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Meson Mass Formulae in $SU(5)$

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Abstract

We present first-order mass formulae for mesons in broken $SU(5)$ and, using the upsilon mass as input, we estimate the masses of the new $J^P = 0^-$, 1^- and 2^+ mesons. We find that the masses of the 0^- states exceed those of the corresponding 1^- states for single b quark mesons, and that a rather large ${}^1S_0$–${}^3S_1$ splitting occurs for the upsilon family.

1. Introduction

The discovery of a new vector meson at 9·46 GeV (Herb *et al.* 1977; Berger *et al.* 1978; Flügge 1978), and its interpretation as a new bound state of a fifth quark, implies a new set of mesons associated with this quark. We label this new flavour with the quantum number b. In the present paper we apply a generalization of the mass breaking formalism of Gell-Mann (1962) and Okubo (1962) to $SU(5)$ and we predict the masses of the mesons exhibiting the new quantum number. To the first order in symmetry breaking, the mass operator in the $SU(5)$ group has the form (Okubo 1975*a*; Okubo *et al.* 1975; Hayashi *et al.* 1976)

$$H = T^8 + \alpha_1 T^{15} + \alpha_2 T^{24}. \tag{1}$$

The operators T^{24}, T^{15} and T^8 transform as components of the adjoint representation of $SU(5)$. The operator T^{24} breaks down the $SU(5)$ symmetry to $SU(4)$, T^{15} breaks down $SU(4)$ symmetry to $SU(3)$, while T^8 governs the breaking within the $SU(3)$ multiplets. Implicit in this form of breaking is a labelling of the states (Jarvis 1978) according to the decomposition

$$SU(n) \supset SU(n-1) \times U_x(1),$$

where $x = b$, c and s for $n = 5$, 4 and 3 respectively. Assigning the quarks to the first fundamental representation of $SU(5)$, the $q\bar{q}$ meson states belong to the **24** and singlet representations of $SU(5)$. The **24** representation has the $SU(4)$ reduction

$$\mathbf{24} = \underset{[b=1]}{\bar{\mathbf{4}}} + \underset{[b=0]}{\mathbf{15}+\mathbf{1}} + \underset{[b=-1]}{\mathbf{4}}\ , \tag{2}$$

with $SU(3)$ reductions

$$\mathbf{4} = \underset{[c=0]}{\mathbf{3}} + \underset{[c=1]}{\mathbf{1}}\ , \tag{3a}$$

$$\mathbf{15} = \underset{[c=1]}{\bar{\mathbf{3}}} + \underset{[c=0]}{\mathbf{8}+\mathbf{1}} + \underset{[c=-1]}{\mathbf{3}}\ . \tag{3b}$$

The mesons with a single b quark lie in an $SU(4)$ **4** representation. We will label the $SU(3)$ triplet states of the **4** by G ($I = \pm\frac{1}{2}$) and H ($I = 0$), and the singlet state by I. The charge conjugate states lie in the conjugate representations.

In the next section we present the mathematical formalism and mass formulae. It is easier to work with $U(5)$, the results being the same as those for $SU(5)$ provided that care is taken in associating the correct baryon number with the representations. In Section 3 we deal in a general way with the mixing of the representation states, and we consider the perennial problem regarding the use of linear or quadratic mass terms in the formulae. Estimates are given for the masses of the G, H and I mesons for the J^P families of 0^-, 1^- and 2^+, as well as the $b\bar{b}$ states analogous to Υ for the $J^P = 0^-$ and 2^+ mesons.

