Aust. J. Phys., 1978, **31**, 219–31

Quantum Electrodynamics of Particles with Arbitrary Spin

H. S. Green

Department of Mathematical Physics, University of Adelaide, P.O. Box 498, Adelaide, S.A. 5001.

Abstract

A generalization of quantum electrodynamics is developed for particles of higher spin, with careful attention to the requirements of consistency, causality, unitarity and renormalizability. It is shown that field equations studied previously by the author are expressible in arbitrarily many different forms, which are equivalent in the absence of electromagnetic interactions, but not when electromagnetic coupling is introduced in a gauge-invariant way. A form is chosen which satisfies the requirements of causality. It is shown how to define a particle density, which is positive-definite in the subspace spanned by solutions of the field equation, and satisfies a Lorentz-invariant conservation law. The quantization and renormalization of the resulting electrodynamics is studied, and is found to require only minor modifications of the existing theory for particles of spin $\frac{1}{2}$.

1. Introduction

The quantum electrodynamics of particles of spin $\frac{1}{2}$ is, at present, certainly the most successful and perhaps the only entirely satisfactory application of quantized field theory. Attempts to generalize the theory for charged particles of higher spin were first made at a time when the physical existence of such particles had not yet been demonstrated experimentally. They met with immediate difficulties, usually manifested as violations of the requirements of self-consistency, renormalizability, causality and unitarity. It was found that, even for particles of spin 1, the perturbation theory generated a series of primitive divergences, which could not be removed by the renormalization technique of Schwinger (1948*a*, 1948*b*) and Dyson (1949*a*, 1949*b*). Lee and Yang (1962) introduced a 'ξ-limiting' formalism for the elimination of these divergences, which amounts to the introduction of intermediate particles of negative energy and is therefore strictly inconsistent with causality or unitarity. As Tucker and Hammer (1971) found, this difficulty is not essentially related to the problem of the compatibility of supplementary conditions. Various attempts have been made recently (see e.g. Bellissard and Seiler 1972; Eeg 1972; Prabhakaran *et al.* 1975; Senjanović 1977) to develop a quantum electrodynamics of particles of spin 3/2; but, as pointed out by Singh and Hagen (1977), the introduction of interactions for such particles has led invariably to serious inconsistencies. The same is true, *a fortiori*, for theories of particles of still higher spin, whether based on the equations proposed by Harish-Chandra (1947) and Shamaly and Capri (1971), by Rarita and Schwinger (1941), by Weinberg (1964), or by Gel'fand and Yaglom (1948; see also Gel'fand *et al.* 1963). The last are the most general, and, as might be expected, exhibit a greater variety of pathological features, noticed by Amar and Dozzio (1972) and Cox (1974*a*, 1974*b*).

Some of the problems encountered in the quantum electrodynamics of particles of higher spin are clearly related to those affecting the interaction of such particles with unquantized fields. Adopting the equations of Rarita and Schwinger (1941), Velo and Zwanzinger (1969*a*, 1969*b*) showed that such interaction leads to acausal wave wave propagation. Troubles with causality, or non-unitary evolution in time, were found by Schroer *et al.* (1970) with a wider class of theories, including those which, like Weinberg's (1964), are restricted to a single representation of the Lorentz group. However, there are also apparently unrelated problems, which appear in attempts to quantize fields for particles of spin 3/2 or more, even in the absence of electromagnetic interactions. Johnson and Sudarshan (1961) pointed to the fact that quantization in accordance with Fermi statistics leads to a contradiction for a wide variety of field theories of higher half-odd-integral spin, in spite of Pauli's theorem that this is the only kind of statistics available for such theories. The contradiction is associated with the difficulty in defining a particle density which is positive-definite for particles of half-odd-integral spin 3/2 or more, and there is a corresponding difficulty in defining an energy density which is positive-definite for particles of higher integral spin.

The combined effect of the previous negative results is to leave only a few possibilities for the development of an acceptable quantum electrodynamics of particles of higher spin. There is the possibility that such particles cannot be regarded as truly elementary and that some sort of nonlocal theory will be required; this will not be considered here. There is also the possibility that the problems of quantization and renormalization might need to be solved within the context of the non-abelian gauge groups which have recently become the subject of a very large literature (see e.g. Costa and Tonin 1975). However, this does not yet relate to particles of higher spin and, if we wish to consider only the electromagnetic interactions of a single kind of particle, it would seem reasonable to insist on gauge invariance only in this restricted domain. Finally, there is the possibility that the field equations and electromagnetic couplings hitherto adopted for particles of higher spin are at fault. This last suggestion is investigated here, where a generalization of quantum electrodynamics is developed, based on the field equations for particles of higher spin studied recently by the author (Green 1977). It will appear that, in the absence of interactions, these equations can be written in an infinite variety of equivalent forms; but that, when electromagnetic interactions are introduced in the usual gauge-invariant way, the different forms are no longer equivalent. The problem is thus reduced to choosing a form which is consistent with the requirements of causality. It will be shown that this can be done, at least at the semiclassical level, so that difficulties of the type studied by Velo and Zwanzinger (1969*a*, 1969*b*) do not arise. The remaining problems are of the type considered by Johnson and Sudarshan (1961), but it will appear that these are of the comparatively mild variety encountered in ordinary quantum electrodymanics, for which several techniques are available.

