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Some Predictions from the Mean Field Equations of Magnetoconvection

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Abstract

Nonlinear magnetic convection is investigated by the mean field approximation. The boundary layer method is used assuming large Rayleigh number R for different ranges of the Chandrasekhar number Q. The heat flux F is determined for wavenumbers α_n which optimize F. It is shown that there are a finite number of modes in the ranges $Q \ll R^{2/3}$ and $R^{2/3} \ll Q \ll R$, and that the number of modes increases with increasing Q in the former range and decreases with increasing Q in the latter range. For $Q = O(R^{2/3})$ there are infinitely many modes, and F is proportional to $R^{1/3}$. While the optimal F is independent of Q for $Q \ll R^{1/2}$, it is found to decrease with increasing Q in the range $R^{1/2} \ll Q \ll R$ and eventually to become of $O(1)$ as $Q \to O(R)$, and the layer becomes stable.

1. Introduction

The specific problem considered in this paper is the effect of a magnetic field on convection between two stress-free horizontal boundaries at large Rayleigh numbers. Magnetic convection is important in many areas of geophysics and astrophysics, and has been observed in nature, e.g. in astrophysics the darkness of sunspots indicates the presence of strong magnetic fields that inhibit convective motions. The reason for the stabilizing effect of a strong magnetic field on convective flows is that the magnetic field imparts a certain rigidity to the fluid, and affects the convective velocities by asserting itself near the boundaries, so making the flow through the boundary layers difficult.

We study here nonlinear magnetic convection, subject to the so-called mean field equations for the magnetic field, momentum and heat. Briefly, these equations are derived by ignoring the interaction between the fluctuating quantities, but the interaction between the mean and the fluctuating quantities is retained. For a more detailed discussion of these equations and their derivation, see Herring (1963) and Busse (1970). Previous studies of these equations for the case of thermal convection have shown that, for moderate or large values of the Prandtl number, the derived results (as far as the statistical properties of the motion are concerned) do not differ appreciably from the experimental results based on the original equations.

In the present study we are interested in finding the solution which maximizes the heat transport F. The flow that maximizes F determines uniquely the horizontal scales of the convective modes and also reduces the complexities of the whole problem. Although in general the F that maximizes for the mean field equations is not necessarily the F that maximizes for the full convection equations, it is known from recent studies of Bernard convection with and without rotation (Chan 1971; Hunter and Riahi 1975) that the flow that maximizes F subject to the mean field equations is the same as that which gives an upper bound to F in the limit of large Prandtl number σ for the full convection equations. The mean field approximation of the magnetoconvection equations is believed to represent adequately the average properties of the flow subject to the full hydromagnetic convection equations at large σ and diffusivity ratio τ (the ratio of magnetic diffusivity to thermal diffusivity), since the dominant nonlinear effect arises only from the modification of the horizontally averaged temperature distribution by the convection heat transport and is retained in the mean field equations. Therefore the flow that maximizes F for the mean field equations is expected to represent adequately the flow which gives an upper bound to F for the full magnetoconvection equations at least in the limit of large σ and τ. The success of previous upper-bound studies of thermal convection, which compare reasonably well with observation, has encouraged me to undertake the present study, which is hoped to provide a deeper insight into the subject of nonlinear magnetoconvection.

The present study is the first attempt to apply the multiboundary layer technique to hydromagnetic convection in order to determine the optimal flow quantities of the maximized fields for sufficiently large values of the Rayleigh number. This technique was first formulated by Busse (1969). In improving the upper bound on the heat flux, Busse (1969) considered a sequence of different boundary layers by adjusting the horizontal scale from its interior value to its boundary value. He supposed the thickness of each boundary layer to be large in comparison with the thickness of the following layer, and the convecting component of the heat flux to be approximately equal to the total heat flux in all but the last of the boundary layers, where it was of the order of the total heat flux. Later Chan (1971, henceforth referred to as Paper I) used Busse's (1969) technique to study turbulent convection at infinite Prandtl number, and obtained the preferred upper bound to the heat transport. Since then this technique has been used by Busse and Joseph (1972), Gupta and Joseph (1973), Chan (1974) and Riahi (1977, 1978). In all such studies, a schematic structure was considered for all the modes. Also, it was assumed that higher modes have shorter length scales, and that coupling among the different modes occurs only between the $(n+1)$th and nth modes in the nth boundary layer. Single- and multimodal regimes and details of the solutions of our governing equations (Section 2) are given in Section 3.

