson 1971) for MHD stability.

In the next section, we introduce the basic model and derive from it expressions describing physical properties of experimental interest. Examples of some configurations are presented in detail in Section 3. We conclude with a brief discussion in Section 4.

2. Basic Model

About any given equilibrium, there are an infinite number of possible neighbouring equilibria. Of all the admissible deformations of the Taylor states, we consider a class which is representable by the simplest polynomials:

$$B_\theta = B_0\{\mathrm{J}_1(\mu r)-\varepsilon\alpha\mu r\}, \qquad B_z = B_0\{\mathrm{J}_0(\mu r)-\varepsilon\mu a(1+r^2/a^2)\}, \tag{1a, b}$$

where we have $0 \leqslant \varepsilon \ll 1$, α is a constant of the order of unity and a is the radius of the plasma column. The possible choices of ε, α and μa are restricted both by stability conditions and by the requirement that the resultant distributions are physically realistic. The polynomial deformation in equation (1b) can be generalized to $C_1+C_2 r^2/a^2$ without much more complication to the subsequent algebra. However, numerical search seems to indicate that stable configurations with highest β occur at $C_1 \approx C_2$. From equation (1b), it can be seen that axial field reversal occurs provided we have $J_0(\mu a)-2\varepsilon\mu a < 0$. The above magnetic field configuration in the plasma column is continued across the plasma boundary at $r = a$ into a vacuum region $a \leqslant r \leqslant b$ with fields given by

$$B_\theta = (aB_0/r)\{\mathrm{J}_1(\mu a)-\varepsilon\alpha\mu a\}, \qquad B_z = B_0\{\mathrm{J}_0(\mu a)-2\varepsilon\mu a\}. \tag{2a, b}$$

A perfectly conducting wall is located at $r = b$.

From the Maxwell equation (in SI units) $\mu_0 \boldsymbol{j} = \nabla\times\boldsymbol{B}$, the nonvanishing components of the current densities read

$$j_\theta = (\mu B_0/\mu_0)\{\mathrm{J}_1(\mu r)+2\varepsilon r/a\}, \qquad j_z = (\mu B_0/\mu_0)\{\mathrm{J}_0(\mu r)-2\varepsilon\alpha\}, \tag{3a, b}$$

where μ_0 is the permeability of free space. The condition for pressure-balanced equilibrium, namely $\nabla p = \boldsymbol{j}\times\boldsymbol{B}$, together with the boundary condition $p(a) = 0$ yields

$$\begin{aligned} p(r) = \varepsilon(B_0^2/\mu_0)\{&\mathrm{J}_0(\mu r)(\mu a-2\alpha+\mu r^2/a)-2\mathrm{J}_0(\mu a)(\mu a-\alpha) \\ &+\alpha\mu\big(r\,\mathrm{J}_1(\mu r)-a\,\mathrm{J}_1(\mu a)\big)+\varepsilon\mu^2(\alpha^2+1)(a^2-r^2)+(\varepsilon\mu^2/2a^2)(a^4-r^4)\}. \end{aligned} \tag{4}$$

The total potential energy W of the system is given by

$$W = W_p + W_B + W_v, \tag{5}$$

where

$$W_p = \int \frac{p\,\mathrm{d}\tau}{\gamma-1}, \qquad W_B, W_v = \int \frac{B^2\,\mathrm{d}\tau}{2\mu_0},$$

W_p being the internal thermal energy of the plasma (with γ designating the ratio of specific heats) and W_B and W_v being respectively the magnetic energies in the plasma and vacuum regions. On simulation of a torus of large aspect ratio as a cylinder of length $2\pi R$, where R is the major radius of the torus, straightforward calculations show

$$W_p = \frac{\varepsilon\pi^2 R(B_0^2/\mu_0)}{3\mu^2(\gamma-1)}\Big(6\mathrm{J}_1(\mu a)(4\mu^2a^2-8-\alpha\mu^3a^3) + 12\mu a\,\mathrm{J}_0(\mu a)(2-\mu^2a^2)+\varepsilon\mu^4a^4(3\alpha^2+5)\Big), \tag{6a}$$