2. Formalism

The mass operator in equation (1) can be written in tensor notation as

$$\mathbf{H} = T_{33} + x\,T_{44} + y\,T_{55}, \tag{4}$$

with

$$x = \tfrac{1}{3}\{1 + 2\surd 2\alpha_1\} \qquad \text{and} \qquad y = \tfrac{1}{6}\{2 + \surd 2\alpha_1 + \surd(30)\alpha_2\}. \tag{5}$$

It has been shown that in $U(n)$ a tensor operator T_{ij} can be expressed in terms of a sum of products of the generators of the group (Okubo 1975*b*, 1977). For the adjoint representation, the highest product term in the sum is second order in the generators. Hence we can write

$$T_{ij} = a + b(e_{ij}) + c\sum_{k=1}^{5} e_{ik}\,e_{kj}, \tag{6}$$

where a, b and c are constants and the e_{ij} (with $i, j = 1, \dots, 5$) are the $U(5)$ group generators. The irreducible representations of $U(5)$ can be described by a set of five integers $[m_{i5}]$ with $i = 1, \dots, 5$. The states of a given representation can be expressed in an economical way by means of the Gelfand–Zetlin scheme (Gelfand *et al.* 1963; Louck 1970). In this scheme, the states are described by a triangular array of integers

$$\begin{bmatrix} m_{15} & & m_{25} & & m_{35} & & m_{45} & & m_{55} \\ & m_{14} & & m_{24} & & m_{34} & & m_{44} & \\ & & m_{13} & & m_{23} & & m_{33} & & \\ & & & m_{12} & & m_{22} & & & \\ & & & & m_{11} & & & & \end{bmatrix}, \tag{7}$$

with the m_{ij} taking all values consistent with the 'betweeness condition'

$$m_{ij} < m_{i\,j-1} < m_{i+1\,j}. \tag{8}$$

The quantum numbers in terms of the m_{ij} are given by

$$3B = \sum_{i=1}^{5} m_{i5}, \qquad b = \sum_{i=1}^{5} m_{i5} - \sum_{j=1}^{4} m_{j4}, \tag{9a, b}$$

$$C = \sum_{i=1}^{4} m_{i4} - \sum_{j=1}^{3} m_{j3}, \qquad Y = m_{12}+m_{22} - \tfrac{2}{3}\sum_{i=1}^{3} m_{i3}, \tag{9c, d}$$

$$I = \tfrac{1}{2}(m_{12}-m_{22}), \qquad I_z = m_{11} - \tfrac{1}{2}(m_{12}+m_{22}), \tag{9e, f}$$

where B is the baryon number. The eigenvalues of the two lowest-order Casimir invariants $I_1^{(n)}$ and $I_2^{(n)}$ (Louck 1970) which we require are given in terms of the m_{ij} by

$$I_1^{(n)} = \sum_{i=1}^{n} p_{in} - \binom{n}{2}, \qquad I_2^{(n)} = \sum_{i=1}^{n} (p_{in})^2 - (n-1)\sum_{j=1}^{n} p_{jn} + \binom{n}{3}, \tag{10a, b}$$

with

$$p_{ij} = m_{ij}+j-i. \tag{11}$$

The quark quantum numbers can be calculated from the equations (9) using the states of the **5** representation with $[m_{i5}]$ given by $[1,0,0,0,0]$. The advantages of this scheme lie in the ease of computing the matrix elements of equation (4) between the states of the **24** representation. Also the state labelling inherent in equation (1) is made transparent in the scheme: the rows $[m_{in}]$ describe the irreducible representations of the $U(n)$ subgroups for $n = 2$, 3 and 4.

The general expression for the mass matrix is

$$\begin{aligned} M = m_0 &+ a_1\{-Y - \tfrac{1}{3}(C+b) + xC + yb\} \\ &+ a_2\{L_1 + \langle e_{34}e_{43}\rangle + \langle e_{35}e_{53}\rangle + x(L_2 + \langle e_{45}e_{54}\rangle) + yL_3\}, \end{aligned} \tag{12}$$

where

$$L_1 = \tfrac{1}{2}[I_2^{(3)} + \tfrac{1}{2}Y^2 - 2I(I+1) + Y\{\tfrac{4}{3}(C+b) - 3\} - \tfrac{1}{9}(C+b)^2], \tag{13a}$$

$$L_2 = \tfrac{1}{2}\{I_2^{(4)} - I_2^{(3)} - I_1^{(4)} + 4C + C^2\}, \tag{13b}$$

$$L_3 = \tfrac{1}{2}\{I_2^{(5)} - I_2^{(4)} - I_1^{(5)} + 5b + b^2\}. \tag{13c}$$