2. Governing Equations

We consider an infinite horizontal layer of fluid of depth d permeated by a magnetic field $\boldsymbol{H}^* = (H_1^*, H_2^*, H_3^*)$. The upper and lower surfaces are maintained at temperatures T_0 and $T_0+\Delta T$ respectively. The magnetic field can be written as $\boldsymbol{H}^* = \langle \boldsymbol{H}^* \rangle + \boldsymbol{h}$, where the angle brackets denote a horizontal average over the layer. Since $\langle \boldsymbol{H}^* \rangle$ is only a function of the vertical height z and $\nabla . \langle \boldsymbol{H}^* \rangle = 0$, we see that $\langle H_3^* \rangle$ must be a constant which takes the value of the impressed field. If this value is defined as the unit of field strength then we have $\boldsymbol{H}^* = \boldsymbol{K}+\boldsymbol{h}$, where $\boldsymbol{K}$ is the unit vector in the vertical direction. The mean field equations of hydromagnetic convection are derived from the Boussinesq equations for momentum, magnetic field and heat by neglecting all nonlinear terms with the exception of that which enters the equation for the horizontally averaged temperature (Busse 1970). The nondimensional steady state forms of these equations, after eliminating the pressure and horizontal velocity components, are

$$\nabla^4 w + R\nabla_1^2 T + \tau Q\nabla^2 \partial h_3/\partial z = 0, \qquad \partial w/\partial z + \tau\nabla^2 h_3 = 0, \tag{1a, b}$$

$$\nabla^2 T + (1 - \langle wT\rangle + [\![wT]\!])w = 0. \tag{1c}$$

Here h_3 and w are the vertical components of the magnetic vector $\boldsymbol{h}$ and the velocity vector respectively, T is the deviation of temperature from its horizontal average, the open brackets denote a further vertical averaging over the whole layer, and $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Also $R = \alpha g\Delta T d^3/K\nu$ is the Rayleigh number, $Q = \langle H_3^*\rangle^2 d^2/\mu\rho_0\nu\eta$ is the Chandrasekhar number and $\tau = \eta/K$ is the ratio of the magnetic diffusivity to the thermal diffusivity; while μ is the magnetic permeability, ρ_0 is the reference density (a constant), ν is the kinematic viscosity, α is the coefficient of thermal expansion and g is the acceleration due to gravity.

We now rescale the dependent variable so that

$$\omega = (FR)^{-\frac{1}{2}} w, \qquad \theta = (R/F)^{\frac{1}{2}} T, \qquad H = (FR)^{-\frac{1}{2}} \tau h_3, \tag{2}$$

where $F = [\![WT]\!]$ is the heat flux. The governing differential equations are now

$$\nabla^4\omega + \nabla_1^2\theta + Q\nabla^2\partial H/\partial z = 0, \qquad \partial\omega/\partial z + \nabla^2 H = 0, \tag{3a, b}$$

$$(FR)^{-1}\nabla^2\theta + (1 - \langle\omega\theta\rangle + F^{-1})\omega = 0, \tag{3c}$$

which are seen to be completely independent of the parameter τ.

We use the following constraint to determine F:

$$F = \frac{1 - R^{-1}[\![|\nabla\theta|^2]\!]}{[\![(1 - \langle\omega\theta\rangle)^2]\!]}. \tag{4}$$

This is derived by multiplying equation (3c) by θ and taking the total average over the whole layer. The boundary conditions to be considered for free surfaces at $z = 0$ and 1 are

$$\omega = \partial^2\omega/\partial z^2 = \theta = H = 0. \tag{5}$$

Condition (5) represents the simplest kind of boundary condition for the problem since, for technical reasons, it is found to simplify considerably the present theoretical investigation. The usual form of cellular structure for the dependent variables is assumed, so that we have

$$(\omega, \theta, H) = \sum_n (\omega_n(z), \theta_n(z), H_n(z))\,\phi_n(x, y), \tag{6}$$

where ϕ_n can be any solution of

$$\nabla_1^2\phi_n(x, y) = -\alpha_n^2\phi_n(x, y)$$

for some horizontal wavenumber α_n. Functions with different wavenumbers are naturally orthogonal, and are chosen here to be orthonormal. This separation of variables leads us to the system of nonlinear ordinary differential equations:

$$\left(\frac{d^2}{dz^2}-\alpha_n^2\right)^2 \omega_n - \alpha_n^2\theta_n + Q\left(\frac{d^2}{dz^2}-\alpha_n^2\right)\frac{dH_n}{dz} = 0, \tag{7a}$$

