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Nuclear Systematics in the Interacting Boson (IBA) Model

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Abstract

Within the framework of the interacting boson approximation, simple analytical formulae are presented for analysing the mass dependence of particle transfer, inelastic scattering and electromagnetic decay from experimental data in nuclear physics. It is shown that the identical boson model provides a useful tool to correlate the results of these experiments and that the extension to equivalent proton and neutron bosons leads to some new and interesting results in the Pt–Hg mass region, namely the possibility of phase transitions and symmetry breaking.

Introduction

The work of Bohr and Mottelson (1953) on the liquid drop model of the nucleus has been particularly useful in the understanding of many nuclear processes. The introduction of the collective variables α_m ($m = +2, 1, 0, -1, -2$) associated with the quadrupole vibration ($L = 2$ phonon or boson) of the nucleus led in a natural way to the description of the low lying vibrational states in nuclei (Bohr and Mottelson 1953; Eisenberg and Greiner 1971).

The Bohr–Mottelson vibrational Hamiltonian, for *small* quadrupole vibrations, has the form

$$H_0 = \tfrac{1}{2}\sum_m (\pi_m \pi^m + \alpha_m \alpha^m) = \hat{n} + \tfrac{5}{2}, \tag{1}$$

where $\hat{n}$ is the phonon (boson) number operator, $\alpha^m = (-)^m \alpha_{-m}$ and π_m is the momentum conjugate to α_m. The α_m and π_m are generators of the group O(5), the orthogonal group in five dimensions, and the eigenstates of H_0 can be labelled (Chácon *et al.* 1976; Chácon and Moshinsky 1977) as $|\, v\Lambda t L M \rangle$ in the subgroup chain U(5) $\supset$ O(5) $\supset$ O(3) where the number v of phonons labels the totally symmetric irreducible representation (irrep) of U(5), Λ labels the irrep of O(5) and L, M labels the angular momentum of the state. This leads to the standard classification of vibrational states in nuclei as $0, 1, 2, \ldots$ phonon states. The states $|\, v\Lambda t L M \rangle$ can equivalently be expanded (Chácon *et al.* 1976; Chácon and Moshinsky 1977) in a basis of intrinsic nuclear states characterized by the shape variables β and γ of the liquid drop model, in analogy with the results of the fermion SU(3) model (Elliott 1958).

For quadrupole vibrations of larger amplitude, the nuclear Hamiltonian H will include terms in which α_m and π_m appear in powers higher than the second. In the model of Greiner and coworkers (Gneuss and Greiner 1971; Hess and Greiner 1977) $H(\alpha_m, \pi^m)$ is taken as a polynomial in α_m and π^m whose degree is computationally limited and whose coefficients are determined by a fit to experimental level spectra,

transition probabilities and quadrupole moments. In this model, v is free to vary and, in general, large calculations ($v \approx 20$) are required to ensure convergence of the nuclear eigenstates (in v) and stability of the coefficients of the empirical Hamiltonian H. From the fitted coefficients, the potential energy surfaces (PES) for the liquid drop can be extracted in terms of the shape parameters β and γ, hence enabling a systematic study of the nuclear shape as a function of mass in medium and heavy nuclei. The description of a nucleus as vibrational, prolate, oblate, γ-soft, triaxial etc. depends only on the nature of the minimum(a) in the PES which has been extracted from the known properties of the nucleus. The model is therefore limited in its predictive power until a large body of calculations have been performed. There is also the problem of whether or not such large values of v are physical, as the bosons must be created from the fermion operators of the standard shell model and hence the Pauli principle should limit the values taken by 'v'.

Various techniques are available to overcome the above objections. One method is to construct the inertia and potential shape functions (PES) of the Bohr–Mottelson collective Hamiltonian in terms of the microscopic fermion interaction and solve the Schrödinger equation in terms of the macroscopic variables (Kumar and Baranger 1968). Alternatively, by studying the fermion–boson mapping, it can be shown that the phonon number v should be limited to about half the number of active fermions in the microscopic basis (Janssen *et al.* 1974) giving the truncated quadrupole phonon model (TQM). The $L = 2$ quadrupole phonon Hamiltonian, as an approximation to the underlying fermion Hamiltonian, could then be expressed in terms of a few parameters which could be obtained empirically or from the fermion Hamiltonian. This model was applied in the transitional region by Janssen *et al.* (1974) to obtain level spectra, transition rates and PES. The more elaborate boson expansion theories (BET) (Sorensen 1970; Kishimoto and Tamura 1972) define the $L = 2$ phonon in terms of a coherent quadrupole two-quasiparticle state and force the phonons to satisfy the correct boson commutation rules to some finite order (4–6) thus obtaining a boson–fermion mapping which then transforms the fermion microscopic Hamiltonian to boson space. This Hamiltonian is then diagonalized in the basis $|\, v\Lambda tLM\rangle$ and nuclear properties, PES etc. are calculated. No cutoff in 'v' is applied as the Pauli principle is incorporated into the basis transformation. It is found, however, that large corrections need to be applied to account for collective–noncollective coupling and that the level spectra calculated are very sensitive to these corrections.

Recently, an interacting boson approximation (IBA) theory has been developed by Arima and Iachello (1976, 1978*a*, 1978*b*) which incorporates $L = 0$ ('s') and $L = 2$ ('d') bosons. The boson creation and annihilation operators $s^+(s)$ and $d_m^+(d_m)$ are generators of the group U(6) and hence the multiboson wavefunctions can be characterized by the irreps of this group and its subgroups, the utility of this depending on the form taken by the collective boson Hamiltonian H_B. There are three limiting forms:

(i) The Hamiltonian H_B in the vibrational limit (Arima and Iachello 1976) is given solely in terms of the generators of the group O(5), namely the $d_m^+(d_m)$, and the states can be labelled, as before, $|\, N, v\Lambda tLM\rangle$ in the chain $U(6) \supset U(5) \supset O(5) \supset O(3)$. The total number of bosons is N and the number of d bosons is v. This is formally equivalent to the TQM (Janssen *et al.* 1974), both being different representations of the group U(6) (Paar 1979). This limit predicts (Arima and Iachello 1976) band

structures of B(E2) decays and, depending on details of the d-boson interaction, level spectra varying from vibrational to rotational.

(ii) The rotational limit H_B (Arima and Iachello 1978*a*) is given in terms of the generators ($\hat{N}$ the number operator, $\hat{Q}$ the quadrupole operator and $\boldsymbol{L}$) of the subgroup U(3). The wavefunctions are labelled $|N(\lambda\mu)KLM\rangle$, where $(\lambda\mu)$ labels the irrep of the group SU(3) and K labels the multiple occurrences of L in a given $(\lambda\mu)$. The energy eigenvalues of H_B are given by

$$E = \alpha C'(\lambda\mu) + \beta L(L+1),$$

where α and β are constants and $C'(\lambda\mu)$ is the Casimir operator for the group SU(3). The spectra generated are those of an axially symmetric rotor with fixed moment of inertia (β) and excited $K = 0, 2$ bands occur in a natural way (α).

(iii) The γ-soft limit H_B (Arima and Iachello 1978*b*) is given in terms of the generators of the subgroups O(6), O(5) and O(3). The closest geometrical analogue is the γ-soft vibrator (Wilets and Jean 1956). This limit appears most appropriate for regions near the end of closed shells, where nuclei should be intermediate between the vibrators and rotors, and it has been applied with success in the platinum (Cizewski *et al.* 1978) and osmium (Casten and Cizewski 1978) regions.