$$W_B = \frac{\pi^2 R(B_0^2/\mu_0)}{6\mu^2}\Big(12\mu^2a^2\{\mathrm{J}_0^2(\mu a)+\mathrm{J}_1^2(\mu a)\} - 12\mu a\,\mathrm{J}_1(\mu a)\,\mathrm{J}_0(\mu a) + 24\varepsilon\mu a\,\mathrm{J}_0(\mu a)(\alpha\mu a-1) - 48\varepsilon\,\mathrm{J}_1(\mu a)\{\alpha\mu a+\mu^2a^2-2\} + \varepsilon^2\mu^4a^4(3\alpha^2+14)\Big), \tag{6b}$$

$$W_v = a^2\pi^2 R(B_0^2/\mu_0)\big(\{\mathrm{J}_1(\mu a)-\varepsilon\alpha\mu a\}^2\ln\theta_c^2 + \{\mathrm{J}_0(\mu a)-2\varepsilon\mu a\}^2(\theta_c^2-1)\big), \tag{6c}$$

where $\theta_c = b/a$ is the compression ratio. The total axial magnetic flux Ψ can be shown to be

$$\Psi = 2\pi\int B_z r\,\mathrm{d}r = \frac{\pi a B_0}{2\mu}\Big(4\mathrm{J}_1(\mu a) - 3\varepsilon\mu^2a^2 + 2\mu a(\theta_c^2-1)\{\mathrm{J}_0(\mu a)-2\varepsilon\mu a\}\Big). \tag{7}$$

Since the axial flux Ψ is trapped inside the perfectly conducting shell, it is an invariant for a given experimental external current set-up. Another invariant of experimental interest is the quantity

$$K_0 = \int \boldsymbol{A}\,.\,\boldsymbol{B}\,\mathrm{d}\tau,$$

where $\boldsymbol{A}$ is the magnetic vector potential (Taylor 1974*a*).

A solution of $\nabla\times\boldsymbol{A} = \boldsymbol{B}$, where $\boldsymbol{B}$ is given by equations (1), reads

$$A_\theta = (B_0/\mu)\{\mathrm{J}_1(\mu r) - \varepsilon\mu^2 a(\tfrac{1}{2}r + \tfrac{1}{4}r^3/a^2)\}, \tag{8a}$$

$$A_z = (B_0/\mu)\{\mathrm{J}_0(\mu r) - \mathrm{J}_0(\mu a) + \tfrac{1}{2}\varepsilon\alpha\mu^2(r^2-a^2)\} + aB_0\ln\theta_c\{\mathrm{J}_1(\mu a)-\varepsilon\alpha\mu a\}, \tag{8b}$$

for the plasma region; the corresponding solution for the vacuum region is

$$A_\theta = (B_0/2r)(r^2-a^2)\{\mathrm{J}_0(\mu a)-2\varepsilon\mu a\} + (aB_0/\mu r)\{\mathrm{J}_1(\mu a)-\tfrac{3}{4}\varepsilon\mu^2a^2\}, \tag{9a}$$

$$A_z = aB_0\ln(b/r)\{\mathrm{J}_1(\mu a)-\varepsilon\alpha\mu a\}. \tag{9b}$$

The integration constants have been chosen in such a way that the tangential components of $\boldsymbol{A}$ are continuous across the plasma–vacuum interface and

$$\int_0^{2\pi}\int_0^b B_z r\,\mathrm{d}r\,\mathrm{d}\theta = \Psi, \qquad A_z(b) = 0. \tag{10}$$

Carrying out the appropriate integrations, we find

$$K_0 = (4\pi^2 R B_0^2/\mu^3)(k_0+k_1+k_2+k_3), \tag{11}$$

where

$$k_0 = \mu^2a^2\,\mathrm{J}_1^2(\mu a)\{1+\ln\theta_c^2\} + \mu^2a^2\,\mathrm{J}_0(\mu a) + \mu a\,\mathrm{J}_0(\mu a)\,\mathrm{J}_1(\mu a)\{\tfrac{1}{2}\mu^2a^2\Omega-2\}, \tag{12a}$$

$$k_1 = \tfrac{1}{2}\varepsilon\mu a\,\mathrm{J}_0(\mu a)\{3\mu^2a^2-8+4\alpha\mu a-\mu^3a^3\Omega\}, \tag{12b}$$

$$k_2 = -\tfrac{1}{2}\varepsilon\,\mathrm{J}_1(\mu a)\{8\mu^2a^2-16+8\alpha\mu a+\mu^3a^3(3\mu a+4\alpha)\ln\theta_c+2\mu^4a^4\Omega\}, \tag{12c}$$