$$\frac{d\omega_n}{dz} + \left(\frac{d^2}{dz^2}-\alpha_n^2\right)H_n = 0, \tag{7b}$$

$$\frac{1}{FR}\left(\frac{d^2}{dz^2}-\alpha_n^2\right)\theta_n + \left(1 - \sum_n \omega_n\theta_n + \frac{1}{F}\right)\omega_n = 0, \tag{7c}$$

with boundary conditions

$$\omega_n = d^2\omega_n/dz^2 = \theta_n = H_n = 0 \quad \text{at} \quad z = 0 \text{ and } 1. \tag{8}$$

The subsequent analysis and solution of the system (7) and (8) suppose throughout that both the Rayleigh number and the heat flux are large. The magnitude of the Chandrasekhar number varies, and different classes of solutions are found for different orders of magnitude of it. In each case, the principal focus is on the unique solution that maximizes F.

3. Single- and Multi-modal Regimes*

(a) Case $Q \ll R^{\frac{1}{2}}$

Magnetic effects do not become significant immediately on Q becoming nonzero; they can be regarded initially as being small perturbations to the solution with no field.

The solution in the range $Q \ll 1$ is essentially that given by Howard (1965) for $Q = 0$. The problem does not have a multiboundary layer structure. Thus there exists a single-mode solution only, and its boundary layer structure is unaffected by the magnetic field. This single mode (represented by α_1) has a nonuniform interior and a thin boundary layer of thickness δ_1 close to each boundary. In the interior of the α_1 mode, the dependent variables are all of order one. In the boundary layer region we find that $\omega_1 \approx \delta_1$, $\theta_1 \approx \delta_1^{-1}$ and $H_1 \approx \delta_1$. The heat flux is independent of Q and is maximized for the wavenumber α_1, which is found to be of order one. The dependence of F and δ_1 on R is the same as in the case with no field, that is, $F \propto R^{\frac{1}{3}}$ and $\delta_1 \propto R^{-\frac{1}{3}}$.

The solution in the range $1 \ll Q \ll R^{\frac{1}{2}}$ is qualitatively the same as in the range $Q \ll 1$, except that α_1 is now in the maximizing range $\alpha_1 = O(Q^{\frac{1}{2}})$. Since α_1 is now large, there exists also an intermediate layer of thickness α_1^{-1}. The interior of the α_1 mode is now uniform and we find that $\omega_1 \approx \alpha_1^{-1}$, $\theta_1 \approx \alpha_1$ and $H_1 \approx 0$ in this region. In the intermediate layer we have $\omega_1 \approx \alpha_1^{-1}$, $\theta_1 \approx \alpha_1$ and $H_1 \approx \alpha_1^{-2}$. In the boundary region we have $\omega_1 \approx \delta_1$, $\theta_1 \approx \delta_1^{-1}$ and $H_1 \approx \delta_1\alpha_1^{-1}$.

* For details of the mathematical analysis of the multi-modal regimes, see Paper I.

(*b*) *Case* $R^{\frac{1}{2}} \ll Q \ll R^{\frac{3}{4}} \ln R$

We consider four regions for each α_n mode ($n = 1, 2, ..., N$): the interior, the inner layer, the intermediate layer and the thermal layer. The interior of each mode coincides with the thermal layer of the previous mode. Coupling among the different modes occurs only between the nth and the $(n-1)$th mode in the $(n-1)$th boundary layer. It is assumed that

$$\delta_N \ll \alpha_N^{-1} \ll \varepsilon_N \ll \delta_{N-1} \ll ... \ll \delta_n \ll \alpha_n^{-1} \ll \varepsilon_n \ll \delta_{n-1} \ll ... \ll \varepsilon_1 \ll 1 \quad \text{as} \quad \alpha_n \to \infty, \tag{9}$$

where ε_n $(= Q^{\frac{1}{2}} \alpha_n^{-2})$, α_n^{-1} and δ_n are respectively the thickness of the inner, intermediate and thermal layers of the α_n mode. Without loss of generality we restrict ourselves to consider the boundary layer structure near the lower boundary.