The IBA model described above therefore, under limiting forms of the boson Hamiltonian, reproduces the three 'simple' geometrical models of nuclear physics. The geometrical models correspond to symmetries in H_B being reflected in appropriate subgroup chains and their irreps. The great advantage of this is that, by using group theoretic techniques developed for the fermion shell model, many closed expressions for energy levels and other nuclear properties can be obtained in terms of the boson number N and the various subgroup labels (Arima and Iachello 1976, 1978*a*, 1978*b*) without any computational effort. Therefore the IBA (and more restrictedly the TQM) enables a quick systematic study of a range of nuclei where one of the limiting symmetries is appropriate. For transitional nuclei, a standard configuration mixing approach in a small '*sd*' boson basis can produce useful results (Cizewski *et al.* 1979*a*). Further, for restricted forms of the boson Hamiltonian H_B, Moshinsky (1980) has recently shown that the IBA model has much in common with that of Greiner and coworkers (Gneuss and Greiner 1971; Hess and Greiner 1977). It is worth while therefore to study nuclei with the simple analytic models assuming that H_B changes only slowly (if at all) over a mass range. This study is particularly worth while for systematics of low lying states in even–even nuclei where the effects of symmetry breaking in H_B and the uncertainties in configuration of states with higher excitation energy do not occur. Furthermore it has been shown by Otsuka *et al.* (1978*a*) that there exists a mapping from selected $(j)^n$ fermion configurations to a basis of s and d bosons, and forms for the boson mappings of fermion operators can be generated. It was subsequently shown by McGrory (1978) in realistic multi-shell fermion shell model calculations that these selected configurations dominate the low lying level spectra and retain a large overlap, for yrast and yrare states, with simple boson configurations. It is to be expected, therefore, that a study of the systematics of low excitation nuclear levels in an appropriate boson basis will provide useful information. In the following sections, some analyses of recent experiments in particle transfer, inelastic scattering and electromagnetic decay are presented.

Particle Transfer

The boson mappings of the $L = 0$ and 2 identical fermion pair transfer operators have been shown to have the forms (Otsuka *et al.* 1978*a*)

$$P_+^{(0)} = \alpha s^+ \left(\frac{\Omega - N - n_d}{\Omega}\right)^{\frac{1}{2}}, \tag{2a}$$

$$P_+^{(2)} = \beta d^+ \left(\frac{(\Omega-N-1)(\Omega-N)}{(\Omega-1)\Omega}\right)^{\frac{1}{2}} + \gamma[d^+ d]^2 \left(\frac{2(\Omega-N-1)}{(\Omega-2)^2}\right)^{\frac{1}{2}} - \beta s^+ s^+ d \left(\frac{1}{\Omega(\Omega-1)}\right)^{\frac{1}{2}}, \tag{2b}$$

where Ω, the underlying fermion pair degeneracy ($\Omega \approx \sum_d (j+\frac{1}{2})$), can be estimated from the shell model, and N and n_d are the total and 'd' boson number operators respectively. The constants α, β and γ are arbitrary and are usually fitted to data. The operators (2) suffice when only one type of boson is active. In general, however, we must consider both proton and neutron bosons. When the Hamiltonian H_B is symmetric under charge (proton–neutron) interchange, it is convenient to introduce F spin, a boson analogue of isospin (Otsuka *et al.* 1978*b*; Morrison 1980).

With the assumption that the underlying boson Hamiltonian is F-spin (charge) independent, the N boson wavefunction which carries the totally symmetric irrep $[N]$ of the group U(12) (dimension 2 for F spin by 6 for the s–d boson space) can be decomposed (Wybourne and Butler 1969)

$$\begin{array}{ccccc} \mathrm{U}(12) & \supset & \mathrm{U}(2) & \otimes & \mathrm{U}(6), \\ & & (F\text{ spin}) & & (\text{space}) \end{array} \tag{3a}$$

$$[N] \quad \rightarrow \sum_{\lambda} \quad [\lambda] \quad \times \quad [\lambda], \tag{3b}$$

where the representation $[\lambda]$ of U(2), having at most two rows, can be written $[N-a, a]$. Denoting the number of proton (neutron) bosons as N_π (N_ν), we have

$$F = \tfrac{1}{2}(N-2a), \qquad M_F = \tfrac{1}{2}(N_\pi - N_\nu), \tag{4}$$

where $N = N_\pi + N_\nu$, the proton (neutron) boson having F-spin projection $+\frac{1}{2}$ ($-\frac{1}{2}$).

The standard IBA models have $a = 0$ and $F \equiv M_F = \frac{1}{2}N$ and hence derive from the irrep $[N]$ of U(6). In general, however, the two-rowed irreps $[N-a, a]$ of U(6) will introduce new states into any spectrum, their importance being dependent on the choice of boson Hamiltonian in the U(6) subgroup chain, e.g. (i) vibrator $\mathrm{U}(6) \supset \mathrm{U}(1) \otimes \mathrm{U}(5)$, (ii) axial rotor $\mathrm{U}(6) \supset \mathrm{SU}(3)$ and (iii) γ-unstable vibrator $\mathrm{U}(6) \supset \mathrm{O}(6) \supset \mathrm{O}(5) \supset \mathrm{O}(3)$. The low lying states of nuclei correspond to maximum F spin ($a = 0$), reflecting maximum spatial symmetry in the boson wavefunction. For transitions involving such states, the U(2) (F-spin) dependence of a general operator $\boldsymbol{O}$ corresponds to the simple scaling (Morrison 1980)

$$\boldsymbol{O}(^{\pi}_{\nu}) = \tfrac{1}{2}(1 + (\pm) M_F/F)\boldsymbol{O}, \tag{5}$$

which allows separation of the proton and neutron boson component. This scaling does not hold for transitions involving states of less than maximum F spin ($a \neq 0$). We have therefore

$$P_{+\pi}^{(0)} = \{(N_\pi+1)/(N+1)\} s^+ (\Omega_\pi - N_\pi - N_\pi n_d/N)^{\frac{1}{2}} \Omega_\pi^{-\frac{1}{2}}, \tag{6}$$

and equivalently for $P^{(2)}_{+\pi(\nu)}$. The value of $\langle n_s \rangle$, the average s boson ground state occupancy in the three IBA limits is

$$\langle n_s \rangle_{\text{vib}} = N, \tag{7a}$$

$$\langle n_s \rangle_{\text{rot}} = N(2N+1)/3(2N-1), \tag{7b}$$

$$\langle n_s \rangle_{\gamma} = N(N+3)/2(N+1), \tag{7c}$$

and hence the intensities $I(N \rightarrow N+1)$ for two-neutron ground state transfers (via $P^{(0)}_{+\nu}$) in the three limits are given by

$$I_{\text{vib}} \approx \alpha_\nu^2 (N_\nu+1)(\Omega_\nu - N_\nu), \tag{8a}$$

$$I_{\text{rot}} \approx \alpha_\nu^2 \frac{(N_\nu+1)(2N+3)}{3(2N+1)} \left(\Omega_\nu - N_\nu - \frac{4(N-1)N_\nu}{3(2N-1)} \right), \tag{8b}$$

$$I_{\gamma} \approx \alpha_\nu^2 \frac{(N+4)(N_\nu+1)}{2(N+2)} \left(\Omega_\nu - N_\nu - \frac{(N-1)N_\nu}{2(N+1)} \right). \tag{8c}$$

The results of an analysis by Cizewski *et al.* (1979*b*) of recent (t, p) and (p, t) two-neutron transfer experiments for the Pt isotopes are compared with experimental results in Fig. 1. Due to the overall scale factor α_ν^2, only the variation of intensities with boson number N_ν is important (N_π is fixed). Although the Pt nuclei are known to be well described in the O(6) model (Cizewski *et al.* 1978, 1979*a*) the mass dependence of the intensities differentiates only marginally against the vibrational limit (SU(5)) and not at all between the rotational (SU(3)) and γ-soft (O(6)) limits. This is due primarily to the experimental data spanning too small a mass range, i.e. the neutron boson number $N_\nu \ll \Omega_\nu$. In the case of the even Sn isotopes (pairing vibrational nuclei), extensive two-neutron transfer experiments have been made and the IBA vibrational limit accurately reproduces the experimental maximum intensity at ^{116}Sn ($N_\nu \approx \frac{1}{2}\Omega_\nu$) (Bés and Broglia 1977). The inclusion of a small quadrupole perturbation in H_{B} also predicts the almost exponential rise (with mass) of the, first-order forbidden, $0^+_{\text{g.s.}} \rightarrow 0^+_2$ transfer strength (Bés and Broglia 1977; Morrison and Smith 1980*b*).