$$k_3 = \tfrac{1}{6}\varepsilon^2\alpha\mu^5a^5\{2+9\ln\theta_c+6\Omega\}, \tag{12d}$$

with $\Omega = \theta_c^2-1-\ln\theta_c^2$. The quantity K_0/Ψ^2 can be interpreted as (stored volt-seconds) per (axial flux) and, from equations (7) and (11), it reads

$$K_0/\Psi^2 = 16\mu R(k_0+k_1+k_2+k_3)/\psi^2, \tag{13}$$

where we have defined $\psi = 2\mu^2\Psi/\pi B_0$.

There are two further quantities of experimental interest. These are the pinch parameter Θ, defined by

$$\Theta = B_\theta(b)/\langle B_z\rangle = \pi b^2B_\theta(b)/\Psi(b), \tag{14a}$$

and the field reversal ratio F_{R}, defined by

$$F_{\mathrm{R}} = B_z(b)/\langle B_z\rangle = \pi b^2B_z(b)/\Psi(b), \tag{14b}$$

where angle brackets denote an averaging over the cross section of the cylinder. Using equations (2) and (7) in the definitions (14), we find

$$\Theta = \mu a\theta_c\{\mathrm{J}_1(\mu a)-\varepsilon\alpha\mu a\}/D, \qquad F_{\mathrm{R}} = \mu a\theta_c^2\{\mathrm{J}_0(\mu a)-2\varepsilon\mu a\}/D, \tag{15}$$

with

$$D = 2\mathrm{J}_1(\mu a)-\tfrac{3}{2}\varepsilon\mu^2a^2+\mu a(\theta_c^2-1)\{\mathrm{J}_0(\mu a)-2\varepsilon\mu a\}.$$

3. Examples

A necessary condition for MHD stability (Suydam 1958) is that the function $F_{\mathrm{S}}(r)$ defined by

$$F_{\mathrm{S}}(r) = (B_\theta\,\mathrm{d}P/\mathrm{d}r)^2+8r\mu_0\,\mathrm{d}p/\mathrm{d}r, \tag{16}$$

where $P = rB_z/B_\theta$, be positive definite everywhere in the interval $0 < r \leqslant a$. Analysis of the central region ($r \approx 0$) shows that $B_\theta\,\mathrm{d}P/\mathrm{d}r = O(r^2)$, whilst $\mathrm{d}p/\mathrm{d}r = O(r)$. Hence the Suydam condition is satisfied only if near the axis we have $\mathrm{d}p/\mathrm{d}r > 0$, which entails the necessary condition $4\alpha > \mu a-4/\mu a$. In any case, explicit evaluation of $F_{\mathrm{S}}(r)$ and other field distributions shows that there are continuous ranges of the parameters ε, α, and μa for which configurations conforming to the equations (1) represent Suydam stable high-β equilibria. Among these, we illustrate the physical properties of one with a very high averaged β ($\beta_{\mathrm{av}} = 1\cdot18$) given by the parametric values $\varepsilon = 0\cdot075$, $\alpha = 0\cdot67$ and $\mu a = 3\cdot0$, where we define

$$\beta_{\mathrm{av}} = \int_0^a (2\mu_0p/B^2a)\,\mathrm{d}r.$$

On account of the large axial field reversal (see Fig. 1*a*), the magnitude of the pitch of the magnetic field at the boundary of the column $|P(a)|$ is much larger than at the centre $P(0)$ (Fig. 1*b*). Should there be a vacuum region ($a \leqslant r \leqslant b$) outside the plasma column, the configuration will be unstable to the current-driven nonlocal kink