In the interior of the α_n mode we define $\zeta_{n-1} = z\,\delta_{n-1}^{-1}$ as the boundary layer variable for the $(n-1)$th mode. Equations (7a) and (7b) then give

$$\alpha_n^2\,\omega_n - \theta_n = Q\,\delta_{n-1}^{-1}\,\mathrm{d}H_n/\mathrm{d}\zeta_{n-1}, \qquad \delta_{n-1}\,\alpha_n\,H_n = \mathrm{d}\omega_n/\mathrm{d}\zeta_{n-1}. \tag{10a, b}$$

Use of the relations (9) in equations (10) yields

$$\alpha_n^2\,\omega_n = \theta_n. \tag{11}$$

In the inner layer of the α_n mode we define $\xi_n = z\,\varepsilon_n^{-1}$ as the variable. Since there is no coupling between the modes in this layer, and the conductive term is not yet important, equation (7c) gives

$$\omega_n\,\theta_n = 1. \tag{12}$$

We then find from equations (7a), (7b) and (12) that as $\xi_n \to 0$ we have

$$\alpha_n\,\omega_n = \xi_n(2\ln \xi_n^{-1})^{\frac{1}{2}}, \qquad \alpha_n\,Q^{\frac{1}{2}}\,H_n = (2\ln \xi_n^{-1})^{\frac{1}{2}}. \tag{13}$$

In the intermediate layer of the α_n mode we define $\eta_n = z\,\alpha_n$ as the variable. Since there is no coupling between the modes in this layer, and conductive terms are not yet important, equation (12) is still valid in this layer. We then find from the governing equations (7a) and (7b), after applying matching conditions (matching solutions to the corresponding solutions in the inner layer), that we have

$$\omega_n = A_n\,\eta_n, \qquad \alpha_n\,H_n = A_n\{1 - \exp(-\eta_n)\}, \tag{14}$$

where

$$A_n = \{Q^{-1}\ln(Q\,\alpha_n^{-2})\}^{\frac{1}{2}}. \tag{15}$$

In the thermal layer of the α_n mode we define $\zeta_n = z\,\delta_n^{-1}$ as the variable. We find from the equations (7), after applying matching conditions (matching the solutions to the corresponding solutions in the intermediate layer) and a procedure similar to that used in Paper I, that

$$\omega_n = B_n\,\zeta_n, \qquad H_n = B_n\,\alpha_n^{-1}\,\zeta_n, \tag{16a, b}$$

$$B_n \theta_n = \tfrac{1}{3} C_n^{\frac{1}{2}} \zeta_n (6^{\frac{1}{2}} - \tfrac{1}{2} C_n^{\frac{1}{2}} \zeta_n^2) \qquad \text{for} \qquad 0 \leqslant \zeta_n \leqslant (6/C_n)^{\frac{1}{4}}, \tag{17a}$$

$$= \zeta_n^{-1} \qquad\qquad (6/C_n)^{\frac{1}{4}} \leqslant \zeta_n, \tag{17b}$$

with

$$n = 1, \ldots, N-1, \tag{18}$$

where

$$B_n = Q^{-\frac{1}{2}} \alpha_n \delta_n \{\ln(Q\alpha_n^{-2})\}^{\frac{1}{2}}, \qquad C_n = Q^{-1} \alpha_n^2 \delta_n^4 \alpha_{n+1}^4 \{\ln(Q\alpha_n^{-2})\}. \tag{19a, b}$$

The expression (17b) for θ_n is valid for the condition (18). For $n = N$, equation (7c) has the solution

$$2B_n \theta_n = D_n \zeta_n \int_0^1 (1-t^2)^{-\frac{1}{4}} \exp(-\tfrac{1}{2} D_n \zeta_n^2)\, \mathrm{d}t, \tag{20}$$

where

$$D_n^2 = Q^{-1} F R \alpha_n^2 \delta_n^4 \ln(Q\alpha_n^{-2}). \tag{21}$$

To determine F we evaluate the expressions $[\![\, |\nabla\theta|^2]\!]$ and $[\![(1 - \langle \omega\theta \rangle)^2]\!]$ in equation (4) and, after a formal procedure to maximize F (see Paper I, Section 5), we obtain:

$$\alpha_n = b_n \exp\{\tfrac{2}{5}(1 - \tfrac{3}{8} \times 6^{-n+1}) \ln R - \tfrac{1}{10}(1 - 6^{-n+1}) \ln Q\}$$

$$\times \prod_{K=1}^{n-1} \{\ln(Q^{3/2}/Rg_K^3)\}^{\frac{1}{2} \times 6^{K-n}}, \tag{22}$$