The values of $I_1^{(n)}$ and $I_2^{(n)}$ are given by equations (10a) and (10b) respectively. The $\langle e_{ij}e_{ji}\rangle$ terms mix the three states at the centre of the weight space and can be calculated explicitly by means of an expression given by Louck (1970). The formulae for the various $SU(3)$ representations, together with the mixing matrix **A** for the $b = C = Y = I_z = 0$ states of the **24** representation, are as follows:

$$b = \pm 1, \quad C = \pm 1, \qquad m(1)_4 = m_0 + 2\beta - \tfrac{3}{2}\beta(x+y); \tag{14a}$$

$$C = 0, \qquad m(3)_4 = m_0 + \tfrac{7}{10}\beta - \tfrac{3}{2}\beta y + \tfrac{9}{15}\beta\{I(I+1) - \tfrac{1}{4}Y^2\}; \tag{14b}$$

$$b=0,\qquad C=\pm 1\qquad m(3)_{15}=m_0+\tfrac{7}{10}\beta-\tfrac{3}{2}\beta x+\tfrac{9}{5}\beta\{I(I+1)-\tfrac{1}{4}Y^2\};\tag{15a}$$

$$C=0\qquad m(8)_{15}=m_0+\beta\{I(I+1)-\tfrac{1}{4}Y^2\},\tag{15b}$$

$$m(1)_{15}=m_0+\tfrac{1}{4}\beta(7-9x)\equiv P,\tag{15c}$$

$$m(1)_{1}=m_0+\tfrac{37}{20}\beta-\tfrac{3}{20}\beta x-\tfrac{12}{5}\beta y\equiv Q;\tag{15d}$$

$$\mathbf{A}=\begin{array}{c} \begin{array}{ccc} (8)_{15} & (1)_{15} & (1)_1 \end{array} \\ \left[\begin{array}{ccc} m_0 & -\beta/\surd 2 & 3\beta/\surd(30) \\ & P & -9\beta(x-\frac{1}{3})/4\surd(15) \\ & & Q \end{array}\right] \end{array}\begin{array}{l} \\ (8)_{15} \\ (1)_{15}\,. \\ (1)_1 \end{array}\tag{16}$$

The subscripts in these formulae indicate the $SU(4)$ multiplets. The sign convention for the operators follows that given by Louck (1970). These techniques have also been used by the present authors to obtain the Clebsch–Gordan coefficients and mass formulae for the baryonium states in $SU(4)$ (Anderson and Joshi 1979*a*, 1979*b*).

3. Mass Predictions

From experience gained in the $SU(4)$ sector (Okubo 1975*a*; Okubo *et al.* 1975) we need to consider the mixing of the $SU(5)$ singlet with the **24** representation in addition to the intra-multiplet mixing given in equations (14)–(16). Hence the physical mass values of the four zero-flavour states will be given by diagonalizing a symmetric mass mixing matrix of the form

$$\mathbf{M}=\left[\begin{array}{ccc|c} & & & k_1 \\ & \mathbf{A} & & k_2 \\ & & & k_3 \\ \hline k_1 & k_2 & k_3 & m_s \end{array}\right],\tag{17}$$

where m_s is the mass of the unmixed singlet and the k_i are the strengths of the coupling of the $SU(5)$ singlet to the centre states of the **24** representation. A possible guide as to whether to use linear or quadratic terms is to test the $SU(4)$ prediction

$$M_{D^*}-M_\rho=M_{F^*}-M_{K^*}\tag{18}$$

with the known $SU(4)$ masses. For vector mesons the agreement is better with linear terms, while for pseudoscalars it is better with quadratic terms. The situation is complicated by the possibility of contributions from higher-order breaking terms (e.g. second-order terms transforming like components belonging to the representations in the product **15** × **15** (Verma and Khanna 1978)). In view of this ambiguity we present both linear and quadratic predictions for the vector mesons, while for the tensor and pseudoscalar mesons we restrict ourselves to quadratic predictions.