An important question, which remains to be answered, is whether the renormalization constants appearing in the theory are finite. It is open to doubt whether any theory with divergent renormalization constants is truly unitary, so this could well be crucial to the ultimate validity of the generalized, as well as the ordinary, quantum electrodymanics. In connection with the latter, Johnson *et al.* (1964, 1967), Adler (1972) and Blaha (1974) have pointed to the possibility that, in the Landau gauge, all the renormalization constants could be made finite, provided that the unrenormalized mass of the particle of spin 1/2 is zero. Das and Freedman (1976) have argued independently that the theory of massless particles of spin 3/2 is free of causal difficulties. The idea that the mass of the electron and other charged particles could be wholly or partly electromagnetic in origin is an old one, and it is interesting to witness its revival in modern quantum electrodynamics. It is hoped to discuss this and other outstanding questions in quantitative terms in a subsequent publication.

2. Field Equations with Electromagnetic Interactions

As shown in the earlier paper (Green 1977), particles of spin s can be represented by solutions of the equation

$$\alpha . p\,\phi = s\mu\phi, \tag{1}$$

where p is the energy–momentum four-vector and $\mu^2 = p^\lambda p_\lambda$, so that μ is the mass in the absence of external interactions. The components α_λ of the four-vector α are Bhabha matrices, satisfying the commutation relations

$$[\alpha_\lambda, \alpha_\mu] = \alpha_{\lambda\mu}, \qquad [\alpha_\lambda, \alpha_{\mu\nu}] = g_{\lambda\mu}\alpha_\nu - g_{\lambda\nu}\alpha_\mu . \tag{2}$$

But, to ensure that the mass is single valued, it is necessary to specify that α_λ and $\alpha_{\mu\nu}$ are generators of a de Sitter group in an irreducible representation labelled (s,s), in terms of highest weights. One way of doing this is to introduce a pseudovector with matrix components β_λ $(= \mathrm{i}\varepsilon_\lambda/(s+1)$, in the notation of the earlier paper), and the pseudoscalar matrix s', defined by

$$\beta_\lambda = \tfrac{1}{2}\mathrm{i}\varepsilon_{\lambda\mu\nu\rho}\,\alpha^{\mu\nu}\alpha^\rho, \qquad s' = -\tfrac{1}{8}\mathrm{i}\varepsilon_{\lambda\mu\nu\rho}\,\alpha^{\lambda\mu}\alpha^{\nu\rho}. \tag{3}$$

The representation (s,s) is then distinguished by the property that $\alpha . p/\mu$, $-\mathrm{i}\beta . p/\mu$ and s' are generators of $(2s+1)$-dimensional representations of SO(3). Finite-dimensional representations of the Lorentz group within the spin algebra are labelled (s,s') in terms of highest weights, and s' has the eigenvalues $-s, -s+1, ..., s$. For convenience, special algebraic relations satisfied by the α_λ and β_λ matrices are derived in the Appendix.

Since $s\mu$ is the maximum eigenvalue of $\alpha . p$ in the representation adopted (as shown in the Appendix), $\beta . p - s'\mu$ increases the eigenvalue of $\alpha . p$ by μ; it follows from equation (1) that

$$\beta . p\,\phi = s'\mu\phi. \tag{4}$$

Also, if θ is any pseudoscalar,

$$(\alpha + \theta\beta) . p\,\phi = (s + \theta s')\mu\phi. \tag{5}$$

Thus, ϕ satisfies not one but a continuum of equations, which are all equivalent in the absence of external interactions. However, they are no longer equivalent when electromagnetic interactions are considered! To satisfy the requirements of gauge invariance, p_λ must be replaced by $p_\lambda - eA_\lambda$ in the presence of an electromagnetic field with potentials A_λ; but as, in the quantum theory, p_λ and A_μ do not commute, this can be done consistently only for a particular or restricted choice of the pseudoscalar θ in equation (5). Our initial problem is therefore to choose θ in such a way that the resulting equation

$$(\alpha + \theta\beta) . (p - eA)\phi = (s + \theta s')m\phi, \tag{6}$$

for particles with unrenormalized mass m, is consistent with the principle of causality.

One way, and probably the only satisfactory way (see Hurley 1974), to secure a causal theory is to ensure that the electromagnetic field links only neighbouring pairs of representations of the Lorentz group, labelled (s, s') and $(s, s'-1)$, for $s' = s, s-2, \ldots$, or (for integral spin) $s' = -s+1, -s+3, \ldots$. To do this, we note that $\alpha_\lambda \pm \beta_\lambda$ are matrices connecting neighbouring representations, so that θ must satisfy

$$\theta^2 = 1, \qquad \theta\alpha_\lambda + \alpha_\lambda \theta = 0. \tag{7}$$