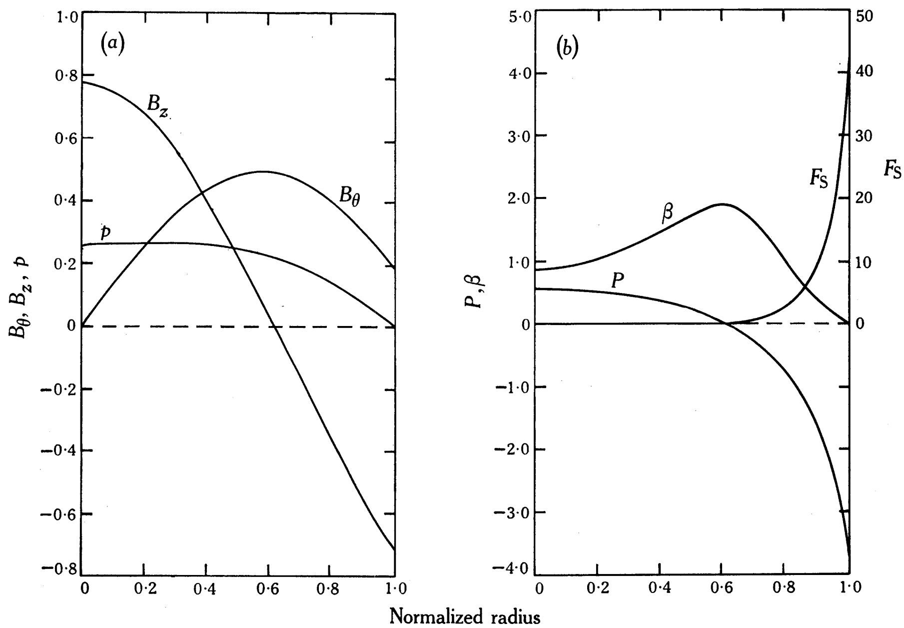

Fig. 1. Profiles of (*a*) the magnetic field B_θ and B_z normalized to B_0 and the pressure p normalized to B_0^2/μ_0; and (*b*) the Suydam factor F_S normalized to B_0^2, the plasma ratio β and the pitch P normalized to the plasma radius a. All curves are for the parameter values $\varepsilon = 0{\cdot}075$, $\alpha = 0{\cdot}67$ and $\mu a = 3{\cdot}0$.

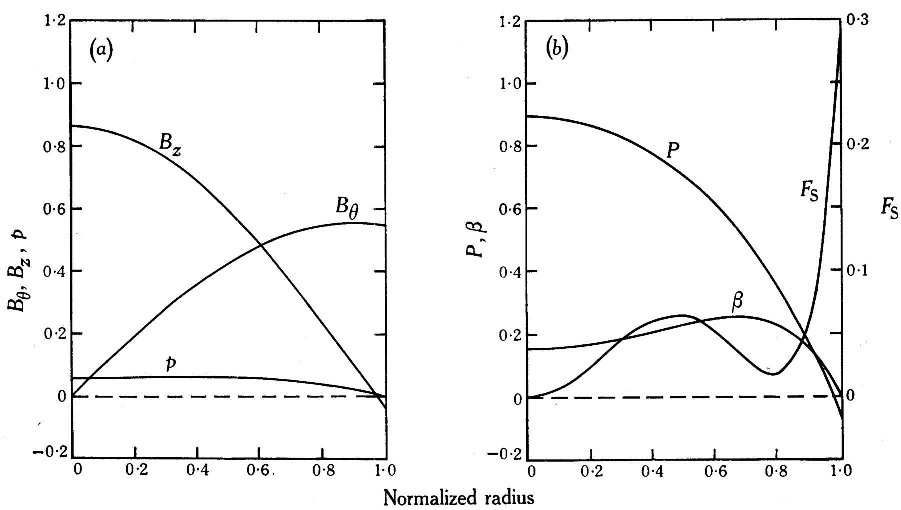

Fig. 2. Profiles of (*a*) B_θ, B_z and p, and (*b*) F_S, β and P as in Fig. 1 except for the case $\varepsilon = 0{\cdot}066$, $\alpha = 0{\cdot}23$ and $\mu a = 2{\cdot}0$.

mode in the wavenumber range specified by $|P(b)|^{-1} < k < |P(a)|^{-1}$, since a necessary condition for stability (Robinson 1971), namely

$$P(b) > -3P(0), \tag{17}$$

is not satisfied. Realistic sufficient conditions for stability do not appear to be known at present. However, Robinson has stated that a good guide for stability of the type of configuration under consideration here is that

$$P(0) > |P(b)|. \tag{18}$$

Since we have $|P(b)| = \theta_c^2 |P(a)|$ for a reversed field pinch, it can be seen that the inequality (18) is more stringent for large compression ratios. Nevertheless, high-β Suydam stable configurations satisfying the condition (18) have been found for θ_c as large as $2 \cdot 5$. Figs 2*a* and 2*b* illustrate an example with $\beta_{av} = 0 \cdot 19$ given by the parameters $\varepsilon = 0 \cdot 066$, $\alpha = 0 \cdot 23$ and $\mu a = 2 \cdot 0$.

