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Spherically Symmetric Charged Dust Distribution in General Relativity. I General Solution

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Abstract

The Einstein–Maxwell field equations characterizing a spherically symmetric charged dust distribution are solved exactly without imposing any mathematical condition on them. The solution is expressed in terms of two arbitrary variables and these can be chosen to correspond to an arbitrary ratio of charge density to mass density, thus allowing the possibility of understanding the interior of the horizon in a more precise manner.

Introduction

In the study of a static charged dust distribution in general relativity, the condition of equality between the magnitudes of charge density and mass density has the advantage of reducing the static Einstein–Maxwell field equations to a single nonlinear equation (Das 1962). The existence of a nonsingular solution to this equation which could be matched with the extreme Reissner–Nordström solution (i.e. where charge Q and mass M densities are equal) led Bonnor (1965) to conclude that the presence of charge can halt gravitational collapse. But it is a well known fact that a static charged dust distribution ultimately collapses under perturbation (De 1968; Hamoui 1969). The theory of gravitational collapse suggests that a material charged sphere, whose exterior is represented by the Reissner–Nordström metric

$$(1-2Mr^{-1}+Q^2r^{-2})^{-1}(\mathrm{d}r)^2+r^2(\mathrm{d}\theta)^2+r^2\sin^2\theta(\mathrm{d}\phi)^2-(1-2Mr^{-1}+Q^2r^{-2})(\mathrm{d}t)^2,$$

can only undergo gravitational collapse if the condition $Q^2 < M^2$ is satisfied, since the cosmic censorship hypothesis (Penrose 1969) rules out the physical validity of the conditions $Q^2 > M^2$ and $Q^2 = M^2$ (Carter 1973, p. 85). In view of this the charged dust sphere in which the magnitude of the ratio of charge to matter density is less than unity warrants a detailed study to understand more fully the interior of the horizon. This means we are required initially to find the spherically symmetric exact solution of the Einstein–Maxwell field equations.

In the present paper the Einstein–Maxwell field equations corresponding to the spherically symmetric static metric are exactly solved. It is shown that the field variables can be expressed in terms of two arbitrarily chosen functions. Since the approach is very general, it includes both the possibilities $|\sigma/\rho| < 1$ and $|\sigma/\rho| > 1$ and in the limiting case leads to $|\sigma/\rho| = 1$, where σ and ρ are the charge and mass densities respectively.

Field Equations

The Einstein–Maxwell field equations characterizing the spherically symmetric static charged dust distribution are

$$\tfrac{1}{2}\beta_{,11}+\tfrac{1}{4}\beta_{,1}^2-r^{-1}\alpha_{,1}-\tfrac{1}{4}\alpha_{,1}\beta_{,1} = 4\pi(e^{-\beta}\psi_{,1}^2-\rho' e^{\alpha}), \tag{1a}$$

$$-1+r^2e^{\alpha}\{r^{-2}+\tfrac{1}{2}r^{-1}(\beta_{,1}-\alpha_{,1})\} = -4\pi(e^{-\beta}\psi_{,1}^2+\rho' r^2), \tag{1b}$$

$$\tfrac{1}{2}\beta_{,11}+\tfrac{1}{4}\beta_{,1}^2+r^{-1}\beta_{,1}-\tfrac{1}{4}\alpha_{,1}\beta_{,1} = 4\pi(e^{-\beta}\psi_{,1}^2+\rho' e^{\alpha}), \tag{1c}$$

$$\psi_{,11}-\tfrac{1}{2}\alpha_{,1}\psi_{,1}+2r^{-1}\psi_{,1}-\tfrac{1}{2}\beta_{,1}\psi_{,1} = \sigma e^{\alpha+\frac{1}{2}\beta}, \tag{1d}$$

where the spherically symmetric static metric is chosen in the canonical form as

$$e^{\alpha}(dr)^2+r^2(d\theta)^2+r^2\sin^2\theta(d\phi)^2-e^{\beta}(dt)^2. \tag{2}$$

The variables α and β are functions only of r and $\rho' = \rho/\sqrt{4\pi}$, and the other variables have their usual meaning. A subscript 1 following a comma in equations (1) denotes differentiation with respect to r.