$$g_n = \exp\{\tfrac{1}{5}(\tfrac{3}{2} + 6^{-n}) \ln Q - \tfrac{3}{10}(\tfrac{2}{3} + 6^{-n}) \ln R\} \{\ln(Q^{3/2}/Rg_n^3)\}^{-1/2}$$

$$\times \prod_{K=1}^{n} \{\ln(Q^{3/2}/Rg_K^3)\}^{6^{K-n-1}}, \tag{23}$$

$$F = K_N \exp\{\tfrac{3}{5}(1 - 6^{-N}) \ln R - \tfrac{2}{5}(1 - 6^{-N}) \ln Q\}$$

$$\times \prod_{K=1}^{N} \{\ln(Q^{3/2}/Rg_K^3)^{\frac{1}{3} \times 6^{K-N}}, \tag{24}$$

where

$$b_{n+1} = \exp[\tfrac{2}{5}\{n - 1 - \tfrac{1}{5}(1 - 6^{-n+1})\} \ln 6 + \tfrac{3}{4}\{1 + \tfrac{1}{5}(1 - 6^{-n+1})\} \ln 2$$

$$- \tfrac{1}{3}\{1 + \tfrac{1}{5}(1 - 6^{-n+1})\} \ln \beta + \tfrac{3}{8}\{1 + \tfrac{1}{15}(1 - 6^{-n+1})\} \ln\{5/(8 \times 6^N - 3)\}],$$

$$\text{for} \quad 0 \leqslant n \leqslant N-1, \tag{25}$$

$$K_N = I^{-4/3} \exp[\tfrac{8}{25}(-6 + 5N + 6^{-N+1}) \ln 6 + \tfrac{18}{5}(1 - 6^{-N}) \ln 2$$

$$- \tfrac{4}{15}(1 - 6^{-N+1}) \ln \beta + \tfrac{1}{10}(16 - 6^{-N+1}) \ln\{5/(8 \times 6^N - 3)\}], \tag{26}$$

with

$$\prod_{K=1}^{0} \equiv 1, \qquad g_n = \alpha_n \delta_n, \qquad I = 1 \cdot 062, \qquad \beta = 1 \cdot 396. \tag{27}$$

To determine the total number of modes we use the relations (9) with equations (16)–(23). We find that for either $R^{1/2} \ll Q \ll R^{9/14}$ or $R^{15/22} \ll Q \ll R^{3/4}$ there is one mode; for either $R^{9/14} \ll Q \ll R^{57/86}$ or $R^{87/130} \ll Q \ll R^{15/22}$ there are two modes. In general, if we ignore the logarithmic terms, a total of l modes exist when either:

$$R^P \ll Q \ll R^q, \tag{28a}$$

where

$$P = 2(1-\tfrac{9}{4}\times 6^{-l})/3(1-6^{-l}), \qquad q = 2(1-\tfrac{9}{4}\times 6^{-l-1})/3(1-6^{-l-1}); \tag{28b, c}$$

or

$$R^{\tilde{P}} \ll Q \ll R^{\tilde{q}}, \tag{29a}$$

where

$$\tilde{P} = (1+\tfrac{3}{2}\times 6^{-l-1})/(\tfrac{3}{2}+6^{-l-1}), \qquad \tilde{q} = (1+\tfrac{3}{2}\times 6^{-l})/(\tfrac{3}{2}+6^{-l}) \tag{29b, c}$$

for a given R. We note that l increases with increasing Q in the range

$$R^{1/2} \ll Q \ll R^{2/3} \tag{30}$$

and decreases with increasing Q in the range

$$R^{2/3} \ll Q \ll R^{3/4}, \tag{31}$$

and that $l \to \infty$ as $Q \to R^{2/3}$, and the lower and upper parts of the inequalities merge. It can also be shown easily that, for a given Q, l is determined uniquely from the inequalities (28a) and (29a). If we include the logarithmic term, it can be shown easily, using the relations (9) and equations (22) and (23), that the solution in the range (31) is also valid in the range

$$R^{2/3} \ll Q \ll R^{3/4}\ln R. \tag{32}$$

(c) Case $R^{3/4}\ln R \ll Q \ll (R\ln R)^{4/5}$

The boundary layer solution (described above) in the range (32) is based essentially on the condition that

$$\alpha_1 \leqslant R^{\frac{1}{4}} \ll \delta_1^{-1}. \tag{33}$$

It is seen readily from equations (22) and (25) that the value of α_1 which maximizes F is indeed