There are just two matrices satisfying these requirements, which differ only in sign; they are given by

$$\theta = \pm \exp\{i\pi(s-s')\}, \tag{8}$$

or an equivalent expression involving a polynomial in s'. For half-odd-integral spin, it is easy to see that the positive sign must be chosen, as otherwise components of ϕ in the representations of the Lorentz group with $s' = \pm s$ are not determined by equation (6). For instance, when $s = \frac{1}{2}$ we have $\alpha_\lambda = \frac{1}{2}\gamma_\lambda$, $\beta_\lambda = \frac{1}{2}\gamma_5 \gamma_\lambda$ and $s' = \frac{1}{2}\gamma_5$, in terms of Dirac matrices, and we must take

$$\theta = \exp\{\tfrac{1}{2}i\pi(1-\gamma_5)\} = \gamma_5$$

to obtain Dirac's equation. But, for integral spin, both signs in equation (8) are needed to represent particles of opposite helicities and to conserve parity, and, moreover, the sign must change under parity conjugation. Thus, for $s = 1$, the α_λ are the 10-dimensional Kemmer matrices and

$$\theta = \pm \exp\{i\pi(1-s')\} = \pm(2s'^2 - 1).$$

The easiest way to accommodate both signs is to adopt a reducible $2(2s+1)^2$-dimensional representation for particles of integral spin s, so that equation (8) assumes the form

$$\theta = \theta_0 \exp\{i\pi(s-s')\}, \tag{9}$$

where θ_0 has eigenvalues ± 1 in the extended representation, when s is integral; but $\theta_0 = 1$ otherwise. With θ chosen in this way, we rewrite equation (6) in the form

$$\gamma . (p-eA)\phi = s_1 m\phi, \tag{10}$$

with

$$\gamma_\lambda = \alpha_\lambda + \theta\beta_\lambda, \qquad s_1 = s + \theta s'.$$

This notation is justified by the fact that the γ_λ are the Dirac matrices when $s = \frac{1}{2}$ and, as shown in the Appendix, for general spin s they satisfy anticommutation relations of the type

$$\gamma_\lambda \gamma_\mu + \gamma_\mu \gamma_\lambda = 2h_{\lambda\mu}, \tag{11}$$

where

$$h_{\lambda\mu} = \{s(s+2) + \theta s'\} g_{\lambda\mu} - \alpha_\lambda{}^\nu \alpha_{\nu\mu} + \alpha_{\lambda\mu}. \tag{12}$$

The matrix $h_{\lambda\mu}$ has the property that, if n_λ is any time-like unit four-vector,

$$h_{\lambda\mu} n^\lambda n^\mu = -\sigma^\lambda \sigma_\lambda - \theta s'(\theta s' - 1), \qquad \sigma_\lambda = \tfrac{1}{2} i \varepsilon_{\lambda\mu\nu\rho} \alpha^{\mu\nu} n^\rho. \tag{13}$$

Here σ_λ is the spin four-vector, when $n_\lambda = p_\lambda/\mu$, so $-\sigma^\lambda\sigma_\lambda$ has eigenvalues of the form $\sigma(\sigma+1)$, where $s-\sigma$ and $\sigma-|s'|$ are non-negative integers. The matrix $h_{\lambda\mu}n^\lambda n^\mu$ is therefore always positive-definite.

We now investigate the causal implications of equation (10) by setting

$$p_\lambda = \mathrm{i}\phi_{,\lambda} = \mathrm{i}\,\partial\phi/\partial x^\lambda, \tag{14}$$

as usual in the coordinate representation (with units chosen so that Planck's constant $h = 2\pi\hbar = 2\pi$). We also substitute

$$\phi = \{\gamma\cdot(p-eA)+s_2 m\}\chi, \tag{15}$$

with

$$s_2 = 2s+1-s_1,$$

where $\gamma_\lambda s_2 = s_1\gamma_\lambda$, and so obtain the second-order equation

$$\{h^{\lambda\mu}(p_\lambda-eA_\lambda)(p_\mu-eA_\mu)-s_1 s_2 m^2-\mathrm{i}eF_{\lambda\mu}\gamma^\lambda\gamma^\mu\}\chi = 0, \tag{16}$$

where the electromagnetic field tensor $F_{\lambda\mu}$ is given by

$$F_{\lambda\mu} = \mathrm{i}[p_\lambda-eA_\lambda, p_\mu-eA_\mu]/e.$$

The commutator $[\gamma^\lambda,\gamma^\mu]$, which is needed to calculate the electric and magnetic moments, is evaluated in terms of the spin tensor in the Appendix. The result (16) is a generalization of the second-order equation derived from Dirac's equation and, because h^{00} is positive-definite, it satisfies the semiclassical criteria advanced by Velo and Zwanzinger (1969*a*, 1969*b*) for a causal theory. In this respect, our equation is similar to that proposed by Hurley (1971, 1974).

We consider next the definition of the particle density and the current density. As usual, we shall need for this purpose a hermitian matrix η such that $\eta\gamma_\lambda$ is also hermitian, i.e.

$$\eta\gamma_\lambda = \gamma^\lambda\eta, \tag{17}$$

when $\alpha_\lambda^* = \alpha^\lambda$, so that $\beta_\lambda^* = -\beta^\lambda$ and $\gamma_\lambda^* = \gamma^\lambda$, provided that η anticommutes with θ. If we define the adjoint field variable by

$$\bar\phi = \phi^*\eta, \tag{18}$$

it will then follow from equation (10) that

$$\bar\phi\,\gamma\cdot(\bar p-eA) = m\bar\phi s_1, \tag{19}$$

where $\bar\phi\,\bar p_\lambda = -\mathrm{i}\bar\phi_{,\lambda}$. It will also follow that the particle current four-vector

$$i_\lambda = \bar\phi\,\gamma_\lambda\phi \tag{20}$$

is real and satisfies the conservation law $i^\lambda{}_{,\lambda} = 0$. Moreover, since

$$[\alpha_{\lambda\mu},\gamma_\nu] = g_{\mu\nu}\gamma_\lambda - g_{\lambda\nu}\gamma_\mu, \tag{21}$$

it will behave correctly under Lorentz transformations. It is well known that η is of the form

$$\eta = \eta_0\exp\{\mathrm{i}\pi(s-\alpha^0)\}, \tag{22}$$

where η_0 commutes with the α_λ, but in this context must anticommute with θ_0 in equation (9), when the spin is integral. A matrix of this kind does not exist naturally in connection with Hurley's (1971, 1974) equations.