We can also discuss α-particle transfer in medium-heavy nuclei with the boson model using the four-particle (two-boson) spectroscopic factor

$$S = \langle N+2, L_{\text{f}} \| A^+(L) \| N, L_{\text{i}} \rangle^2, \tag{9}$$

where $A^+(L) = (P^{(L_\pi)}_{+\pi} \times P^{(L_\nu)}_{+\nu})^L$ is the α creation operator. Using the expressions for $P^{(L)}_+$, we again can write down reaction intensities in terms of Ω and N. Recent experiments (Janëcke *et al.* 1979) in the Sn, Cd and Te isotopes (where the vibrational limit is appropriate) have been analysed by Morrison and Smith (1980*a*) and some results are presented in Fig. 2. The mass dependence of S is accurately reproduced in all reactions and only two strengths (α and β from equations 2) are required to normalize correctly. It is particularly interesting that the level and mass dependence of the intensity maxima (or minima) are reproduced.

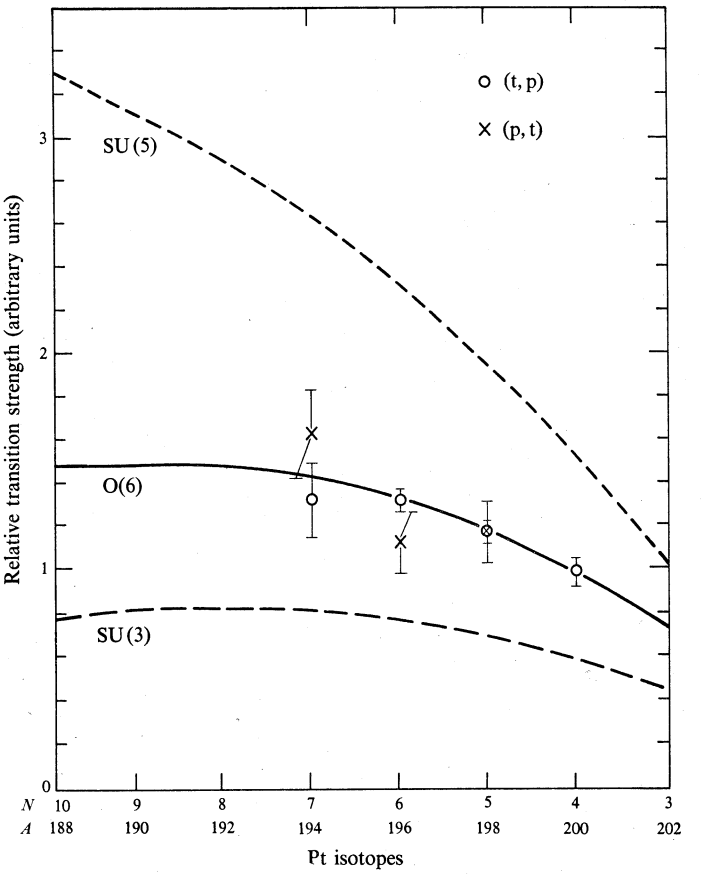


Fig. 1. Relative (t, p) and (p, t) experimental two-neutron transfer intensities for the Pt isotopes compared with the results of a model analysis (Cizewski *et al.* 1979*b*). The O(6) curve is the γ-soft prescription, the SU(5) is the vibrator and the SU(3) is the rotor. The proton boson number N_π is fixed at 2. The three theoretical predictions have been displaced vertically for clarity; only the shapes of the curves are important.

Inelastic Scattering

In this section we discuss inelastic nucleon (and electron) scattering to states in nuclei described by a boson model. In the distorted wave approximation the measurables associated with direct reaction inelastic nucleon scattering leading to discrete final states of nuclei are all related to transition amplitudes of the form (Satcher 1966)

$$T_{\mathrm{fi}} = \langle \chi_{\mathrm{f}}^{(-)}(0) \,|\, \langle J_{\mathrm{f}} M_{\mathrm{f}} \,|\, t \,|\, J_{\mathrm{i}} M_{\mathrm{i}} \rangle \,|\, \chi_{\mathrm{i}}^{(+)}(0) \rangle , \tag{10}$$

where the $\chi^{(\pm)}$ are the distorted waves describing the relative motion of the continuum nucleon–target system and the $|\, JM \rangle$ are the initial and final states of the target nucleus. The transition operator t is a two-body operator modelling the interaction between the continuum particle and a particle bound in the target. Choosing our bound particle to be a boson, we can cast the transition amplitude T_{fi} in the form (Geramb and Amos 1971)

$$T_{\mathrm{fi}} = \sum_{\lambda_1 \lambda_2 \mu_1 \xi_2} \langle J_{\mathrm{f}} M_{\mathrm{f}} \,|\, b^{+}_{\lambda_2 \mu_2} b_{\lambda_1 \mu_1} \,|\, J_{\mathrm{i}} M_{\mathrm{i}} \rangle M_{\mathrm{fi}} , \tag{11}$$

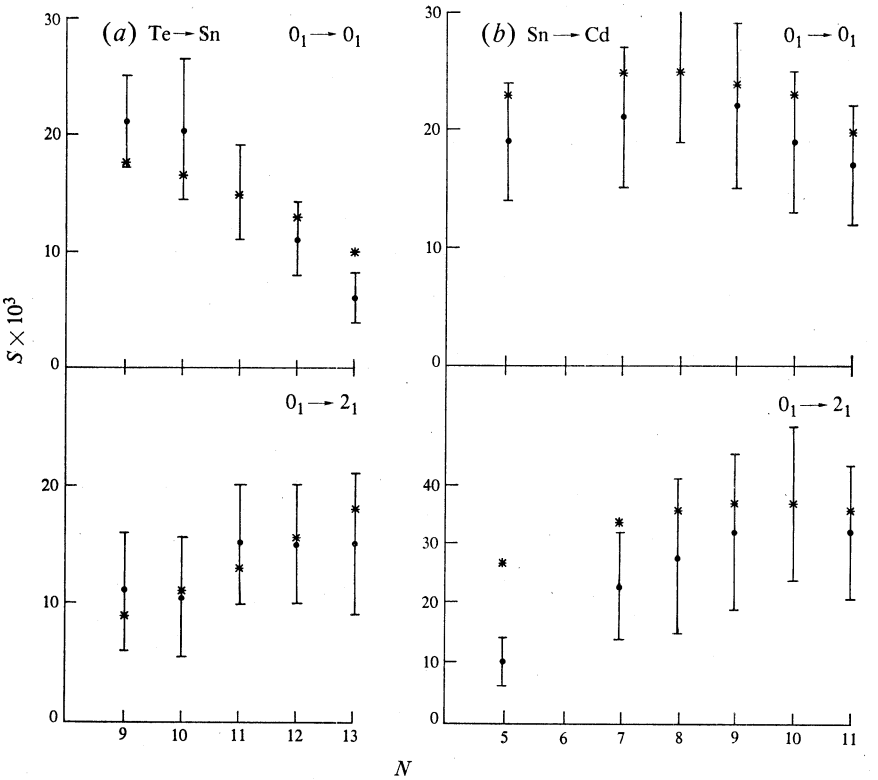


Fig. 2. Comparison of experimental results (points; Janëcke *et al.* 1979) with model analyses (asterisks) for the spectroscopic factors S in (*a*) Te → Sn and (*b*) Sn → Cd reactions as a function of the residual nucleus neutron boson number N (i.e. 9–13 in (*a*) corresponds to ^{118}Sn–^{126}Sn and 5–11 in (*b*) to ^{108}Cd–^{120}Cd).