Solution

As mentioned in the Introduction, our object is to solve the field equations (1) without imposing any mathematical conditions on them. To effect this we introduce an auxiliary function F such that

$$e^{\beta} = 4\pi e^{F}(\psi\pm\sqrt{2})^2. \tag{3}$$

In view of equation (3), β is eliminated in terms of F and ψ and the equations (1a)–(1d) reduce to respectively

$$\frac{\psi_{,11}}{\psi\pm\sqrt{2}}-r^{-1}\alpha_{,1}-\frac{\alpha_{,1}\psi_{,1}}{2(\psi\pm\sqrt{2})} = e^{-F}\left(\frac{\psi_{,1}}{\psi\pm\sqrt{2}}\right)^2-\tfrac{1}{2}F_{,11}-\tfrac{1}{4}F_{,1}^2-\frac{F_{,1}\psi_{,1}}{\psi\pm\sqrt{2}} +\tfrac{1}{4}\alpha_{,1}F_{,1}-4\pi\rho' e^{\alpha}, \tag{4a}$$

$$r^{-2}(1-e^{\alpha})+\frac{r^{-1}\psi_{,1}}{\psi\pm\sqrt{2}}-\tfrac{1}{2}r^{-1}\alpha_{,1} = -e^{-F}\left(\frac{\psi_{,1}}{\psi\pm\sqrt{2}}\right)^2-\tfrac{1}{2}r^{-1}F_{,1}-4\pi\rho' e^{\alpha}, \tag{4b}$$

$$\frac{\psi_{,11}}{\psi\pm\sqrt{2}}+\frac{2r^{-1}\psi_{,1}}{\psi\pm\sqrt{2}}-\frac{\alpha_{,1}\psi_{,1}}{2(\psi\pm\sqrt{2})} = e^{-F}\left(\frac{\psi_{,1}}{\psi\pm\sqrt{2}}\right)^2-\tfrac{1}{2}F_{,11}-\tfrac{1}{4}F_{,1}^2-r^{-1}F_{,1} +\tfrac{1}{4}\alpha_{,1}F_{,1}-\frac{F_{,1}\psi_{,1}}{\psi\pm\sqrt{2}}+4\pi\rho' e^{\alpha}, \tag{4c}$$

$$\frac{\psi_{,11}}{\psi\pm\sqrt{2}}-\frac{\alpha_{,1}\psi_{,1}}{2(\psi\pm\sqrt{2})}+\frac{2r^{-1}\psi_{,1}}{\psi\pm\sqrt{2}}-\frac{F_{,1}\psi_{,1}}{2(\psi\pm\sqrt{2})}-\left(\frac{\psi_{,1}}{\psi\pm\sqrt{2}}\right)^2 = \pm\sigma\sqrt{(4\pi)}\,e^{\alpha+\frac{1}{2}F}. \tag{4d}$$

For convenience in further calculation and to highlight the significance of the variable F, we subtract equation (4d) from (4c) to give

$$\tfrac{1}{2}F_{,11}+\tfrac{1}{4}F_{,1}^{2}+r^{-1}F_{,1}+\frac{3F_{,1}\psi_{,1}}{2(\psi\pm\sqrt{2})}-\tfrac{1}{4}\alpha_{,1}F_{,1}=\left(\mathrm{e}^{-F}-1\right)\left(\frac{\psi_{,1}}{\psi\pm\sqrt{2}}\right)^{2}$$
$$+4\pi\rho'\mathrm{e}^{\alpha}\mp\sigma\sqrt{(4\pi)}\,\mathrm{e}^{\alpha+\frac{1}{2}F}. \tag{5}$$

It can be verified that for $F=0$, equation (5) suggests that the absolute value of the ratio of charge to matter density is unity, whereas equation (3) reduces to the well-known relationship between the 4–4 component of the static metric and the electrostatic scalar ψ obtained by De and Raychaudhuri (1968). In view of equation (5), we drop equation (4d) and consider equations (4a), (4b), (4c) and (5) for our further calculation. Subtracting equation (4a) from (4c), we get

$$r^{-1}\alpha_{,1}+\frac{2r^{-1}\psi_{,1}}{\psi\pm\sqrt{2}}+r^{-1}F_{,1}=8\pi\rho'\mathrm{e}^{\alpha}. \tag{6}$$