Now, to satisfy the causal criteria of Johnson and Sudarshan (1961), the particle density i_0 must be positive, and $\eta\gamma^0$ should therefore be positive-definite, at least in the vector subspace spanned by solutions of the field equations. In investigating this matter, we can disregard the electromagnetic field and can also choose a frame in which the particle is at rest ($\boldsymbol{p} = 0$). The field equations (1) and (10) show that, when $\alpha_0 = \pm s$, we have $\gamma_0 = \pm s_1$, where $s_1 = s+\theta s'$ is positive, so that we must have also

$$\eta_0 \phi = \phi. \tag{23}$$

This is a Lorentz-invariant condition, since η_0 commutes with $\alpha_{\lambda\mu}$, and is sufficient to ensure that $\eta\gamma^0$ is positive-definite in the subspace spanned by solutions of the field equations, which correspond to particles of spin s. However, this property does not extend to the subspaces with spins differing from s by an odd integer. It is evident from equations (11) and (13) that the eigenvalues of γ_0 are in general of the form $\pm(s_0+\theta s')$ and $\pm(s_0+1-\theta s')$, where s_0 is the nonrelativistic spin. But, as is seen in the Appendix, the sign of the eigenvalue does not change when the spin changes by one unit, so that $\eta\gamma^0$ has a negative eigenvalue, for instance, when $s_0 = s-1$. It follows that i_0, as defined by equation (20), will be negative if ϕ is an eigenvector of $\gamma.p$ corresponding to the eigenvalues $\pm(\sigma+\theta s')$ and $\pm(\sigma+1-\theta s')$, when the spin σ has the unphysical value $s-1$, which is in fact inconsistent with equation (1) above. The same state of affairs is encountered in the theory of the electromagnetic field, where the Lorentz condition is needed to eliminate field components corresponding to unphysical particles with spin 0 and negative energies or probabilities. However, the Lorentz condition, and supplementary conditions in general, are notoriously difficult to reconcile with Lorentz-invariant quantization procedures and, as the unphysical component of the four-vector potential of the electromagnetic field does not interact with charge, it is not strictly necessary to eliminate it. So we shall not impose the Lorentz condition in the following. A similar disregard for the strict requirements of causality may be adopted in dealing with the unphysical particles of spin $s-1$ etc. in the present theory, on the understanding that they do not interact with the electromagnetic field.

3. Quantization and Renormalization Procedures

We shall now develop a generalized quantum electrodynamics, based on the field equations of the previous section, as far as necessary to discuss the consequential problems of causality and unitarity. It will appear that only minor modifications of ordinary quantum electrodynamics are required.

We adopt the renormalized Lagrangian density

$$L = z(\mathrm{i}\bar{\phi}\gamma^\lambda\phi_{,\lambda} - m\bar{\phi}s_1\phi - e\bar{\phi}\gamma^\lambda A_\lambda\phi) - \tfrac{1}{2}yA^{\lambda,\mu}(A_{\lambda,\mu} - uA_{\mu,\lambda}), \tag{24}$$

where z and y are renormalization constants and u is a gauge constant. This yields the field equations (10) and (19), and the variant

$$y\,\partial_\mu(A^{\lambda,\mu} - uA^{\mu,\lambda}) = ez\bar{\phi}\gamma^\lambda\phi, \tag{25}$$

where $\partial_\mu = \partial/\partial x^\mu$, of Maxwell's equations. The energy of the electromagnetic field is not positive-definite, unless the Lorentz condition $A^\lambda{}_{,\lambda} = 0$ or an indefinite metric is introduced. But, as explained above, there is no need to insist on either of these devices. Because of the conservation equation satisfied by the current defined by equation (20), any part of the field that does not satisfy $A^\lambda{}_{,\lambda} = 0$ cannot interact with the charge, and the associated scalar 'photons' with negative energy cannot produce any observable violation of the principle of causality.

In quantization, we regard the field variables ϕ, $\bar{\phi}$ and A_λ as linear operators defined on a Hilbert space. But, since linear operators do not commute in general, it is necessary to apply an ordering convention in passing from the classical to the quantal theory. We agree that the factors of a product of field variables, however written, are to be reordered, in the reverse of the order of their time variables, with a change of sign if this involves an odd permutation of field variables representing particles of half-odd-integral spin. Products of field variables with equal times are to be interpreted as averages of the distinct limiting values for unequal times. Thus, when $t = t'$, $\phi_q(x)\,\bar{\phi}_r(x')$ will be understood to mean

$$\tfrac{1}{2}\phi_q(x)\,\bar{\phi}_r(x') + \tfrac{1}{2}(-1)^{2s}\,\bar{\phi}_r(x')\,\phi_q(x),$$

in which the subscripts q and r designate components of the field variables. We use the notation $[\phi, \chi]$ to denote the commutator in general, but the anticommutator when both ϕ and χ are products of an odd number of field variables of particles with half-odd-integral spin. Since such a bracket would vanish if the time-ordering convention were applied, we exempt the first factor of a commutator or an anticommutator from the convention.