where the $b^+(b)$ are the bound boson creation (annihilation) operators and M_{fi} is the matrix element of the transition operator between continuum and bound state particles and has the form

$$M_{\mathrm{fi}} = \langle \chi_{\mathrm{f}}^{(-)}(0)\, \phi_{\lambda_2\mu_2}(1) \,|\, t(0,1) \,|\, \chi_{\mathrm{i}}^{(+)}(0)\, \phi_{\lambda_1\mu_1}(1) \rangle , \tag{12}$$

where the $\phi_{\lambda\mu}$ are the bound state single-boson wavefunctions. It is convenient to define the operator

$$B_{In}(\lambda_1 \lambda_2) = \sum_{\mu_1\mu_2} (-)^{\lambda_1-\mu_1} \langle \lambda_1 \lambda_2 \mu_1 - \mu_2 \,|\, I - n \rangle\, b^+_{\lambda_2\mu_2}\, b_{\lambda_1\mu_1} \tag{13}$$

and an associated reduced spectroscopic factor

$$S(\lambda_1 \lambda_2 J_{\mathrm{i}} J_{\mathrm{f}}; I) = \langle J_{\mathrm{f}} \,\|\, B_I \,\|\, J_{\mathrm{i}} \rangle . \tag{14}$$

The boson spectroscopic factor (14) carries the many-body nuclear structure information and can be computed in the IBA using the eigenstates from Hamiltonians which have successfully described the energy level, B(E2) and $Q(2_1^+)$ systematics in the mass region. If we are to use the IBA nuclear structure information to predict inelastic nucleon scattering cross sections, however, we must introduce an ansatz

for the single-boson transition matrix (M_{fi} in equation 12). The two-body interaction can be written quite generally as a multipole expansion of the form

$$t(0,1) = \sum_{LM} A_L(r_0, r_1)\, \mathrm{Y}_{LM}(\Omega_0)\, \mathrm{Y}^*_{LM}(\Omega_1). \tag{15}$$

If the two-body interaction is of Yukawa form, as are a number of the effective nucleon–nucleon interactions in use in current distorted wave programs, then the radial dependence is exactly separable in the nucleon (r_0) and boson (r_1) coordinates (Brink and Satchler 1970). We take such separability as a convenient ansatz for the nucleon–boson interaction. If we then equate the target bosons with the surface phonons, the standard vibrating potential model approach (Rowe 1970) will identify the proton coordinate dependence as a surface-peaked form, i.e.

$$A_L(r_0, r_1) = r_0(\partial U(r_0)/\partial r_0) G_L(r_1), \tag{16}$$

where $U(r_0)$ is usually taken to have a Woods–Saxon form. The functional form of the $G_L(r_1)$ boson form factor is at present unspecified, but will be handled by a fitting procedure described below.

The bound boson wavefunction $\phi_{\lambda\mu}$ is written separately in its radial and angular coordinates as

$$|\phi_{\lambda\mu}(1)\rangle = R_\lambda(r_1)\, \mathrm{Y}_{\lambda\mu}(\Omega_1), \tag{17}$$

and standard angular momentum coupling techniques reduce the inelastic scattering amplitude to the form

$$\begin{aligned} T_{\mathrm{fi}} = \sum_{\lambda_1\lambda_2, In, L_1J_1L_2J_2} & S(\lambda_1\lambda_2 J_{\mathrm{i}} J_{\mathrm{f}}; I) \langle J_{\mathrm{i}} I M_{\mathrm{i}} n \,|\, J_{\mathrm{f}} M_{\mathrm{f}}\rangle (4\pi \hat{f} \hat{\lambda}_1 \hat{J}_1/\hat{J}_{\mathrm{f}})^{\frac{1}{2}} \hat{L}_1\, \mathrm{Y}_{L_2M_2}(\Omega_{\mathrm{sc}}) \\ & \times (-)^{I+n+L_2+J_1+\frac{1}{2}} \langle L_1 \tfrac{1}{2} 0 \nu_1 \,|\, J_1 \nu_1\rangle \langle L_2 \tfrac{1}{2} M_2 \nu_2 \,|\, J_2 M_2+\nu_2\rangle \\ & \times \langle J_1 I \nu_1 - n \,|\, J_2 M_2+\nu_2\rangle \left(\begin{smallmatrix} IL_1L_2 \\ \frac{1}{2}J_2J_1 \end{smallmatrix}\right) \langle L_1 I 00 \,|\, L_2 0\rangle \\ & \times \int r_0^2\, \mathrm{d}r_0\, F^{(-)}_{L_2J_2}(k_{\mathrm{f}} r_0) F^{(+)}_{L_1J_1}(k_{\mathrm{i}} r_0) r_0 \frac{\partial U(r_0)}{\partial r_0} \langle \lambda_1 I 00 \,|\, \lambda_2 0\rangle K_{I\lambda_1\lambda_2}, \end{aligned} \tag{18a}$$

with

$$K_{I\lambda_1\lambda_2} = \int r_1^2\, \mathrm{d}r_1\, R_{\lambda_2}(r_1) R_{\lambda_1}(r_1) G_I(r_1). \tag{18b}$$

Here the $F^{(\pm)}_{LJ}$ are the solutions of the radial optical model equation for the incoming and exit proton, $\hat{C}$ denotes $(2C+1)$ and the $\nu_{1,2}$ are the entrance–exit proton spins.

The boson overlaps strengths $K_{I\lambda_1\lambda_2}$ defined by equation (18b) cannot be determined *a priori*. However, if inelastic scattering to a range of nuclei has been carried out experimentally, the $K_{I\lambda_1\lambda_2}$ can be obtained by a least squares fit to a few data and then the formalism applied over the full mass range to see if the IBA spectroscopic information correctly describes the level and isotope dependence of the scattering data. Such a program has been carried out (Morrison and Smith 1980*b*) in the $110 < A < 130$ mass region where extensive data are available. It should be noted that, with s and d bosons, two parameters K_{202} and K_{222} then determine the direct scattering to all 2^+ states in this mass region. Similarly K_{422} determines direct scattering cross sections to all 4^+ states. As stated above, we are then testing the assumption of our model, namely that the state and isotope dependence of the scattering cross

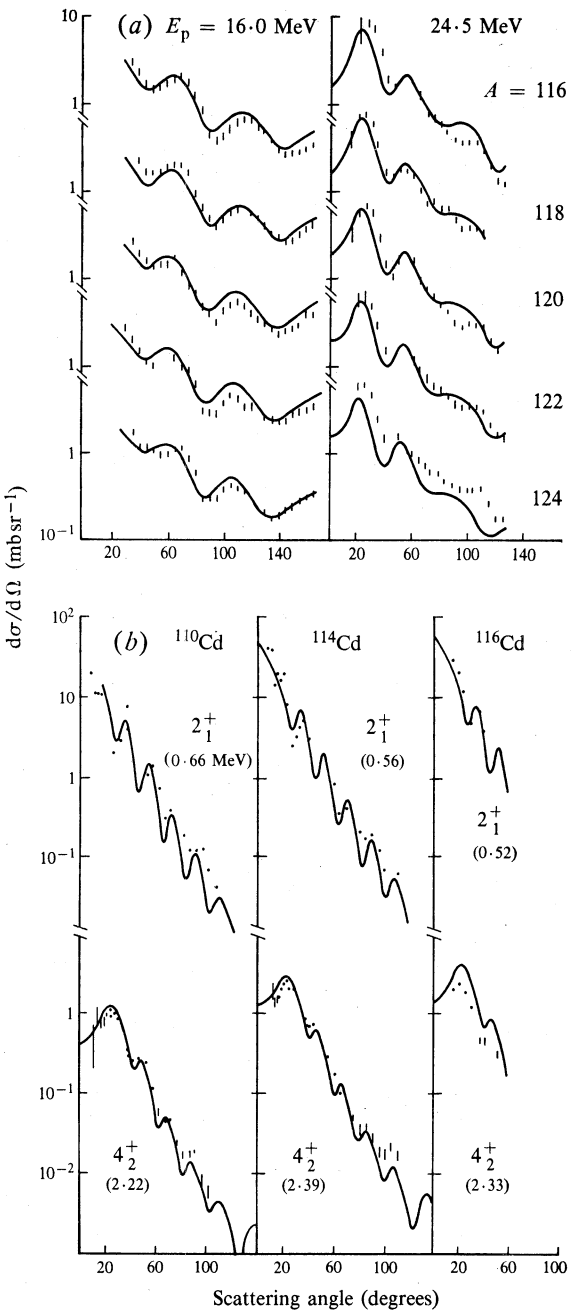


Fig. 3. Comparison with experiment of differential cross section predictions from the IBA model for the inelastic scattering of (*a*) 16·0 and 24·5 MeV protons to the 2_1^+ states of ^{116}Sn–^{124}Sn and (*b*) 55·16 MeV protons to the 2_1^+ and 4_2^+ states of ^{110,114,116}Cd.