Ideal Mixing

If we consider the situation of the physical states being pure quark states then the mixing of the representation states is given by

$$\frac{1}{\surd 2}|p\bar{n}-n\bar{p}\rangle = \frac{1}{\surd 3}|\mathbf{15}_4,\mathbf{8}_3\rangle + \frac{1}{\surd 6}|\mathbf{15}_4,\mathbf{1}_3\rangle - \frac{1}{\surd(10)}|\mathbf{1}_4,\mathbf{1}_3\rangle + \frac{\surd 2}{\surd 5}|\mathbf{1}\rangle, \tag{19a}$$

$$|s\bar{s}\rangle = \frac{2}{\surd 6}|\mathbf{15}_4,\mathbf{8}_3\rangle + \frac{1}{2\surd 3}|\mathbf{15}_4,\mathbf{1}_3\rangle - \frac{1}{2\surd 5}|\mathbf{1}_4,\mathbf{1}_3\rangle + \frac{1}{\surd 5}|\mathbf{1}\rangle, \tag{19b}$$

$$|c\bar{c}\rangle = \frac{\surd 3}{2}|\mathbf{15}_4,\mathbf{1}_3\rangle + \frac{1}{2\surd 5}|\mathbf{1}_4,\mathbf{1}_3\rangle - \frac{1}{\surd 5}|\mathbf{1}\rangle, \tag{19c}$$

$$|b\bar{b}\rangle = \frac{2}{\surd 5}|\mathbf{1}_4,\mathbf{1}_3\rangle + \frac{1}{\surd 5}|\mathbf{1}\rangle, \tag{19d}$$

where $|\boldsymbol{\mu}_4,\mathbf{v}_3\rangle$ belongs to the $SU(5)$ **24** representation, with $\boldsymbol{\mu}$ indicating the $SU(4)$ representation and $\mathbf{v}$ indicating the $SU(3)$ representation, while $|\mathbf{1}\rangle$ represents the $SU(5)$ singlet. The condition of $\mathbf{M}$ being diagonal in this basis leads to the mass relations

$$2M_{ij} = M_{ii}+M_{jj} \qquad \text{for} \qquad i \neq j; \tag{20a}$$

with

$$i = p\,(=n),\, s,\, c; \qquad j = s,\, c,\, b; \tag{20b}$$

$$\mathbf{M}((p\bar{p}+n\bar{n})/\surd 2) = \mathbf{M}((p\bar{p}-n\bar{n})/\surd 2). \tag{20c}$$

Vector Mesons ($J^P = 1^-$)

For vector mesons, in the case of general mixing, we have five unknowns (k_i, for $i = 1$, 2 and 3, and m_s and y) with four input masses. The ρ, K* and D* (Particle Data Group 1978) masses determine the parameters β, m_0 and x. Since two of the parameters (m_s and y) occur only in the diagonal elements, the condition that the trace of $\mathbf{M}$ be invariant and equal to the sum of the ω, ϕ, ψ and Υ masses enables a reduction of the number of unknowns to four. Using a ρ mass of 0·776 GeV, a numerical iteration of the eigenvalue equation gives the following quadratic and linear predictions for y:

$$y(\text{quadratic}) = 211{\cdot}65 \qquad \text{and} \qquad y(\text{linear}) = 38{\cdot}87. \tag{21}$$

The value of y is strongly dependent on the ρ mass input. In Fig. 1 we plot the variation of y(quadratic) with the ρ mass over one standard deviation. It is significant that the maximum y(quadratic) value occurs with the ρ mass equal to the average ρ^0 mass given from phase-shift analysis, and also that this is the y value closest to that estimated from ideal mixing. The masses for the $b = \pm 1$ mesons are shown in Table 1.