On eliminating $\alpha_{,1}$ from equations (4b) and (6), we find e^{α} to be fully expressed in terms of ψ, F and their derivatives as

$$r^{-2}\mathrm{e}^{\alpha}=\left(r^{-1}+\tfrac{1}{2}F_{,1}+\frac{\psi_{,1}}{\psi\pm\sqrt{2}}\right)^{2}+\mathrm{e}^{-F}\left(\frac{\psi_{,1}}{\psi\pm\sqrt{2}}\right)^{2}-\left(\frac{\psi_{,1}}{\psi\pm\sqrt{2}}+\tfrac{1}{2}F_{,1}\right)^{2},$$

which can be reduced to the very compact form

$$\mathrm{e}^{\alpha}=r^{2}(K^{2}+MN), \tag{7}$$

where

$$K=r^{-1}+\tfrac{1}{2}F_{,1}+\frac{\psi_{,1}}{\psi\pm\sqrt{2}},\qquad M=\frac{\mathrm{e}^{-\frac{1}{2}F}\psi_{,1}}{\psi\pm\sqrt{2}}+\tfrac{1}{2}F_{,1}+\frac{\psi_{,1}}{\psi\pm\sqrt{2}},$$
$$N=\frac{\mathrm{e}^{-\frac{1}{2}F}\psi_{,1}}{\psi\pm\sqrt{2}}-\tfrac{1}{2}F_{,1}-\frac{\psi_{,1}}{\psi\pm\sqrt{2}}.$$

It can be verified that $F=0$ implies $N=0$. In view of equations (6) and (7), we drop equations (4a) and (4b) and instead consider (4c), (5), (6) and (7) for our further calculation. Using equation (7) to eliminate α in terms of K, M and N from equations (4c) and (6), and then adding, we get

$$r^{2}(K^{2}+MN)K_{,1}-\tfrac{1}{2}Kr^{2}(2KK_{,1}+NM_{,1}+MN_{,1})=r^{2}MN(K^{2}+MN),$$

which reduces to

$$K^{2}+K_{,1}-\tfrac{1}{2}KL_{,1}=\mathrm{e}^{L}, \tag{8}$$

where the function L is defined by $\mathrm{e}^{L}=K^{2}+MN$. In terms of L, equations (5) and (6) become respectively

$$\tfrac{1}{2}F_{,11}-\tfrac{1}{4}F_{,1}^{2}+\tfrac{1}{2}KF_{,1}+K^{2}-\tfrac{1}{4}L_{,1}F_{,1}=\mathrm{e}^{L}\{1+4\pi\rho'r^{2}\mp\sigma\sqrt{(4\pi)}\,r^{2}\mathrm{e}^{\frac{1}{2}F}\}, \tag{9}$$

$$K+\tfrac{1}{2}L_{,1}=4\pi\rho'r^{3}\mathrm{e}^{L}. \tag{10}$$

Thus our field equations reduce to equations (8), (9) and (10). In view of equation (8), we introduce a variable p defined by

$$\mathrm{e}^{\alpha} = r^2\mathrm{e}^{L} = r^2K^2\mathrm{e}^{p}, \tag{11}$$

so that equation (8) reduces simply to

$$2K = p_{,1}/(1-\mathrm{e}^{p}) \qquad (p \neq 0). \tag{12}$$

Substituting this value of K into

$$K = r^{-1} + \tfrac{1}{2}F_{,1} + \psi_{,1}/(\psi \pm \sqrt{2})$$

and integrating, gives us the direct relationship between e^{β} and p,

$$\mathrm{e}^{\beta} = 4\pi A^2\mathrm{e}^{p}/r^2(1-\mathrm{e}^{p}), \tag{13}$$

where A is an arbitrary constant of integration. It is worth noting that equation (13) places a restriction on p such that $\mathrm{e}^{p} < 1$. In terms of K and p, equations (9) and (10) reduce to respectively

$$\tfrac{1}{2}F_{,11} - \tfrac{1}{4}F_{,1}^2 + \tfrac{1}{2}F_{,1}K\mathrm{e}^{p} + K^2 - \tfrac{1}{2}K_{,1}F_{,1}K^{-1} = K^2\mathrm{e}^{p}\{1 + 4\pi\rho' r^2 \mp \sigma\sqrt{(4\pi)}\,r^2\mathrm{e}^{\frac{1}{2}F}\}, \tag{14}$$

$$K^2 + K_{,1} + \tfrac{1}{2}Kp_{,1} = 4\pi\rho' r^3K^3\mathrm{e}^{p}. \tag{15}$$