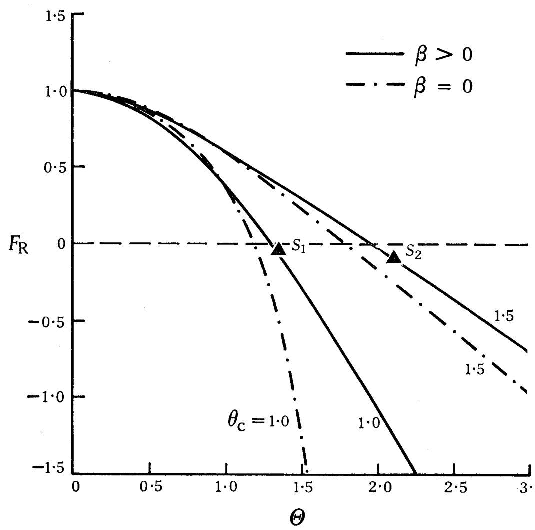

Fig. 3. Relationship between the field reversal ratio F_R and the pinch ratio Θ for the case $\varepsilon = 0 \cdot 066$, $\alpha = 0 \cdot 41$ and two compression ratios $\theta_c = 1 \cdot 0$ and $1 \cdot 5$. The solid triangles S_1 and S_2 indicate Suydam stable configurations with $\mu a = 2 \cdot 0$ and $\beta_{av} = 0 \cdot 14$.

In Fig. 3, the relationship between the field reversal ratio F_R and the pinch parameter Θ, defined by equations (14) and (15), is shown for the case $\varepsilon = 0 \cdot 066$, $\alpha = 0 \cdot 41$ and two compression ratios $\theta_c = 1 \cdot 0$ and $1 \cdot 5$. The two solid triangles S_1 and S_2 in the figure indicate Suydam stable configurations with $\mu a = 2 \cdot 0$ and $\beta_{av} = 0 \cdot 14$. Since β_{av} is the β value averaged over the plasma column, the configuration represented by S_2 actually has a lower average β than that of S_1, provided the average is taken over the whole tube. This observation indicates that the position of a configuration on the Θ–F_R diagram depends not only on β_{av} but also on θ_c, and that the deviations from the Taylor states, which are represented by the curve on the extreme left in Fig. 3, do not necessarily indicate the size of β_{av}. Moreover, for any given ε and α not all points on the Θ–F_R curves represent physically realizable configurations, as they might imply unrealistic (e.g. negative) potential energies or pressure distributions, which can be calculated from equations (4)–(6). The invariants Ψ and K_0 also limit the accessibility to certain regions of the Θ–F_R space.

4. Discussion

The examples given in Figs 1 and 2 above are extreme cases for the purpose of illustration only. Numerical searches indicate that there are an infinite number of high-β equilibria of the general form given by equations (1), satisfying Suydam's (1958) criterion (16) and Robinson's (1971) condition (17). Many of these have field profiles which closely resemble those obtained experimentally (see e.g. Bodin 1975) and they satisfy the stability guideline (18) which indicates likely stability. However, without a complete stability analysis, it has not been possible to specify exactly the critical wall positions for stability with respect to the ideal MHD kink mode and the resistive tearing mode. The object of a forthcoming report is to present the needed stability analysis. The example given above (see Fig. 3) also shows the significant effect that the vacuum region has on the field reversal ratio and the pinch parameter.

In the pressureless case ($\varepsilon = 0$), without a vacuum region ($\theta_c = 1$), the invariants K_0 and Ψ can be used (Taylor 1974*b*) to determine μa and hence the pinch parameter Θ and field reversal ratio F_R. From equations (7), (11) and (12), it can be seen that both finite pressure effects and a vacuum region complicate this simple procedure.

The transition from the cylindrically symmetric state to the helical state has yet to be discussed for a finite-β plasma. Predictions for such transitions are made on the basis of the principle of minimum potential energy. The expressions (6a)–(6c) derived in Section 2 here should be valuable for this purpose.

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