sections is carried solely by the IBA spectroscopic weight S. The predicted differential cross sections obtained for

$$K_{202} = K_{220} = 0{\cdot}100 \pm 0{\cdot}025, \qquad K_{222} = 0{\cdot}6 \pm 0{\cdot}2, \qquad K_{422} = 0{\cdot}9$$

are compared with experiment (Beer *et al.* 1970; Makopke *et al.* 1968) in Fig. 3. Overall the agreement is good, showing that the use of boson spectroscopies in the analysis of inelastic hadron scattering to low lying collective levels of an isotopic group offers a better alternative to standard collective model approaches which assign a deformation parameter β_L to each level excited in the scattering process. By a simple extension of the '*sd*' boson model to include an $L = 3$ octupole boson, the mass dependence of the cross section for exciting the 3^- and 5^- levels in Sn isotopes is also correctly predicted (Morrison and Smith 1980*b*).

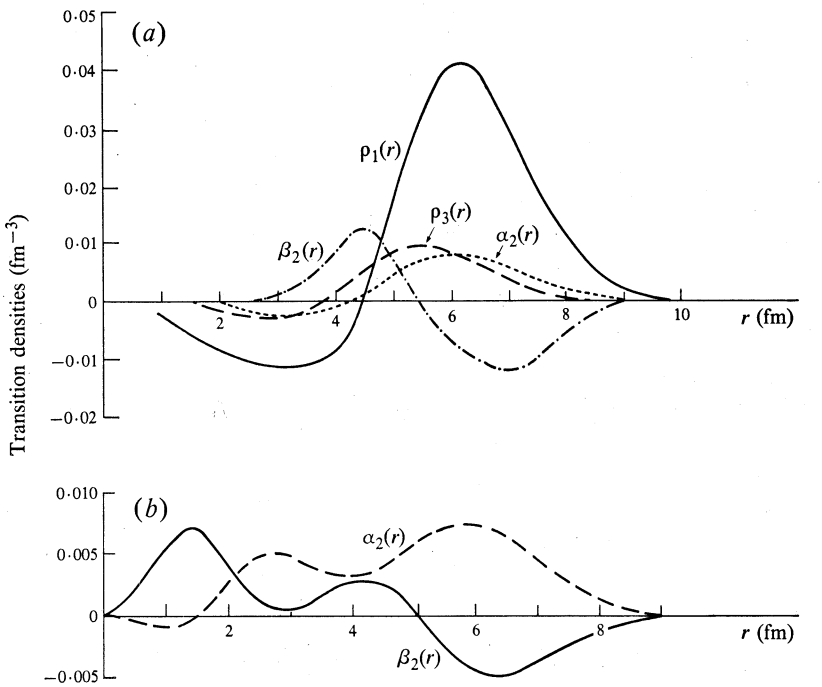


Fig. 4. Calculated boson transition densities $\alpha_2(r)$ and $\beta_2(r)$ for (*a*) ^{150}Nd, using the transition densities $\rho_1(r)$ and $\rho_3(r)$ for the $0_1 \rightarrow 2_1^+$ and $0_1 \rightarrow 2_3^+$ transitions, and (*b*) the Nd–Sm mass region.

An analysis of electron scattering to low lying states can be made by again appealing to the collective model. This has been done for the excitation of 2^+ states in medium mass nuclei (Dieperink 1979) by a generalization of the boson E2 operator. Defining

$$T_\mu(\mathrm{E2}) = \alpha_2(s^+d + d^+s)^2_\mu + \beta_2(d^+d)^2_\mu \tag{19}$$

as the E2 operator, we have in the long wavelength limit

$$T_\mu(\mathrm{E2}) \equiv \lim_{q\to 0} q^{-2} \int \mathrm{d}r\, r^2 j_2(qr)\, \hat{\rho}^2_\mu(r),$$

where $\hat{\rho}^2_\mu$ is the multipole density operator. As the electron scattering cross section can be written in terms of the form factor (Überall 1971)

$$F_\lambda(q) = \int \mathrm{d}r\, r^2 j_\lambda(qr) \langle \psi_\mathrm{f} \| \hat{\rho}^\lambda(r) \| \psi_\mathrm{i} \rangle, \tag{20}$$

we can obtain our $L = 2$ multipole transition densities by generalizing equation (19) to

$$\hat{\rho}^2_\mu(r) = \alpha_2(r)(s^+d+d^+s)^2_\mu + \beta_2(r)(d^+d)^2_\mu . \tag{21}$$

For any given transition, the density $\rho^2(r) = A\alpha_2(r) + B\beta_2(r)$ where A and B contain the isotope and level information and α_2 and β_2 are, as yet unknown, functions of 'r'. In analogy with the inelastic hadron scattering analyses, we make the same ansatz that *all* mass and level dependence is contained in A and B, but whereas α_2 and β_2 were constant in the hadron case they are now functions. Therefore a large quantity of electron scattering data must be transformed, assuming that A and B are *known* from other sources, to obtain $\rho_2(r)$ and hence $\alpha_2(r)$ and $\beta_2(r)$. The extraction of the basic parameters of the model is therefore an order of magnitude more difficult than the hadron case, and it was concluded by Dieperink (1979) that the available data did not allow a quantitative comparison with other levels, although some transition densities were calculated. These are shown for the Nd–Sm region in Fig. 4.

The relative success of the inelastic hadron scattering model has shown that the isotope and level dependence of the reaction mechanism can be absorbed by the weights S due to the constancy of $K_{I\lambda_1\lambda_2}$ over the mass range. The similarity of the electron scattering model suggests that the parametrization of $\hat{\rho}$ in terms of mass-independent radial functions $\alpha_2(r)$ and $\beta_2(r)$ is a realistic one. It is clear that electron scattering can be used to investigate boson models although it can be best applied where a large amount of data is available over an extended mass range.