The equal-time commutation relations are chosen so as to make the total energy and momentum of the fields, derived from the Lagrangian density in the usual way, universal generators of translations in time and space. They include

$$z[\phi_q(x), \bar{\phi}(x')\,\gamma_{0r}] = \varepsilon_{qr}\,\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{26a}$$

$$y[A_\lambda(x), -A_{\mu,0}(x') + u\,A_{0,\mu}(x')] = \mathrm{i}g_{\lambda\mu}\,\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{26b}$$

(for $t = t'$ only), where ε is the minimal idempotent matrix satisfying

$$\varepsilon\phi = \phi. \tag{27}$$

Any attempt to restrict the application of the commutation rules to the physical particles of spin s would lead to unnecessary complications, so, as already explained, we shall tolerate particles with spin $s-1$ and negative energies, just as we tolerate the unphysical photons which do not satisfy the Lorentz condition. With half-odd-integral spin s, this means defining ϕ_r^* as the negative, instead of the positive, hermitian conjugate of ϕ_r in states with spin differing from s by an odd integer. The unphysical particles are not, of course, allowed to interact with the electromagnetic field.

To set up a matrix representation, it is usual to make use of the techniques of the interaction representation. A unitary transformation is established between the field variables $z^{\frac{1}{2}}\phi$, $z^{\frac{1}{2}}\bar{\phi}$ and $y^{\frac{1}{2}}A_\lambda$, and corresponding field variables in a theory without interactions. The known matrix representations for the latter enable matrix representations for the former to be determined. As we shall be concerned only with vacuum expectation values, however, we shall not need to make explicit reference to the interaction representation.

4. Field Propagators

It remains to verify that the particle propagators are consistent with causality, and that the renormalization procedure, implicit in the adoption of the Lagrangian density (24), is effective. Let us define the vacuum expectation values

$$S(x-x') = \langle\phi(x)\,\bar{\phi}(x')\rangle\,, \tag{28}$$

that is,

$$S_{rs}(x-x') = \langle\phi_r(x)\,\bar{\phi}_s(x')\rangle\,,$$

and

$$S_\lambda(x-x', x'-x'') = \langle\phi(x)\,A_\lambda(x')\,\bar{\phi}(x'')\rangle\,, \tag{29a}$$

$$D_{\lambda\mu}(x-x') = \langle A_\lambda(x)\,A_\mu(x')\rangle\,. \tag{29b}$$

Because of the time-ordering convention, the particle propagator $S(x)$ is discontinuous at $t = 0$. Using the field equation (10) and the equal-time commutation relation of (26), we have

$$(\mathrm{i}\,\gamma^\lambda\,\partial_\lambda - s_1\,m)S(x) = e\gamma^\lambda S_\lambda(0,x) + \mathrm{i}\varepsilon\,\delta(x)/z\,, \tag{30}$$

where $\delta(x)$ is the four-dimensional δ function. Similarly, the photon propagator $D_{\lambda\mu}(x)$ satisfies

$$y\,\partial^\nu[\partial_\nu\,D_{\lambda\mu}(x) - u\,\partial_\lambda\,D_{\nu\mu}(x)] = ez\,\mathrm{tr}[\gamma_\lambda\;S_\mu(x, -x)] + \mathrm{i}g_{\lambda\mu}\,\delta(x)\,, \tag{31}$$

where tr denotes the trace of the matrix indicated. Because of the conservation law satisfied by the current i^λ of equation (20), it follows on further differentiation that

$$y(1-u)\,\partial^\nu\,\partial_\nu\,\partial^\lambda\,D_{\lambda\mu}(x) = \mathrm{i}\,\partial_\mu\,\delta(x)\,. \tag{32}$$

The same conservation law can be used to obtain an equation satisfied by $S_\lambda(x'-x, x)$. Using our variant (25) of Maxwell's equations, we have

$$y\,\partial^\nu[\partial_\nu\,S_\lambda(x'-x,x) - u\,\partial_\lambda\,S_\nu(x'-x,x)] = ze\langle\phi(x')\,\bar{\phi}(x)\,\gamma_\lambda\,\phi(x)\,\bar{\phi}(0)\rangle\,, \tag{33}$$

and hence, on further differentiation,

$$y(1-u)\,\partial^\nu\,\partial_\nu\,\partial^\lambda\,S_\lambda(x'-x, x) = e[\delta(x)\,S(x'-x) - \delta(x'-x)\,S(x)]\,. \tag{34}$$

In integrating the above equations for $S(x)$ and $D_{\lambda\mu}(x)$, we make use of the property of the vacuum in eliminating field components with negative frequencies; it is well known that this is achieved by Feynman's precription of giving a small negative imaginary mass to all particles. Thus, the required solution of equation (30) is

$$S(x) = z^{-1}\,S_{\mathrm{F}}(x) - \mathrm{i}e\int S_{\mathrm{F}}(x-x')\,\gamma^\lambda\,S_\lambda(0,x')\,\mathrm{d}^4x'\,, \tag{35}$$

where

$$S_{\mathrm{F}}(x) = \frac{\mathrm{i}}{(2\pi)^4}\int\frac{(s_1\,s_2)^{-1}(\gamma^\lambda\,p_\lambda + ms_2)\exp(-\mathrm{i}p\,.\,x)\,\mathrm{d}^4p}{p^2 - m^2 + \mathrm{i}\delta}\,, \tag{36}$$