$$\alpha_1 = (\tfrac{1}{9}R)^{\frac{1}{4}}. \tag{34}$$

Now as Q further increases beyond the range (32) we have a new condition that

$$\alpha_1 \leqslant \delta_1^{-1} \leqslant R^{\frac{1}{4}}. \tag{35}$$

Consequently F is now maximized by the largest possible value of α_1. These results indicate that there exists a new boundary layer solution for the case in which $\delta_1 = O(\alpha_1^{-1})$, and it turns out to be a single-mode solution only. The interior and the inner layer of this new boundary layer structure are the same as those discussed for the range (32). In the thermal layer of the α_1 mode (there does not exist an intermediate layer for the maximizing solution here) we define $\zeta_1 = z/\delta_1$ as the variable in this layer. We find from the equations (7), after applying matching conditions (matching the solutions to the corresponding solutions in the inner layer), that the equations (16) hold ($n = 1$), where

$$\alpha_1 = \delta_1^{-1}, \tag{36}$$

and that $\hat{\theta}_1$ (where $\hat{\theta}_1 = B_1\theta_1$) must be found as the solution of

$$\mathrm{d}^2\hat{\theta}_1/\mathrm{d}\zeta_1^2 - (1+\zeta_1^2)\hat{\theta}_1 = -\zeta_1, \tag{37}$$

for which

$$\hat{\theta}_1(0) = 0 \qquad \text{and} \qquad \hat{\theta}_1(\infty) \approx \zeta_1^{-1}. \tag{38}$$

Specifically, we have

$$2B_1\theta_1 = \zeta_1 \int_0^2 (1+t)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}\zeta_1^2 t)\,\mathrm{d}t. \tag{39}$$

To determine F we evaluate the expressions $[\![\,|\nabla\theta|^2]\!]$ and $[\![(1-\langle\omega\theta\rangle^2]\!]$ in equation (4) and find that

$$\alpha_1 = \varsigma^{-1} RQ^{-1}\ln(Q^{3/2}R^{-1}), \qquad F = (2\varsigma^2)^{-1} RQ^{-1}\ln(Q^{3/2}R^{-1}), \tag{40a, b}$$

where

$$\varsigma = [\![\hat{\theta}_1^2]\!] + [\![(\mathrm{d}\hat{\theta}_1/\mathrm{d}\zeta_1)^2]\!] + [\![(1-\zeta_1\hat{\theta}_1)^2]\!] = 1{\cdot}77. \tag{41}$$

The present analysis assumes that

$$\alpha_1^{-1} \ll Q^{\frac{1}{2}}\alpha_1^{-2} \ll 1. \tag{42}$$

Using the results (35), (36), (40a) and (42) we find that the boundary layer structure is valid provided

$$R^{3/4}\ln R \leqslant Q \ll (R\ln R)^{4/5}. \tag{43}$$

(d) Case $(R\ln R)^{4/5} \ll Q \ll R$

For Q larger than $O(R\ln R)^{4/5}$, the condition (42) is no longer valid, and we must then have a new condition in which

$$\alpha_1^{-1} \ll \delta_1 \ll Q^{\frac{1}{2}}\alpha_1^{-2}. \tag{44}$$

The analysis for the condition (44) (though omitted from the present paper) shows that the boundary layer structure has a single mode only and that the value of α_1 which maximizes F is $O(Q^{\frac{1}{4}})$.

The α_1 mode has three regions: the interior, the thermal layer and a thinner layer of thickness α_1^{-1}. In the interior, z is $O(1)$, and the equations (7) give the following solution as $z \to 0$:

$$Q^{\frac{1}{2}}\omega_1 = \alpha_1 z\{2\ln(Q^{\frac{1}{2}}z^{-1}\alpha^{-2})\}^{\frac{1}{2}}, \qquad \theta_1 = \omega_1^{-1}, \tag{45a, b}$$

$$\alpha_1 Q^{\frac{1}{2}} H_1 = \{2\ln(Q^{\frac{1}{2}}z^{-1}\alpha^{-2})\}^{\frac{1}{2}}. \tag{45c}$$

In the thermal layer we define $\zeta_1 = z\delta_1^{-1}$ as the variable, and find from the equations (7), after using the condition (44), that

$$\omega_1 = E_1\zeta_1, \qquad \theta_1 = \zeta_1/E_1(1+\zeta_1^2), \qquad \delta_1\alpha_1^2 H_1 = E_1, \tag{46}$$

where

$$E_1 = \alpha_1\delta_1\{2Q^{-1}\ln(Q^{\frac{1}{2}}\delta_1^{-1}\alpha_1^{-2})\}^{\frac{1}{2}}. \tag{47}$$