Electromagnetic Transitions

The utility of the standard (one-boson type) IBA model in reproducing experimental E2 decay patterns, both in the limiting symmetries and with symmetry breaking, is well documented. In particular there have been extensive studies of vibrators (Arima and Iachello 1976) and γ-soft nuclei (Cizewski *et al.* 1978; Bolotin *et al.* 1980) with E2 decay preferences in the Pt–Os transition region (Cizewski *et al.* 1979*a*) showing the evolution from γ-soft nuclei to rotational nuclei as the quadrupole force begins to dominate the 'd' boson energy. It is also instructive to study the $B(\mathrm{E2}; 2_1 \rightarrow 0_1)$ systematics in the $190 \leqslant A \leqslant 204$ mass region using the equivalent p–n boson model in the O(6) limit (Morrison 1980). The eigenstates of H_B can be labelled $|[N-a, a](\omega_1\,\omega_2\,\omega_3)(\tau_1\,\tau_2)LM\rangle$ corresponding to the subgroup chain

$$\begin{array}{ccccccc} \mathrm{U(6)} & \supset & \mathrm{R(6)} & \supset & \mathrm{R(5)} & \supset & \mathrm{R(3)}, \\ [N-a, a] & & (\omega_1\,\omega_2\,\omega_3) & & (\tau_1\,\tau_2) & & L \end{array} \tag{22}$$

where the irrep labels are given for the various groups. Assuming, as usual (Arima and Iachello 1976, 1978*a*, 1978*b*), that the boson Hamiltonian H_B can be written in terms of the Casimir operators of the group chain, we have

$$H_\mathrm{B} \equiv C_1\,G(\mathrm{U(6)}) + C_2\,G(\mathrm{R(6)}) + C_3\,G(\mathrm{R(5)}) + C_4\,G(\mathrm{R(3)}). \tag{23}$$

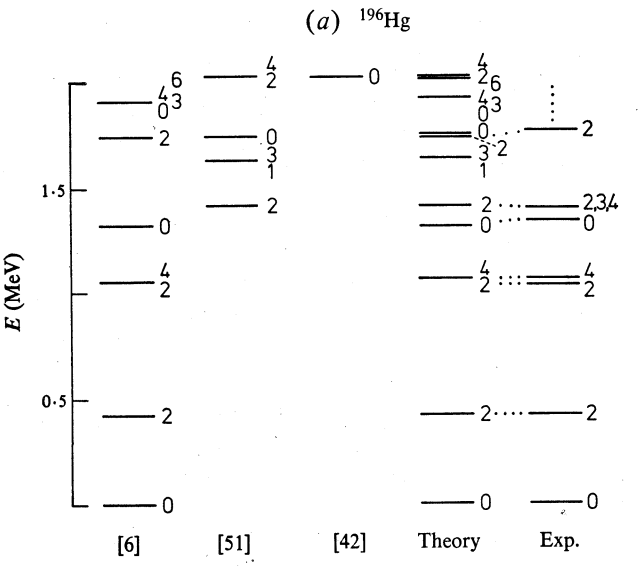
The two-rowed irreps $[N-a, a]$ of U(6) generate irreps $(\omega_1\,\omega_2\,\omega_3)$ of R(6) with $\omega_3 = 0$ (hence R(6) $\equiv$ O(6)) and on expanding the Casimir operators (Feneuille 1967) we find that the eigenstate $|[N-a, a](\omega_1\,\omega_2\,0)(\tau_1\,\tau_2)L\rangle$ of H_B has energy

$$\begin{aligned} E = {} & A\{(N-a)(N-a-1)+a(a-3)\} + B\{\omega_1(\omega_1+4)+\omega_2(\omega_2+2)\} \\ & + C\{\tau_1(\tau_1+3)+\tau_2(\tau_2+1)\} + D\,L(L+1). \end{aligned} \tag{24}$$

The spectra generated in the identical boson O(6) limit (Arima and Iachello 1978*b*) are a subset of equation (24), obtained on the restriction $a = 0$ (hence $\omega_2 = \tau_2 = 0$). The parameters B, C and D have the same physical interpretation, namely pairing strength, 'd' phonon energy and angular momentum shift. The parameter A serves to split those irreps $(\omega_1\, \omega_2\, 0)(\tau_1\, \tau_2)$ which occur in different irreps of U(6) and hence can be viewed as an F-spin splitting since the first term in equation (24) can be written

$$A\{\tfrac{1}{2}N(N-4)+2F(F+1)\}. \tag{25}$$

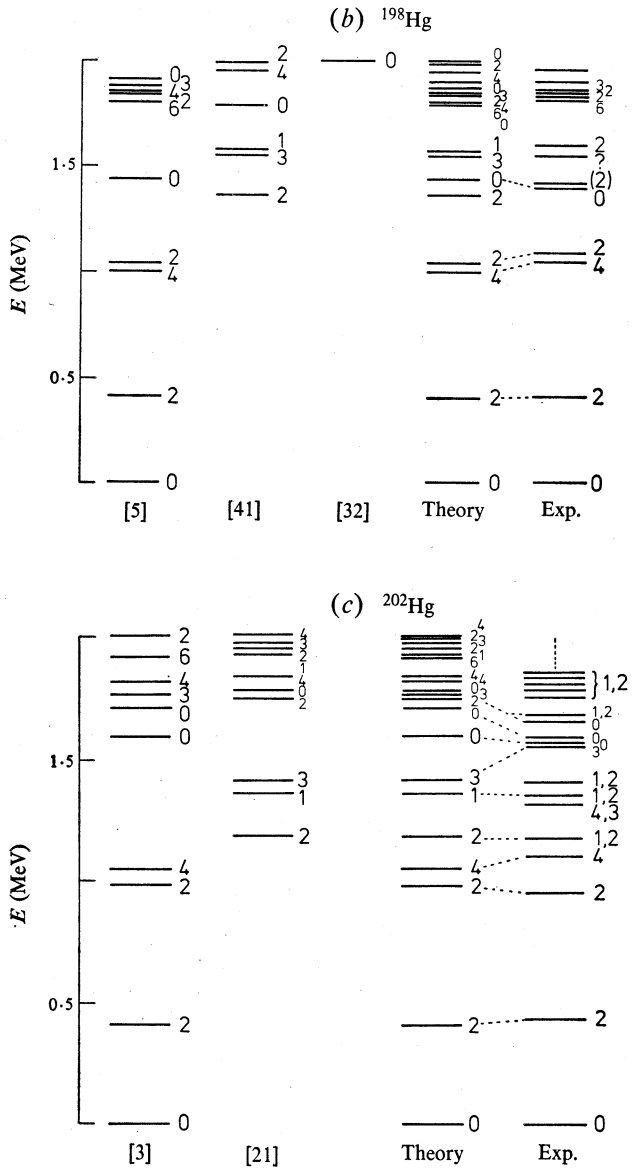
This expression is equivalently generated by assuming a residual Majorana-type space exchange interaction P^S_{12} (and hence $+P^F_{12}$) between the bosons.



Figs 5*a*–5*c*. Comparison with experiment of predicted level spectra for the even Hg isotopes. The corresponding model parameters (equation 24) are:
(*a*) ^{196}Hg, $[A, B, C, D] = [-0{\cdot}034, -0{\cdot}047, 0{\cdot}106, 0]$;
(*b*) ^{198}Hg, $[A, B, C, D] = [-0{\cdot}034, -0{\cdot}060, 0{\cdot}106, -0{\cdot}0025]$;
(*c*) ^{202}Hg, $[A, B, C, D] = [-0{\cdot}030, -0{\cdot}100, 0{\cdot}095, 0{\cdot}005]$.

The spectra of the even Hg isotopes calculated in this limit are compared with experiment in Figs 5*a*, 5*b* and 5*c* (Breitig *et al.* 1975) where it can be seen that this limiting symmetry is especially appropriate in discussing these nuclei. Not only does the maximum F-spin representation ($a = 0$) reproduce the low lying levels but representations with lower F spin ($a \neq 0$) produce extra states at >1 MeV excitation for which there exist possible experimental counterparts. The mapping of Fig. 5 is not complete due to ambiguities in experimental spin assignments, but three average features are 'explained' by inclusion of the $[N-1, 1]$ states:

(1) extra states in the 1–1·5 MeV region;
(2) very high density of states $\gtrsim 1{\cdot}6$–$1{\cdot}7$ MeV;
(3) three 0^+ states at $\sim 1{\cdot}6$ MeV.