and the limit $\delta \to +0$ is intended. We have made use here of the requirement that only particles of spin s interact with the electromagnetic field, so that, according to equations (13), we have $h_{\lambda\mu}p^\lambda p^\mu = s_1 s_2 p^2$. The above 'bare particle' propagator is not more singular on the light cone than the Feynman propagator for the electron, and has similar causal implications. We shall therefore not encounter the difficulties found in the use of Weinberg's (1964) propagator. The solution of equation (31) is

$$y D_{\lambda\mu}(x) = g_{\lambda\mu} D_F(x) - i \int D_F(x-x') \{ez \operatorname{tr}[\gamma_\lambda S_\mu(x', -x')] - u(1-u)^{-1} \partial'_\lambda \partial'_\mu D_F(x')\} \, d^4x', \tag{37}$$

where

$$D_F(x) = \frac{1}{(2\pi)^4 i} \int \frac{\exp(-ik.x) \, d^4k}{k^2 + i\delta} \tag{38}$$

is the usual Feynman photon propagator.

We now transcribe the above results in the momentum representation, by writing

$$S(p) = -i \int S(x) \exp(ip.x) \, d^4x, \tag{39a}$$

$$D_{\lambda\mu}(k) = -i \int D_{\lambda\mu}(x) \exp(ik.x) \, d^4x, \tag{39b}$$

$$S_\lambda(p,q) = -i \int S_\lambda(x,y) \exp\{i(p.x + q.y)\} \, d^4x \, d^4y, \tag{39c}$$

and defining the vertex function $\Gamma_\lambda(p,q)$ by

$$S_\lambda(q,p) = ie D_{\lambda\mu}(p-q) S(q) \Gamma^\mu(q,p) S(p). \tag{40}$$

Then it follows from equation (35) that

$$S(p) = z^{-1} S_F(p) \{1 + \Sigma(p) S(p)\}, \tag{41}$$

where

$$\Sigma(p) = \frac{ie^2 z}{(2\pi)^4} \int \gamma^\lambda D_{\lambda\mu}(p-q) S(q) \Gamma^\mu(q,p) \, d^4q. \tag{42}$$

This is a precise analogue of Dyson's (1949*a*, 1949*b*) equation for the electron propagator. If we introduce a pseudo-inverse $S^{-1}(p)$ of $S(p)$, we obtain from equation (41)

$$S^{-1}(p) = z S_F^{-1}(p) - \Sigma(p) = z(\gamma^\lambda p_\lambda - m s_1) - \Sigma(p), \tag{43}$$

and this is obviously amenable to the usual renormalization techniques. Moreover, in the momentum representation the identities (32) and (34) reduce to

$$y(1-u)k^2 k^\lambda D_{\lambda\mu}(k) = -k_\mu \tag{44}$$

and

$$y(1-u) k^2 k^\lambda S_\lambda(p, p+k) = ie[S(p) - S(p+k)]. \tag{45}$$

Hence, on substitution from equation (40),

$$k^\lambda \Gamma_\lambda(p, p+k) = S^{-1}(p+k) - S^{-1}(p). \tag{46}$$

Thus the generalized Ward identity, which was first obtained by the author (Green 1953) for ordinary electrodynamics, continues to hold for particles of arbitrary spin, and is, moreover, independent of the gauge. The number of independent renormalization constants is, consequently, independent of the spin.

The finiteness of the renormalization constants probably has an important bearing on the strict unitarity of the theory, and this is a question which deserves further investigation. The work of Johnson *et al.* (1964, 1967) and others suggests that z at least may be finite in the Landau gauge, if the bare mass m is allowed to approach zero; and that there is some possibility that y could be finite also, if the fine structure satisfies an eigenvalue condition. However, it would seem unrealistic to calculate the polarization of the vacuum due to particles of spin $\frac{1}{2}$ in isolation, so that the contributions of particles of higher spin may be important in this connection. It is hoped to study this question in a future publication.

