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Hyperbolic Motion in Gravitational Theories

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Abstract

The equation of motion of a high speed test particle in the field of a spherical mass is discussed for a parametric range of gravitational theories, including general relativity. It is shown how in principle such hyperbolic orbits may discriminate between these theories.

Introduction

In analysing gravitational field theories to determine the paths of test particles, elliptic orbits have provided the main interest in the case of the field due to a spherical mass. A well-known example is the prediction in general relativity (GR) that an elliptic orbit in the Sun's field will undergo a rotation of perihelion, by about 43″ arc century^{-1} in the case of the planet Mercury. Usually, hyperbolic orbits have been considered of special significance only in the limit as the orbital speed $V \to c$ (e.g. optical or radio photons in trajectories grazing the Sun). In the GR case the Sun is predicted to cause a deflection of the photon path through $1 \cdot 75''$ arc at the grazing incidence.

Hyperbolic orbits have of course been included in general theoretical analyses of free particle motions in the GR Schwarzschild field. Comprehensive exact treatments have been given by Hagihara (1931), Darwin (1958, 1961) and Mielnik and Plebanski (1962). These analyses deal only with the GR case, and the solutions are given in terms of elliptic functions from which it is not always easy to make a comparison with Newtonian theory or other gravitational theories.

Approximate equations of motion of the bodies or fluid creating the field have also been derived in GR for a general system of gravitating material by the post-Newtonian and the so-called PPN formalisms (Einstein and Infeld 1949; Chandrasekhar 1965, 1969; Chandrasekhar and Nutku 1969). These analyses depend on expansions of functions in powers of v/c ($\ll 1$), or their equivalent, v being a typical material velocity.

The purpose of this paper is to add a small feature complementary to these previous efforts. Hyperbolic orbits of test particles are considered for a Schwarzschild-type metric appropriate to a parametric range of gravitational theories, including the GR case. The analysis is carried to an adequate approximation in m/r (m being the gravitational radius of the central body) to produce the post-Newtonian character of the orbits: terms of order m^2/r^2 are neglected and for the particle of nonzero rest mass this will require that V^4/c^4 is negligible, V being its speed at infinity. The photon

case when $V = c$ is dealt with independently. Finally it is shown that escape orbits provide in principle independent means to determine the parameters of the gravitational theory in best agreement with observation.

The Metric

Attention will be restricted to gravitational theories that have the following properties:

(1) compatibility with special relativity in local frames of reference;
(2) fulfilment of the principle of equivalence at least in the case of test particles of small mass, so that $m_i = m_g$;
(3) reduction to Newtonian theory as a first approximation where there is a gravitational field.

We follow Eddington (1957), Robertson (1962), Misner (1969) and Thorne and Will (1970) in supposing that theories satisfying properties 1, 2 and 3 can be set in a Riemannian geometric form, signature ± 2, in which at least test particles, if not massive bodies, move on geodesics of the Riemannian space–time. Thorne and Will (1970) have described the situation as follows: Each such theory (satisfying properties 1, 2 and 3) attributes to space–time a unique metric g_{ij} whose geodesics are the trajectories of freely falling test bodies. An infinite number of such theories is possible, each differing from the others by the manner in which the matter of the universe generates the metric, and by the manner in which non-test-particle fields respond to the metric.

We know from the Einstein, Infeld and Hoffman (EIH) theory that in GR massive bodies move on geodesics of the space–time created by the rest of the matter in accordance with Einstein's equations. The Brans–Dicke (1961) theory obeys the equivalence principle and has the geodesic property in the limit of small mass, that is, $m_i \to m_g$ as $m_i \to 0$, as shown by Nordtvedt (1968). Additionally it is known that if the mass in a gravitational theory depends on its space–time position, $m(x^i)$, and the equation of motion may be derived from a variational principle of the form

$$\delta \int m(g_{ij} u^i u^j)^{\frac{1}{2}} \, \mathrm{d}s = 0 ,$$

then by a conformal transformation the metric of a new space–time may be obtained in which free masses do move on geodesics (Brans and Dicke 1961).

In the present paper we shall be dealing with test particles only, and from the foregoing discussion we shall assume that we may take their world lines to be geodesics of the metric.