Figs 5*b* and 5*c* [see caption on facing page]

However, by themselves, level spectra are not conclusive and it is possible to also study E2 decay patterns. Writing the E2 operator, in general, as

$$T(\mathrm{E2}) = e_{\mathrm{p}}(s^{+}d+d^{+}s)^{2}_{\mathrm{p}}+e_{\mathrm{n}}(s^{+}d+d^{+}s)^{2}_{\mathrm{n}}+e'_{\mathrm{p}}(d^{+}d)^{2}_{\mathrm{p}}+e'_{\mathrm{n}}(d^{+}d)^{2}_{\mathrm{n}}, \tag{26}$$

the e_{p} (e_{n}) being effective proton (neutron) transition strengths, the very weak crossover decay $2^{+}_{2}\rightarrow 0^{+}_{1}$, which could proceed via the $\Delta\tau = 0, \pm 2$ $(d^{+}d)^{2}$ operator (where $\tau = \tau_1+\tau_2$), forces $e'_{\mathrm{p}} \approx e'_{\mathrm{n}} \approx 0$. Then

$$T(\mathrm{E2}) = e_{\mathrm{p}}(s^{+}d+d^{+}s)^{2}_{\mathrm{p}}+e_{\mathrm{n}}(s^{+}d+d^{+}s)^{2}_{\mathrm{n}}. \tag{27}$$

In the case $e_p = e_n$ the expression (27) reverts to that of Arima and Iachello (1978*b*) with the E2 selection rules: (i) $T(E2)$ will not couple different U(6) and R(6) irreps, and (ii) $\Delta\tau = \pm 1$; hence the Majorana states cannot decay to the $[N]$ ground state band. However, it is unreasonable to expect $e_p = e_n$ even if H_B is charge independent (cf. the fermion case). If $e_p \neq e_n$ then only the condition (ii) above holds and the Majorana states can decay to the ground state band. Values for both e_p and e_n can be unambiguously extracted from $B(E2, 2_1 \rightarrow 0_1)$ since the following relationship holds for the ground state band:

$$S \equiv \{B(E2, 0_1 \rightarrow 2_1)/N(N+4)\}^{\frac{1}{2}} = e_s + e_v M_F/F, \tag{28}$$

where e_s (e_v) is the F-scalar (F-vector) transition strength. A plot of S against M_F/F over a range of N values will give e_s (intercept), e_v (slope) and hence e_p and e_n. From the results of Bockisch *et al.* (1979) for ^{204,202,200}Hg, a linear dependence is indeed obtained which gives $e_p = 23{\cdot}86\, e\,\mathrm{fm}^2$ and $e_n = 13{\cdot}82\, e\,\mathrm{fm}^2$. The ratio e_n/e_p (0·58) is identical with that obtained in fermion shell model calculations (Ma and True 1973) in this mass region. Extension of the above technique to lighter Hg and Pt isotopes gives empirical indication of phase transitions in ^{190}Pt and ^{200}Hg (Morrison and Spear 1980). A plot of S against M/F is shown for the even Hg isotopes in Fig. 6*a*. The $B(E2)$ $0_1 \rightarrow 2_1$ values are taken from experiment (Breitig *et al.* 1975). A sharply defined 'knee' is apparent at $A = 200$. In the region $A = 204$–200, we obtain $e_p = 23{\cdot}86\, e\,\mathrm{fm}^2$ and $e_n = 13{\cdot}82\, e\,\mathrm{fm}^2$, giving $e_n/e_p = 0{\cdot}58$. However, in the region $A = 200$–196, we obtain $e_p = 39{\cdot}55\, e\,\mathrm{fm}^2$ and $e_n = 8{\cdot}55\, e\,\mathrm{fm}^2$, giving $e_n/e_p = 0{\cdot}21$. Thus, there is a sharp change in the ratio of effective neutron and proton transition strengths at ^{200}Hg, and hence in the neutron and proton distributions.

It is noteworthy that a similar plot in Fig. 6*b* for the ^{196}Pt–^{186}Pt isotopes shows a continuation of the $A = 200$–196 line with another 'knee' at mass 190 where the well-known shape transition occurs (Cline 1978). Weeks and Tamura (1980) have shown recently from a microscopic boson-expansion calculation that this transition arises from competition between proton and neutron shapes as a function of proton and neutron boson number. The results of the present elementary IBA calculation suggest that the observed discontinuity at ^{200}Hg has a similar cause. It should be noted that the standard single-boson IBA model cannot predict this 'phase' transition.

Using the above values for e_p and e_n in ^{202}Hg, we find that the transitions from the Majorana levels to the ground state band remain weak ($B(E2) \approx 30\, e^2\,\mathrm{fm}^4$) in comparison with the $B(E2, 2_1 \rightarrow 0_1)$ value of $1233\, e^2\,\mathrm{fm}^4$, and hence they do not differ significantly from the $e_p = e_n$ value of zero. The general E2 decay pattern is in fair agreement with experiment (Breitig *et al.* 1975) but it should be noted that the experiment assigned $B(E2)$ rates, normalized to a theoretical model, to tentative level spins and therefore detailed comparison is not possible. However, for the Majorana states ($a \neq 0$), the model predicts:

(1) $2_3 \rightarrow 2_2, 0_1, 4_1$ (weak);

(2) $1_1 \rightarrow 2_3$ (strong), 2_1 (weak);

(3) $3_1 \rightarrow 2_3$ (strong), 2_1 (weak);

and these features can generally be seen. The decays of the 2_2, 4_1 and 0_2 states are adequately explained by normal selection rules applied to the $[N]$ irrep. There is an obvious need for more experimental work on E2 rates in the Hg isotopes, particularly

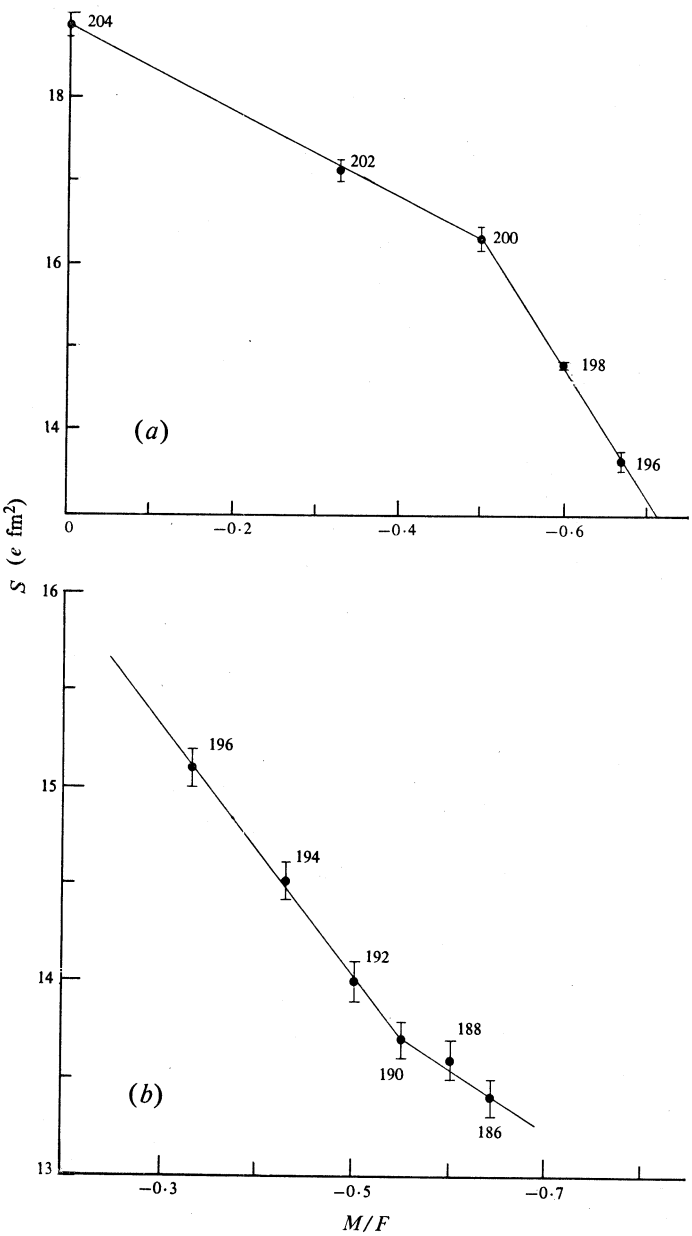


Fig. 6. Plots of the 'reduced' B(E2) factor S versus M/F for (*a*) the even isotopes ^{196}Hg–^{204}Hg and (*b*) the even isotopes ^{186}Pt–^{196}Pt.

for states in the 1–2 MeV region, if the (p–n) O(6) model is to be tested even in its symmetry-conserving limit.