Tensor Mesons ($J^P = 2^+$)

To proceed for the tensor mesons, two assumptions are needed; one regarding the x value, the other concerning f_b, the state analogous to Υ. To overcome the lack of a mass value for f_b, the minimal assumption we can impose is that f_b consists solely of a $b\bar{b}$ state. This leads to the following condition on one of the eigenvalues

$$f_b = 4(f+f'+f_c-Q-m+k_3). \qquad (22)$$

For an x value we have used the vector meson value of 17·71. Since x and y are related to the quark masses through

$$x = (M_c-M_p)/(M_s-M_p), \qquad y = (M_b-M_p)/(M_s-M_p),$$

we would expect them to be common for all families of J^P values. However as the x value differs for pseudoscalar and vector mesons by 10%, a similar variation can be expected between vector and tensor mesons. We have assigned f_c the value 3·55 GeV. The prediction for f_b lies within the $\Upsilon-\Upsilon'$ mass range (Lederman 1978), a result expected from potential calculations (Martin 1977; Grosse 1977; Brosse and Martin 1978).

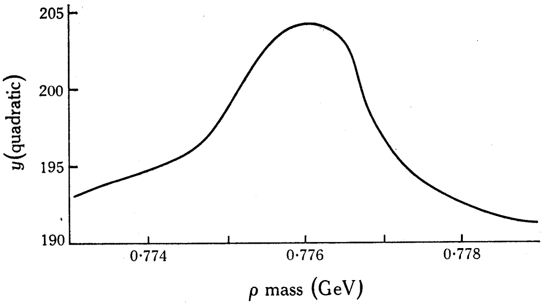

Fig. 1. Dependence of the quadratic prediction for y on the ρ mass.

Table 1. Mass predictions for mesons containing b quarks

Italicized quantities indicate input values. Masses are given in GeV

J^P	x	y	G($p\bar{b}$) mass	H($s\bar{b}$) mass	I($c\bar{b}$) mass	
1^-(quadratic)	*17·7*	211·6	6·45	6·47	6·71	Υ = *9·46*
1^-(linear)	*10·6*	38·97	5·29	5·41	6·53	Υ = *9·46*
2^+	*17·7*	177·8	7·69	7·71	8·05	f_b = 9·95
0^-	*15·3*	201·0	6·75	6·77	7·01	η_b = 8·77

Pseudoscalar Mesons ($J^P = 0^-$)

For pseudoscalar mesons the calculations follow in a similar manner to those for the tensor mesons, except that the D and F mesons enable an x value for the pseudoscalars to be calculated. An inversion of the masses for the G, H and I mesons compared with those for the vector masses has occurred. This is largely due to the increase in the β term for the pseudoscalars. The masses presented here are tentative estimates, since the assumption (22) is questionable for the pseudoscalars.

We conclude with some general comments: (i) The analysis presented here is a pure group-theoretical one, stemming solely from the operator (1), and is independent of any potential assumed for the $q\bar{q}$ interaction. (ii) The mixing analysis expressed in equation (17) is quite general. We have made no assumptions regarding the relationships between the k_i, which previous symmetry-breaking treatments based on the form (1) have used (Okubo *et al.* 1975; Hayashi *et al.* 1976). In our notation these assumptions have the form $k_2 = -\alpha_1 k_1$ and $k_3 = \alpha_2 k_1$. (iii) Improved mass values could be obtained by considering higher-order mass breaking terms, although at the expense of greater complexity. (iv) Our approach has the difficulty that the eigenvalues of equation (17) are highly nonlinear functions of y and the k_i, and hence there exist numerous solutions leading to the same eigenvalues. Fortunately, since different y and k_i lead to different eigenvectors, we have placed an additional constraint on the y and k_i. For the vector and tensor mesons this has taken the form of selecting a set of y and k_i which were close to and continuous with the solutions expected for ideally mixed eigenvectors, while for the pseudoscalar meson the y and k_i were chosen to be close to and continuous with an $SU(3)$ mixing angle of 10·4°.

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