For a spherically symmetric field we take the metric in the expansion form

$$\mathrm{d}s^2 = \left(1 - \frac{2\alpha m}{r} + \frac{2\beta m^2}{r^2} + \ldots\right)\mathrm{d}t^2 - \left(1 + \frac{2\gamma m}{r} + \ldots\right)\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2 \sin^2\theta\,\mathrm{d}\phi^2 , \quad (1)$$

where α, β, γ are constant parameters, $m = GM/c^2$ is the gravitational radius of the mass M generating the field, and the velocity of light c has been set equal to unity. For all theories characterized by such parameters we have to take $\alpha = 1$ for conformity at the Newtonian level of approximation. For the GR case we have $\alpha = 1$, $\beta = 0$ and $\gamma = 1$. In the Brans–Dicke (1961) theory $\alpha = 1$, $\beta = 1/(\omega+2)$ and $\gamma = (\omega+1)/(\omega+2)$, where ω is the dimensionless constant of the theory. Note that

these constants are different from those that would apply if we had taken the Schwarzschild-type metric in the isotropic form. We can convert the metric (1) to the isotropic form by the transformation

$$r = R(1 + \gamma m R^{-1} + \ldots), \tag{2}$$

giving

$$ds^2 = \left(1 - \frac{2\alpha' m}{R} + \frac{2\beta' m^2}{R^2} + \ldots\right)dt^2 - \left(1 + \frac{2\gamma' m}{R} + \ldots\right)\left(dR^2 + R^2\,d\theta^2 + R^2 \sin^2\theta\, d\phi^2\right), \tag{3}$$

where

$$\alpha' = \alpha, \qquad \beta' = \alpha\gamma + \beta, \qquad \gamma' = \gamma. \tag{4}$$

In the metric (3), it is well known that GR has for a spherical mass $\alpha' = 1$, $\beta' = 1$, $\gamma' = 1$ and this agrees, in virtue of equations (4), with the parameters given earlier for the standard expansion form. In the Brans–Dicke theory, in agreement with equations (4), we have

$$\alpha' = \beta' = 1, \qquad \gamma' = (\omega + 1)/(\omega + 2).$$

In the present paper we prefer the standard type of expansion given by the metric (1) since the analysis is less complicated, although entirely equivalent to that following equations (3) and (4) through the transformation (2).

Equation of Hyperbolic Motion

We set $\alpha = 1$ in accordance with the previous section and write the usual equations for a test particle moving on an ordinary geodesic:

$$\phi = \tfrac{1}{2}\pi, \tag{5}$$

$$r^2\, d\theta/ds = H, \tag{6}$$

$$\left(1 - \frac{2m}{r} + \frac{2\beta m^2}{r^2} + \ldots\right)\frac{dt}{ds} = A, \tag{7}$$

H and A being constants. To these equations we add the integral provided by the metric

$$\left(1 - \frac{2m}{r} + \frac{2\beta m^2}{r^2} + \ldots\right)\left(\frac{dt}{ds}\right)^2 - \left(1 + \frac{2\gamma m}{r} + \ldots\right)\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\theta}{ds}\right)^2 = 1. \tag{8}$$

Note that no term in m^2/r^2 is specifically required in the coefficient of dr^2 in the metric (1), in contrast to the coefficient of dt^2, in order that the equation of motion be finally correct up to terms in m/r.

Combining equations (6), (7) and (8) we obtain

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{\{A^2/(1 - 2mr^{-1} + 2\beta m^2 r^{-2} + \ldots) - 1\} - H^2 r^{-2}}{H^2(1 + 2\gamma m r^{-1} + \ldots)r^{-4}}. \tag{9}$$

Let $r = \rho$ be the point of closest approach to the central body, where $dr/d\theta = 0$.

It follows that

$$H^2 = \rho^2\left(\frac{A^2}{1-2m\rho^{-1}+2\beta m^2\rho^{-2}+\ldots}-1\right). \tag{10}$$

Substitution in equation (9) then gives

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = \frac{r^4}{\rho^2(1+2\gamma mr^{-1}+\ldots)}\left\{\frac{(1-2mr^{-1}+2\beta m^2r^{-2}+\ldots)^{-1}-A^{-2}}{(1-2m\rho^{-1}+2\beta m^2\rho^{-2}+\ldots)^{-1}-A^{-2}}-\frac{\rho^2}{r^2}\right\}. \tag{11}$$

If V is the speed of the particle at $r = \infty$ then

$$A^{-2} = 1 - V^2. \tag{12}$$

If we substitute from equation (12) into (11) and expand the expressions in m/r and m/ρ inside the braces, we obtain

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = \frac{r^4}{\rho^2(1+2\gamma mr^{-1}+\ldots)}\left\{\frac{V^2+2mr^{-1}+(4-2\beta)m^2r^{-2}+\ldots}{V^2+2m\rho^{-1}+(4-2\beta)m^2\rho^{-2}+\ldots}-\frac{\rho^2}{r^2}\right\}. \tag{13}$$

Write now

$$a = mc^2/V^2 = GM/V^2, \tag{14}$$

so that in the Newtonian theory a would be the semimajor axis of the hyperbolic orbit. Then equation (13) becomes

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = \frac{r^4}{\rho^2(1+2\gamma mr^{-1}+\ldots)}\left\{\frac{1+2ar^{-1}+(4-2\beta)amr^{-2}+\ldots}{1+2a\rho^{-1}+(4-2\beta)am\rho^{-2}+\ldots}-\frac{\rho^2}{r^2}\right\}. \tag{15}$$

In equation (15) only terms containing m explicitly are small, i.e. of the order of m/r (or m/ρ); all other terms are Newtonian terms for the hyperbolic orbit.