The solutions for ω_1 and θ_1 satisfy the required boundary conditions at $\zeta_1 = 0$. A thinner layer is then needed to adjust the solution to satisfy the correct boundary condition on H_1. This is a layer of thickness α_1^{-1}, with $\eta_1 = z\alpha_1$ as its variable. We then find from the equations (7) that

$$H_1 = E_1\delta_1^{-1}\alpha_1^{-2}\{1-\exp(-\eta_1)\}. \tag{48}$$

To determine F we evaluate the expressions $[\![\,|\nabla\theta|^2]\!]$ and $[\![(1-\langle\omega\theta\rangle)^2]\!]$ in equation (4) and find that

$$\delta_1 = \tfrac{1}{2}\pi Q\{R\ln(R/Q)\}^{-1}, \qquad F = (2/\pi^2)Q^{-1}R\ln(R/Q). \tag{49a, b}$$

Using the condition (44) for the maximizing wavenumber α_1 and equation (49a) for δ_1, we find the range of the validity of the analysis to be

$$(R\ln R)^{4/5} \ll Q \ll R. \tag{50}$$

As for the calculations of Hunter and Riahi (1975), it is found that the maximizing α_1 is of the form

$$\alpha_1 = 2.29\,Q^{\frac{1}{4}}. \tag{51}$$

4. Discussion

The above boundary layer analysis shows that it is appropriate to divide the parameter space into three different regions. For a weak magnetic field ($Q \ll R^{\frac{1}{2}}$) we find $F \propto R^{\frac{1}{3}}$, and the solution has a single mode only. The stabilizing effect of the field is so small that the maximizing flow behaves as if there is no field. According to Malkus's (1954) principle, the convective flow organizes itself in such a way as to transport the maximum amount of heat. In the thermal convection zone at large Rayleigh and Prandtl numbers (Chan 1970; Paper I), it was found that the maximum amount of heat is transported (that is, $F \propto R^{\frac{1}{3}}$) by the flow which has a single mode for free boundaries or infinitely many modes for rigid boundaries. Since the present study considers free boundaries which have no stabilizing effect on the convective motion, the flow structure contains a single mode only (for a weak field) as is expected. From the results of Section 3*a* and also the transformations (2), we arrive at the following conclusions regarding the magnitudes W_n, T_n and H_n of the vertical dependence of the vertical velocity, the temperature fluctuation and the vertical component of the magnetic fluctuation respectively within each region of the α_n mode: (i) W_1 and H_1 increase with R; (ii) T_1 decreases with increasing R in all regions except the thermal layer, where $T_1 = O(1)$; (iii) H_1 decreases with increasing τ; (iv) for $1 \ll Q \ll R^{\frac{1}{2}}$, W_1 decreases with increasing Q in the interior and the intermediate layer only; (v) for $1 \ll Q \ll R^{\frac{1}{2}}$, T_1 increases with Q in the interior and the intermediate layer only; (vi) for $1 \ll Q \ll R^{\frac{1}{2}}$, H_1 is zero in the interior and decreases with increasing Q in the intermediate and thermal layers.

For a moderately strong magnetic field ($R^{\frac{1}{2}} \ll Q \ll R^{\frac{2}{3}}$) there are a finite number of modes, although the number of modes increases with Q in this range. The flux F is a decreasing function of Q and increases with R. Note that the total number of modes N is determined essentially from the condition $\alpha_N \ll Q^{\frac{1}{2}}$, where N is the maximum positive integer that satisfies this inequality. It can be shown that this condition is equivalent to the condition that $F \ll R^{\frac{1}{3}}$ (the relations (28a) can be derived from either of these conditions), where F is the heat transport for the flow structure having N modes. This implies that the convective flow tends to counteract and oppose the stabilizing effect of the field by transporting the maximum amount of heat (which is as close as possible to $R^{\frac{1}{3}}$) and retaining the smallest scale of motion (which is as close as possible to the length scale proportional to $Q^{-\frac{1}{2}}$ in the nonmagnetic case). Thus the convective flow fixes N once this goal is reached. Although the expression for F (equation 24) shows that the heat transport is a decreasing function of Q, the flow structure has a sufficient number of modes for the heat transport to approach its nonmagnetic value $R^{\frac{1}{3}}$ as close as possible. Chan (1974) obtained similar results from his investigation of turbulent convection under a rotational constraint for a moderately large Taylor number. He concluded that the flow arranges itself so as to tend to offset the stabilizing effect of the rotational constraint, at least in so far as the heat transport is concerned. Following Veronis (1959) we may also conclude from the present study that, for a high Prandtl number flow, the effect of a moderately strong or strong magnetic field is always to suppress the convective motion so that the heat transport is always less than in the nonmagnetic case. From equation (22) we can see that the wavenumbers α_n decrease with increasing Q, although the number of the small scales of motion associated with these wavenumbers increases with Q. Increasing the thickness of the thermal layer of each α_n mode is also a consequence of the stabilizing effect of the field. From the equations (2) and the results of Section 3*b* we can make the following conclusions for the magnitudes W_n, T_n and H_n which are valid for $R^{\frac{1}{2}} \ll Q \ll (R \ln R)^{4/5}$: (i) W_n and H_n increase with R and decrease with increasing Q; (ii) H_n decreases with increasing τ; (iii) T_n increases with R and decreases with increasing Q in all but the thermal layer, where $T_n = O(1)$ (within logarithmic terms).