References

Adler, S. (1972). *Phys. Rev.* D **5**, 3021.
Amar, V., and Dozzio, V. (1972). *Nuovo Cimento* B **9**, 53.
Bellissard, J., and Sieler, R. (1972). *Lett. Nuovo Cimento* **5**, 221.
Blaha, S. (1974). *Phys. Rev.* D **9**, 2246.
Bracken, A. J., and Green, H. S. (1971). *J. Math. Phys.* (*New York*) **12**, 2099.
Costa, G., and Tonin, M. (1975). *Riv. Nuovo Cimento* **5**, 29.
Cox, W. (1974*a*). *J. Phys.* A **7**, 1.
Cox, W. (1974*b*). *J. Phys.* A **7**, 2249.
Das, A., and Freedman, D. Z. (1976). *Nucl. Phys.* B **114**, 271.
Dyson, F. (1949*a*). *Phys. Rev.* **75**, 486.
Dyson, F. (1949*b*). *Phys. Rev.* **75**, 1736.
Eeg, J. O. (1972). *Lett. Nuovo Cimento* **5**, 591.
Gel'fand, I. M., Minlos, R. A., and Shapiro, Ya. (1963). 'Representations of the Rotation and Lorentz Groups and their Applications' (Pergamon: New York).
Gel'fand, I. M., and Yaglom, A. M. (1948). *Zh. Eksp. Teor. Fiz.* **18**, 703, 1096, 1105.
Green, H. S. (1953). *Proc. Phys. Soc. London* A **66**, 873.
Green, H. S. (1977). *Aust. J. Phys.* **30**, 1.
Harish-Chandra (1947). *Phys. Rev.* **71**, 7933.
Hurley, W. H. (1971). *Phys. Rev.* D **4**, 3605.
Hurley, W. H. (1974). *Phys. Rev.* D **10**, 1185.
Johnson, K., Baker, M., and Willey, R. (1964). *Phys. Rev.* **136**, B1111.
Johnson, K., and Sudarshan, E. C. G. (1961). *Ann. Phys.* (*New York*) **13**, 126.
Johnson, K., Willey, R., and Baker, M. (1967). *Phys. Rev.* **163**, 1699.
Lee, T. D., and Yang, C. N. (1962). *Phys. Rev.* **128**, 885.
Prabhakaran, J., Seetharaman, M., and Mathews, P. M. (1975). *J. Phys.* A **8**, 560.
Rarita, W., and Schwinger, J. (1941). *Phys. Rev.* **60**, 61.
Schroer, B., Seiler, R., and Swieca, J. A. (1970). *Phys. Rev.* D **2**, 2927.
Schwinger, J. (1948*a*). *Phys. Rev.* **73**, 416.
Schwinger, J. (1948*b*). *Phys. Rev.* **74**, 1439.
Senjanović, G. (1977). *Phys. Rev.* D **16**, 307.
Shamaly, A., and Capri, A. Z. (1971). *Nuovo Cimento* B **2**, 236.
Singh, L. P. S., and Hagen, C. R. (1977). *Phys. Rev.* D **16**, 347.
Tucker, R. H., and Hammer, C. L. (1971). *Phys. Rev.* D **3**, 2448.
Velo, G., and Zwanzinger, D. (1969*a*). *Phys. Rev.* **186**, 1337.
Velo, G., and Zwanzinger, D. (1969*b*). *Phys. Rev.* **188**, 2218.
Weinberg, S. (1964). *Phys. Rev.* **133**, B1318.

Appendix

We prove here a number of algebraic relations which have been used in the text.

The labelling (s, s') of irreducible representations of the Lorentz group in terms of highest weights implies that s is the maximum eigenvalue of α_{01} in the representation, and that s' is the maximum eigenvalue of the spin component $i\alpha_{23}$ when α_{01} attains its maximum. Within an irreducible representation (s, s) of the Poincaré group, s' takes the values $-s, -s+1, ..., s$; the value of s is of course fixed. A vector such as α_λ normally has four components, which change the labels s and s' of the Lorentz group by ± 1 (Bracken and Green 1971). But, as s is fixed, in the representations adopted we have

$$\alpha_\lambda = \alpha_{\lambda+} + \alpha_{\lambda-}, \tag{A1}$$

where $\alpha_{\lambda+}$ and $\alpha_{\lambda-}$ change the label of s' by $+1$ and -1 respectively:

$$s'\alpha_{\lambda+} = \alpha_{\lambda+}(s'+1), \qquad s'\alpha_{\lambda-} = \alpha_{\lambda-}(s'-1). \tag{A2}$$

Now, it is easily verified from equations (3) in Section 2 that

$$[s', \alpha_\lambda] = \beta_\lambda, \tag{A3}$$

so

$$\beta_\lambda = \alpha_{\lambda+} - \alpha_{\lambda-} \qquad \text{and} \qquad [s', \beta_\lambda] = \alpha_\lambda. \tag{A4}$$

Also, from equations (3),

$$[\alpha_0, \beta_0] = -s' \tag{A5}$$

and, by Lorentz invariance it follows that, if p_λ is any four-vector,

$$[\alpha.p, \beta.p] = -s'p^2. \tag{A6}$$

From equations (A3), (A4) and (A5) we infer that, as stated, $\alpha.p/\mu$, $-i\beta.p/\mu$ and s' are generators of representations of SO(3), when $\mu^2 = p^2 = p^\lambda p_\lambda$. Also, $\beta.p - s'\mu$ increases the eigenvalue of $\alpha.p$ by μ.

It follows, from equations (A5) and (A6) that

$$\alpha_{\lambda+} = \tfrac{1}{2}(\alpha_\lambda + \beta_\lambda), \qquad \alpha_{\lambda-} = \tfrac{1}{2}(\alpha_\lambda - \beta_\lambda) \tag{A7}$$

are the operators which change the eigenvalue of s' by ± 1 and, as

$$\gamma_\lambda = (1+\theta)\alpha_{\lambda+} + (1-\theta)\alpha_{\lambda-}, \tag{A8a}$$

we have

$$\gamma_\lambda \gamma_\mu = 2(1+\theta)\alpha_{\lambda+}\alpha_{\mu-} + 2(1-\theta)\alpha_{\lambda-}\alpha_{\mu+}, \tag{A8b}$$

so that $\gamma_\lambda \gamma_\mu$ leaves the eigenvalue of s' unchanged. Thus γ_λ couples only pairs of representations of the Lorentz group, labelled $s' = s$ and $s-1$, $s-2$ and $s-3$, etc. Also, $\gamma_\lambda \gamma_\mu$ must be expressible in terms of the generators $\alpha_{\lambda\mu}$ of the Lorentz group. To determine $\gamma_\lambda \gamma_\mu$, we note first that $\alpha^\lambda \alpha_\lambda$ is the difference between the quadratic invariants