To proceed further we bring the expression in the braces under one denominator, divide out by the factor $1+2a\rho^{-1}$ of order unity and take the square root of the whole expression. We now retain terms to order m/r only, and find that

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{r}{\rho^{\frac{1}{2}}(\rho+2a)^{\frac{1}{2}}}\,\frac{\{(r+a)^2-(\rho+a)^2\}^{\frac{1}{2}}}{1+(2-\beta)\,a(\rho+2a)^{-1}\,m\rho^{-1}+\gamma mr^{-1}}. \tag{16}$$

On inversion equation (16) may be written to the same order:

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{1+(2-\beta)\,a(\rho+2a)^{-1}\,m\rho^{-1}+\gamma mr^{-1}}{r^2[(\rho^{-1}-r^{-1})\{(\rho+2a)^{-1}+r^{-1}\}]^{\frac{1}{2}}}. \tag{17}$$

The post-Newtonian terms are those containing m explicitly and in view of the standard integral of the Newtonian equation, we set

$$l/r = 1+e\cos\eta, \tag{18}$$

where as in Newtonian theory

$$l = \rho(\rho+2a)/a, \qquad e = (\rho+a)/a. \tag{19}$$

At the Newtonian level η would be the same as θ, and then l would be the semi-latus-rectum and e the eccentricity (>1) of the hyperbolic orbit. The substitution leads to the exact integral of equation (17):

$$\theta = \eta + ml^{-1}\{(2+\gamma-\beta)\eta+\gamma e\sin\eta\}, \tag{20}$$

where θ is measured from perihelion at $\eta = 0$. Our required post-Newtonian solution is therefore equation (20) combined with (18). A particular orbit depends on the geometrical parameters l and e, or in view of equations (19), ρ and a. We note that ρ and a are unambiguously related to the dynamical constants of integration H and A. Thus a is given by equation (14) where V is defined in terms of A by equation (12), while ρ is connected to A and H through equation (10). In turn A and H can be identified unambiguously in the flat space–time geometry at $r \to \infty$. For A is measured by the velocity V through equation (12), and H is given by equation (6) which can be expressed as

$$H = pV/(1-V^2)^{\frac{1}{2}}, \tag{21}$$

where p is the impact parameter for the orbit at $r \to \infty$. Thus ρ and a, and hence l and e, are determined by the observables V and p at $r \to \infty$.

Case of a Photon Trajectory

For a particle of zero rest mass we have $V = c$ and then a as defined by equation (14) becomes equal to m. Our previous analysis depended on a being of order r, m/r small and hence m/a being small. Since by equation (14) $m/a = V^2/c^2$, the neglect of terms $O(m^2/r^2)$ or $O(m^2/a^2)$ implies that V^4/c^4 is negligible. Both equations (17) and (18) must therefore be modified to apply to the photon case. The safest procedure is to start with the null geodesic equations of the metric field (1) using a nonzero affine parameter in place of s. Although a is now equal to m we can make expansions in powers of m/r or m/ρ as before, and instead of equation (17) we then obtain to order m/r:

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\rho}{r(r^2-\rho^2)^{\frac{1}{2}}}\left\{1 + m\left(\frac{1}{\rho} - \frac{1}{\rho+r} + \frac{\gamma}{r}\right)\right\}. \tag{22}$$

It is consistent that this is also what we obtain from equation (17) if we set $a = m$ in that result and expand appropriately the terms containing a. It will be noticed that to order m/r the term containing β does not survive.

The appropriate substitution is now

$$r^{-1} = (\cos\eta)/\rho, \tag{23}$$

and the exact integral of equation (22) is

$$\theta = \eta + m\rho^{-1}(\tan\tfrac{1}{2}\eta + \gamma\sin\eta). \tag{24}$$

We could of course have obtained equations analogous to (20) and (24) if we had used the isotropic form (3) of the metric. It may be shown that calculation of the geodesic equations in that metric, although less elegant, gives precisely the same results as substituting the transformation equations (2) and (4) into (17) and (22), as indeed must follow from the covariance of the geodesic equations. In the latter process we have to note that in the analogue of equations (19) the parameter a is invariant while the perihelion coordinate becomes ρ' given by $\rho = \rho'(1+\gamma m/\rho')$. In the simpler case of the null geodesic, the analogue of equation (22) is

$$\frac{\mathrm{d}\theta}{\mathrm{d}R} = \frac{\rho'}{R(R^2-\rho'^2)^{\frac{1}{2}}}\left\{1 + (\gamma+1)m\left(\frac{1}{\rho'} - \frac{1}{\rho'+R}\right)\right\}, \tag{25}$$

while the analogues of equations (23) and (24) are

$$R^{-1} = (\cos\eta')/\rho', \qquad \theta = \eta' + m\rho'^{-1}(\gamma+1)\tan\tfrac{1}{2}\eta'. \tag{26}$$