For $Q = O(R^{\frac{2}{3}})$ the flow has almost infinitely many modes and we have $F \propto R^{\frac{1}{3}}$. This is expected because the stabilizing effect of the magnetic field is sufficiently strong that the flow retains infinitely many modes so that F can reach its nonmagnetic value.

For a strong magnetic field ($R^{\frac{2}{3}} \ll Q \ll R$) there are three subranges, for which separate discussions are needed:

In the subrange $R^{2/3} \ll Q \ll R^{3/4} \ln R$ there are a finite number of modes, although the number of modes decreases as Q increases in this range. The total number of modes for this case is determined essentially by the condition $F \gg \alpha_N$, where N is the maximum positive integer that satisfies this inequality. Since the stabilizing effect of the field thickens the thermal layer, F decreases with increasing Q. Therefore N decreases according to the above condition ($F \gg \alpha_N$). The rest of the discussion for the case of a moderately strong field is also applicable to this subrange.

In the subrange $R^{3/4} \ln R \ll Q \ll (R \ln R)^{4/5}$, the magnetic field is so strong that the flow is left with only one mode α_1. The intermediate layer is now merged with the thermal layer, and $F = O(\alpha_1)$.

In the subrange $(R \ln R)^{4/5} \ll Q \ll R$, the thermal layer is thicker than the intermediate layer, and $F \ll \alpha_1$. The flow structure contains a single α_1 mode only, and $\alpha_1 = O(Q^{\frac{1}{4}})$. The length scale for the α_1 mode is now a decreasing function of Q. The magnitudes W_1, T_1 and H_1 now have the following properties: (i) W_1, T_1 and H_1 increase with R; (ii) H_1 decreases with increasing τ or Q; (iii) T_1 decreases with increasing Q; (iv) W_1 decreases with increasing Q in the interior. However, W_1 increases with Q in the thermal layer. Since the subrange under discussion is that immediately prior to the stability region ($Q \geqslant R$), and it is found that the convective motion is amplified by the magnetic field in the thermal layer for this subrange, then the reason for a qualitative agreement with the finding of Busse (1975) is apparent. Busse (1975) investigated the effect of a weak vertical magnetic field on two-dimensional steady convection of small finite amplitude. He found, for example, that the influence of the magnetic field decreases with increasing amplitude of convection, so that finite amplitude onset of steady convection becomes possible at R values considerably less than those predicted by linear theory.

It is seen from equation (49b) that, as $Q \to R$, F approaches order unity, which is consistent with the linear theory (Chandrasekhar 1961). However, we must note that the present study is supposed to be valid asymptotically for strong convective flow (large R and F). Therefore, for slow convective motion, we should not expect to obtain quantitative agreement with what actually happens.

Van der Borght *et al.* (1972) considered the effect of a vertical magnetic field on convection using the mean field approximation. They assumed the flow to be laminar and to have a single mode only, with a current-free adjoining medium. They investigated two special cases: Case I, $Q = O(1)$ and $\alpha_1 = O(1)$; Case II, $Q = O(R)$. For Case I, they found $F \propto R^{\frac{1}{3}}$, just as in the nonmagnetic case, although the proportionality constant decreases monotonically as Q increases. Case II was found to be a marginal instability case. Their general result that steady convection seems to persist in the face of magnetic inhibition is in agreement with the present study.

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