$$2s(s+2) = \tfrac{1}{2}\alpha^\lambda_\mu \alpha^\mu_\lambda + \alpha^\lambda \alpha_\lambda \qquad \text{and} \qquad s(s+2) + s'^2 = \tfrac{1}{2}\alpha^\lambda_\mu \alpha^\mu_\lambda$$

of SO(4, 1) and SO(3, 1) respectively:

$$\alpha^\lambda \alpha_\lambda = s(s+2) - s'^2.$$

By commuting this result with s' we see also that

$$\beta^{\lambda}\alpha_{\lambda} = -\alpha^{\lambda}\beta_{\lambda}, \qquad \beta^{\lambda}\beta_{\lambda} = -\alpha^{\lambda}\alpha_{\lambda}$$

and, if we notice that $\beta^{\lambda}\alpha_{\lambda} = 2s'$, we have

$$\gamma^{\lambda}\gamma_{\lambda} = 2[s(s+2)-s'^2]+4\theta s'. \tag{A9}$$

Next, since

$$\{\alpha^{\mu\lambda},\alpha_{\lambda}\} = [s'^2,\alpha^{\mu}] = 2s'\beta^{\mu}-\alpha^{\mu},$$

we can verify easily that

$$\alpha_{\lambda}{}^{\mu}\alpha_{\mu} = \alpha_{\lambda}+s'\beta_{\lambda}, \qquad \alpha_{\lambda}{}^{\mu}\beta_{\mu} = \beta_{\lambda}+s'\alpha_{\lambda},$$

and consequently

$$\alpha_{\lambda}{}^{\mu}\gamma_{\mu} = (1+\theta s')\gamma_{\lambda}, \qquad \gamma_{\mu}\alpha_{\lambda}{}^{\mu} = \gamma_{\lambda}(1+\theta s'). \tag{A10}$$

In view of the characteristic identity (Bracken and Green 1971)

$$[(\alpha-1)^2-(s+1)^2][(\alpha-1)^2-s'^2] = 0$$

satisfied by the matrix α whose elements are the generators $\alpha_{\lambda}{}^{\mu}$ of the Lorentz group, we see that

$$\gamma_{\lambda}\gamma_{\mu} = \{(\alpha-1+\theta s')[(s+1)^2-(\alpha-1)^2]\}_{\lambda\mu}/(\theta s'), \tag{A11}$$

with a normalization determined from equation (A9). If β is the matrix with element $\beta_{\lambda}{}^{\mu}$, where

$$\beta_{\lambda\mu} = \tfrac{1}{2}\mathrm{i}\varepsilon_{\lambda\mu\nu\rho}\alpha^{\nu\phi}, \tag{A12}$$

the characteristic identity is derivable from

$$(\alpha-1)^2+\beta^2 = (s+1)^2+s'^2,$$

$$(\alpha-1)\beta = \beta(\alpha-1) = s'(s+1).$$

Hence

$$s'(s+1)\beta = (\alpha-1)[(s+1)^2+s'^2-(\alpha-1)^2]$$

is antisymmetric; and it follows from equation (A11) that

$$[\gamma_{\lambda},\gamma_{\mu}] = 2\alpha_{\lambda\mu}+2\theta[(s+1)\beta_{\lambda\mu}-s'\alpha_{\lambda\mu}], \tag{A13a}$$

$$\{\gamma_{\lambda},\gamma_{\mu}\} = 2(\theta s'+s^2+2s)g_{\lambda\mu}-2\alpha_{\lambda}{}^{\nu}\alpha_{\nu\mu}+2\alpha_{\lambda\mu}. \tag{A13b}$$

In particular, from equation (A13b) we have

$$\begin{aligned}\gamma_0{}^2 &= s(s+2)+\theta s'-\alpha_0{}^{\lambda}\alpha_{\lambda 0}\\ &= \tfrac{1}{2}\alpha_{lm}\alpha_{ml}-s'(s'-\theta)\\ &= s_0(s_0+1)-\theta s'(\theta s'-1),\end{aligned} \tag{A14}$$

where s_0 is the nonrelativistic spin. Since γ_{λ} transforms as a four-vector, it has components $\gamma_{\lambda}{}^{0}$, $\gamma_{\lambda}{}^{+}$ and $\gamma_{\lambda}{}^{-}$ which commute with s_0, and change the eigenvalue of s_0 by $+1$ and -1 respectively. These components are projected by pairs of factors of the characteristic identity

$$(\alpha'-1)(\alpha'-s_0-1)(\alpha'+s_0) = 0$$

satisfied by the matrix α' with elements α_l^m which are the generators of SO(3). From equations (A13) it is readily verified that

$$\gamma_0\,\gamma_l^{\,0}+\gamma_l^{\,0}\,\gamma_0 = 0, \tag{A15a}$$

$$\gamma_0\,\gamma_l^{\,+}(s_0+\theta s') = \gamma_l^{\,+}\,\gamma_0(s_0+2-\theta s'), \tag{A15b}$$

$$\gamma_0\,\gamma_l^{\,-}(s_0+1-\theta s') = \gamma_l^{\,-}\,\gamma_0(s_0-1+\theta s'). \tag{A15c}$$

Thus the sign of the eigenvalue of $\eta\gamma_0$ is not changed by $\gamma_l^{\,0}$, but is changed by the spin-changing components $\gamma_l^{\,+}$ and $\gamma_l^{\,-}$ of γ_l, as stated in Section 2.

Manuscript received 27 February 1978