It can be easily verified that $F = 0$ implies $p = 0$ (via $N = 0$), such that equation (12) is identically satisfied for all values of K and equation (14) implies

$$\sigma/\rho = \pm 1,$$

whereas equation (15) reduces to

$$K^2 + K_{,1} = 4\pi\rho' r^3K^3,$$

which admits Bonnor's (1965) nonsingular solution. Thus for $p \neq 0$ (which implies $F \neq 0$), the final set of equations governing the equilibrium configuration of the spherically symmetric static charged dust distribution comprises equations (12), (14) and (15). Since these three equations involve the five unknown parameters K, p, ρ', σ and F, yet to be determined, any two of them can be chosen arbitrarily. It is convenient to choose p and F arbitrarily as this choice does not require us to integrate equations (12), (14) and (15) to determine the rest of the unknowns; rather by simply substituting the chosen values of p and F, the parameters σ, ρ' and K are determined. Thus, the original field variables $(\alpha, \beta, \psi, \rho, \sigma)$ can be expressed in terms of p and F with the help of equations (11), (13), (3), (15) and (14) as

$$\mathrm{e}^{\alpha} = \tfrac{1}{4}r^2\mathrm{e}^{p}p_{,1}^2/(1-\mathrm{e}^{p})^2, \qquad \mathrm{e}^{\beta} = 4\pi A^2\mathrm{e}^{p}/r^2(1-\mathrm{e}^{p}),$$

$$\psi = \mp\sqrt{2} \pm A\mathrm{e}^{\frac{1}{2}(p-F)}/r\sqrt{(1-\mathrm{e}^{p})}, \qquad \rho = \frac{r^{-3}\mathrm{e}^{-p}}{\sqrt{\pi}}\left(\frac{1-\mathrm{e}^{p}}{p_{,1}}\right)^2\left(\frac{2p_{,11}}{p_{,1}} + \frac{p_{,1}(2+\mathrm{e}^{p})}{1-\mathrm{e}^{p}}\right),$$

$$\sigma = \pm\frac{r^{-2}\mathrm{e}^{-(p+\frac{1}{2}F)}}{\sqrt{(4\pi)}}\left(\frac{1-\mathrm{e}^{p}}{p_{,1}}\right)^2\left\{F_{,1}^2 - 2F_{,11} + F_{,1}\left(\frac{2p_{,11}}{p_{,1}} + \frac{\mathrm{e}^{p}p_{,1}}{1-\mathrm{e}^{p}}\right)\right.$$
$$\left. + \frac{2}{r}\left(\frac{2p_{,11}}{p_{,1}} + \frac{2+\mathrm{e}^{p}}{1-\mathrm{e}^{p}}p_{,1}\right) - \frac{p_{,1}^2}{1-\mathrm{e}^{p}}\right\}.$$

The expression for the ratio of charge to matter density then reduces to

$$\frac{\sigma}{\rho} = \pm\tfrac{1}{2}r\,\mathrm{e}^{-\frac{1}{2}F}\left\{F_{,1}^{2} - 2F_{,11} + F_{,1}\left(\frac{2p_{,11}}{p_{,1}} + \frac{\mathrm{e}^{p}p_{,1}}{1-\mathrm{e}^{p}}\right)\right.$$

$$\left. + \frac{2}{r}\left(\frac{2p_{,11}}{p_{,1}} + \frac{2+\mathrm{e}^{p}}{1-\mathrm{e}^{p}}p_{,1}\right) - \frac{p_{,1}^{2}}{1-\mathrm{e}^{p}}\right\}\Big/\left(\frac{2p_{,11}}{p_{,1}} + \frac{2+\mathrm{e}^{p}}{1-\mathrm{e}^{p}}p_{,1}\right).$$

It is interesting to observe that when the space–time variables α and β and the matter variable ρ are completely determined by p, the electromagnetic variables ψ and σ require both p and F to be completely determined. This reflects the physical character of these auxiliary functions p and F, which separate the space–time–matter part from the electromagnetic part so distinctly.

Conclusions

The most general solution to Einstein's field equations characterizing the static charged dust sphere has been presented. This solution generates a class of particular solutions when the arbitrary functions p and F are suitably assigned. Only the solution of this class that can be smoothly matched to the Reissner–Nordström metric at the surface has been considered for the physical study of the interior. In a forthcoming paper, this aspect of the solution shall be studied in detail.

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