The extension required to discuss symmetry breaking in H_B can best be divided into two areas:

(i) F-spin conserving but R(6) breaking, e.g. via a $\boldsymbol{Q}.\boldsymbol{Q}$ quadrupole force; this is necessary for $Q(2_1^+)$ predictions;

(ii) F-spin breaking.

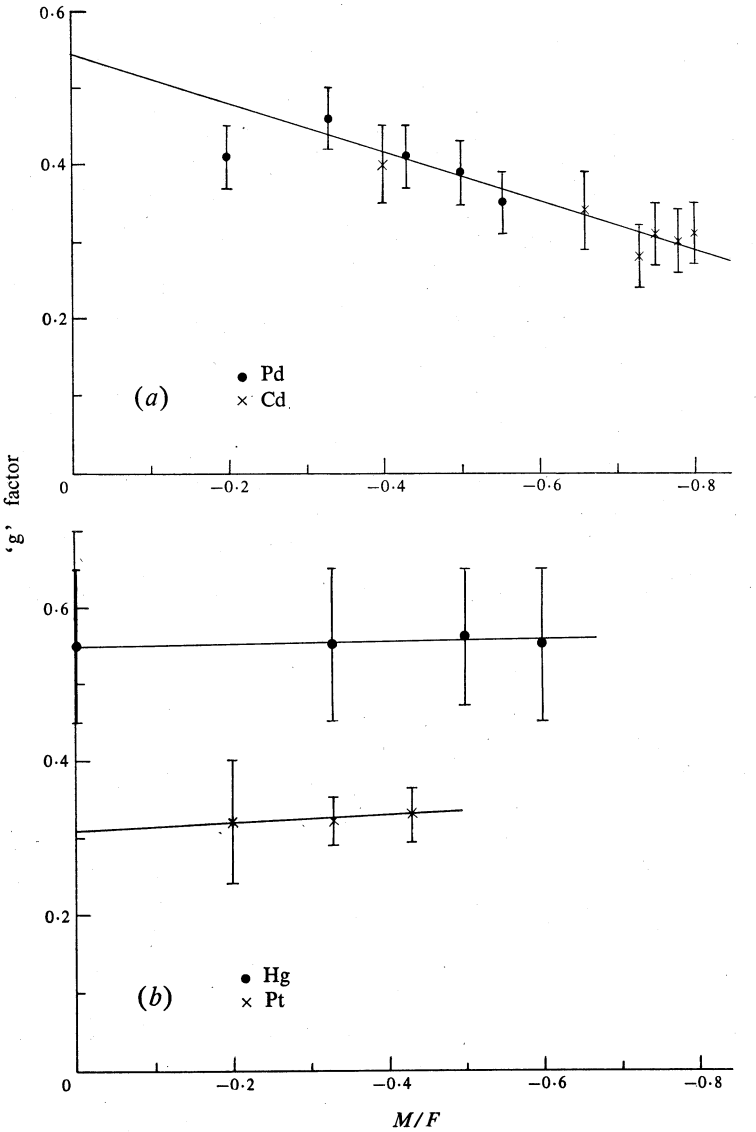


Fig. 7. Plots of the experimental 'g' factors versus M/F for the 2_1^+ states in (*a*) even Pd and Cd isotopes and (*b*) even Pt and Hg isotopes.

From the facts that the model appears to work for the Hg isotopes ($N_p = 1$, $N_n = 1$–5) *and* that the spectra of ^{196,198}Pt ($N_p = 2$) are *not* exactly identical with ^{196,198}Hg, we conclude that the p–n and n–n boson forces are identical but that the p–p force differs. Symmetry breaking via the condition (ii) would therefore appear more appropriate, although again more experimental work is required in the even–even $196 < A < 204$ mass region.

For a discussion of M1 decay, the appropriate boson operator is (Arima and Iachello 1976)

$$T(\mathrm{M1}) = m'(d^+d)' = g\boldsymbol{L} \tag{29}$$

(where g is a parameter). Hence, since $\boldsymbol{L}$ is diagonal in any representation, even if the symmetries of H_B are broken there can be no M1 decays in the $(s+d)$ boson model. Similarly from the form of equation (29) the 'g' factors of all the nuclear levels are identical in magnitude and sign. Variations to the above can only occur if we allow both symmetry breaking in H_B and second-order corrections to $T(M1)$ (Arima and Iachello 1976).

The predictions of the (p–n) boson model are different in that M1 decays and differences in g factors occur in first order in $T(M1)$. Writing

$$T(M1) = g_p \boldsymbol{L}_p + g_n \boldsymbol{L}_n, \tag{30}$$

we find, using the F-spin variables, for transitions involving the highest F-spin states

$$T(M1) = \{g^+ + (M/F)g^-\}\boldsymbol{L}, \tag{31}$$

where $g^\pm = \frac{1}{2}(g_p \pm g_n)$. This is the simple scaling result used in the E2 systematics. Therefore for the standard model (highest F-spin states) again no M1 decays are allowed (in reasonable agreement with experiment) and g factors are constant. However, we now have that the lower-F states have a different g factor and can M1 decay to the maximum-F levels in first order. In analogy with the $B(E2)$ systematics, we see from equation (31) that for the 2_1^+ level its g factor should have an isotope dependence given by M/F. A plot of the 2_1^+ g factor against M/F for nuclei near the vibrational limit is compared with experiment (Brennan *et al.* 1980) in Fig. 7*a*. It can be seen that the correct mass dependence is obtained and that the experimental results are consistent with the standard collective model value $g \approx Z/A$ to which equation (30) reduces if $g_p \approx 1$ and $g_n \approx 0$. In the (p–n) boson model, therefore, differences in g factors for low lying states are interpreted in terms of mixing of states of different F spin, that is, F-spin symmetry breaking. The corresponding g-factor plot for the even Hg and Pt nuclei is compared with experiment (Lederer and Shirley 1978) in Fig. 7*b*. The experimental results for both Hg and Pt are consistent within experimental errors with the simple scaling model. To enable further work to be done in evaluating the degree of F-spin breaking in the Hg–Pt region, lower experimental errors are required in g-factor measurements which should be extended to other excited states, possibly the 2_2^+ and 4_1^+ levels. For the nuclei plotted, we obtain:

Pd, Cd	$g_p = 0{\cdot}87$	$g_n = 0{\cdot}22$
Hg	$0{\cdot}543$	$0{\cdot}618$
Pt	$0{\cdot}226$	$0{\cdot}364$

Therefore while we expect that Hg and Pt should be similar, this is not the case.

Conclusions

In comparison with other boson models of the nucleus, and the fermion shell model, the IBA model has been shown to provide a realistic basis for analysing experimental data. The mathematical framework of the model (given by the appropriate subgroup chains) enables simple analytical formulae to be obtained, for the matrix elements of standard nuclear operators, which clearly exhibit isotope and level dependences. This is particularly useful for the study of the systematics of low excitation levels in even–even nuclei where any symmetry breaking in the boson Hamiltonian should be limited.

The utility of the model in providing estimates of particle transfer intensities, inelastic scattering cross sections and nuclear transitions and moments has been discussed in some detail and from this study it would appear that the Pt–Hg mass region is an interesting one due to the presence of 'shape' transitions and the possibility of F-spin breaking (as indicated by level spectra and 'g' factor measurements). The extension of the model to include the dipole ($L = 1^-$) boson, and hence to examine the mass dependence of shape (quadrupole) and isospin (F-spin) splitting of the giant dipole resonance, is relatively straightforward and such an extension is being investigated.

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