Determination of β and γ

Particular gravitational theories in the present model are characterized by the parameters β and γ. If we let $r \to \infty$, equation (18) shows that $\cos\eta \to -e^{-1}$, so that $\eta \to \pm(\pi - \arccos e^{-1})$. The total change in η during the whole motion is therefore

$$[\eta] = \pi + 2\delta, \tag{27}$$

where

$$\operatorname{cosec}\delta = e. \tag{28}$$

At the Newtonian level 2δ is the total angular deviation of the motion of a particle whose orbital parameters are l and e as in equation (18). At the post-Newtonian level we find from equation (20) that

$$[\theta] = \pi + 2\delta', \tag{29}$$

where

$$2\delta' = 2\delta + ml^{-1}\{(2+\gamma-\beta)(\pi+2\delta) + 2\gamma\cot\delta\}. \tag{30}$$

This quantity depends only on the orbital parameters l and e, together with the unknowns β and γ.

To see the order of magnitude of the post-Newtonian term in the GR case, consider the limiting situation when the particle just grazes the central body and choose V so that classically the asymptotes of the orbit would be at right angles. Then $e = \sqrt{2}$ and $l = a$ in accordance with equations (28) and (19). If we suppose the body is the Sun then we can calculate the post-Newtonian term by setting (Allen 1973)

$$\rho = R_\odot = 6{\cdot}960\times10^5\ \text{km}, \qquad m = 1{\cdot}476\ \text{km}, \qquad c = 2{\cdot}998\times10^5\ \text{km s}^{-1}. \tag{31}$$

We find that $a = 1{\cdot}680\times10^6$ km (and $V = 281\ \text{km s}^{-1}$). Hence for the GR case one finds

$$2\delta' = 90^\circ + 2{\cdot}92''. \tag{32}$$

The post-Newtonian effect would therefore be an extra angular deviation equal to $2{\cdot}92''$ arc.

Turning to the case of the photon, we find that when $r \to \infty$ equation (23) gives $\eta \to \pm\frac{1}{2}\pi$. Hence by equation (24)

$$[\theta] = \pi + 2\delta'', \tag{33}$$

where

$$2\delta'' = 2m\rho^{-1}(1+\gamma). \tag{34}$$

For the GR case we get the familiar result $4m/\rho$ for the angular deviation of an optical or radio photon whose perihelion coordinate is ρ.

It is to be noted that, in equations (30) and (34), δ' and δ'' are observational quantities measurable in principle in the flat space–time at spatial infinity. The right-hand sides of these equations give the predicted values of δ' and δ'' if any gravitational theory (values of β and γ) is specified, along with particular orbital parameters l and

e (related to observables V and p at spatial infinity). For such a specification, δ' and δ'' are invariant quantities independent of the metric style chosen. If we had chosen the isotropic form of metric, say, then although the right-hand side of equation (30) would be formally different it would be numerically the same result for orbital parameters l' and e' corresponding to l and e of the standard metric. In the case of the deflected photon it is clear from equations (26) that the isotropic metric form leads to precisely the same value of δ'' as in equation (34).

With regard to the practical measurement of V and p, it is not in the spirit of the present paper to elaborate on the necessary observational techniques. But it is envisaged that these could be determined from an artificial satellite at great solar distance in a circular orbit using radar methods when the test projectile was at much greater distance on an asymptote of its orbit.

Concluding Remarks

The analysis of this paper has provided specific post-Newtonian approximations as integrals to the equations of hyperbolic motion in the centrally symmetric case. These are of some interest in themselves since in the GR case the concern has usually been with elliptic orbits. In addition it has been shown that the possibility exists in principle of finding a best fitting metric of the form (1) compatible with observation using escape orbits only. Thus γ can be determined in the well-known way by observations involving radio or optical photons, in accordance with equation (34). Then the difference $\gamma-\beta$ may be obtained in principle by experiments with small projectiles on hyperbolic orbits, in accordance with equation (30). In the latter experiments the theoretical parameters l and e of the orbit may be derived in principle unambiguously by measurements of V and p at great distance.

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