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Enumeration of Group Invariant Quartic Polynomials in Higgs Scalar Fields

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Abstract

The properties of Schur functions are used to enumerate the terms that must appear in a G-invariant symmetric quartic polynomial constructed from a set of Higgs fields. The importance of classifying Higgs fields as to their orthogonal, symplectic or complex representations is emphasized. The peculiar difficulties associated with symplectic irreps are noted. The groups SU_5 and SO_{10} are used to provide examples of the methods developed.

Introduction

The Higgs mechanism plays an essential role in introducing spontaneous symmetry breaking in grand unified theories that attempt to unify weak, electromagnetic and strong interactions. An essential part of the implementation of the Higgs mechanism is the construction of the Higgs potential for the group G associated with a particular unification scheme. In essence, sets of Higgs scalar fields that transform under G according to various irreducible representations (irreps) of G are introduced and a G-invariant quartic polynomial is constructed. An essential part of the problem is the enumeration of the number of independent G-invariant quartic polynomials that may be constructed from a chosen set of Higgs scalar fields. In this paper we give a systematic, and general, method for making such an enumeration.

We consider a set of Higgs scalar fields $\phi \equiv \{\phi_{\lambda_1}, \phi_{\lambda_2}, ..., \phi_{\lambda_r}\}$ with the D_{λ_i} components ϕ_{λ_i} transforming as the D_{λ_i} dimensional irrep λ_i of a group G. From these scalar fields we may construct a Higgs potential $P(\phi)$ as a symmetric quartic polynomial that is invariant at least with respect to G. In some cases $P(\phi)$ may admit a higher symmetry group $\bar{G} \supset G$ (see Weinberg 1972). The number n_{H} of independent parameters appearing in the Higgs potential $P(\phi)$ depends on the number of independent polynomials $p(\phi)$ of order 4 or less that can be constructed out of the set of scalar fields. In this paper we outline some simple methods for classifying the possible Higgs potentials and determining the numbers n_{H}. Finally, we sketch the identification of those cases where the Higgs potential may exhibit an invariance with respect to a higher symmetry group $\bar{G} \supset G$.

Integrity Bases and Polynomial Invariants

It is well known (Weyl 1946) that every G-invariant polynomial in ϕ may be expressed as a polynomial function of a minimal set of G-invariant polynomials. This minimal set is said to form an integrity basis. A general prescription for constructing integrity bases for finite groups has been given by McLellan (1974, 1979), while Judd *et al.* (1974) have investigated aspects of compact groups with special attention being given to SU_3. These methods all involve the construction of generating functions and would yield a complete solution to the construction of G-invariant polynomials. Renormalizability requirements limit the Higgs potential $P(\phi)$ to at most fourth order in the fields ϕ. This restriction makes it unnecessary to construct a complete integrity basis which will normally involve at least some polynomials of greater than fourth order in the ϕ. Thus the problem of constructing and enumerating the terms in the Higgs potential is a lesser problem than the construction of a complete integrity basis.

A number of alternative constructions and enumeration methods exist. Thus Cvitanovic (1976) has developed a method of group-theoretic weights coupled with a diagrammatic technique to treat a number of cases of interest. Here we shall make use of the properties of Schur functions (S-functions) (Littlewood 1950; Wybourne 1970, 1974) to enumerate the various possible Higgs potentials $P(\phi)$ for various Higgs scalar fields ϕ.

S-functions and Group Invariants

It is well known that the characters of the classical groups A_n, B_n, C_n and D_n may all be expressed in terms of S-functions and that this can lead directly to the enumeration of Kronecker products and branching rules (see e.g. Wybourne 1970; King 1975). S-functions can also play a vital role in determining the properties of the exceptional groups G_2, F_4, E_6, E_7 and E_8 (Wybourne and Bowick 1977; Wybourne 1978, 1979; King and Al-Qubanchi 1981). These methods rely heavily on the Littlewood–Richardson (1934) rule for forming outer products of S-functions and the plethysm of S-functions (Littlewood 1943).

A given S-function will usually be labelled by a partition of integers (λ). Frequently we shall omit the parentheses for the partition and just write λ. The operation of S-function plethysm plays a key role in what follows. The operation of the plethysm $\otimes$ is distributive on the *right*, but not necessarily so on the *left*, with respect to addition, subtraction and multiplication. Six rules that lead to a complete calculus of S-function plethysm may be stated as follows (Wybourne 1970):

$$A\otimes(BC) = (A\otimes B)(A\otimes C), \tag{1}$$

$$A\otimes(B\pm C) = A\otimes B \pm A\otimes C, \tag{2}$$

$$(A\otimes B)\otimes C = A\otimes(B\otimes C), \tag{3}$$

$$(A+B)\otimes\lambda = \sum \Gamma_{\mu\nu\lambda}(A\otimes\mu)(B\otimes\nu), \tag{4}$$

$$(A-B)\otimes\lambda = \sum (-1)^r \Gamma_{\mu\nu\lambda}(A\otimes\mu)(B\otimes\tilde{\nu}), \tag{5}$$

$$(AB)\otimes\lambda = \sum g_{\mu\nu\lambda}(A\otimes\mu)(B\otimes\nu). \tag{6}$$

In equations (4), (5) and (6), $\Gamma_{\mu\nu\lambda}$ is the coefficient of λ in the S-function outer product $\mu\nu$, $\tilde{\nu}$ is a partition of r conjugate to ν, and $g_{\mu\nu\lambda}$ is the coefficient of λ in the S-function inner product $\mu\circ\nu$.

Suppose that we have $\bar{G}\supset G$ and that under $\bar{G}\to G$ the vector irrep of $\bar{G}$, designated $[1]$, decomposes as

$$[1] \to \{\alpha\} + \{\beta\} + \dots + \{\omega\}; \tag{7}$$

then the representation $\{\lambda\}$ of $\bar{G}$ decomposes into the representations $\{\rho\}$ of G according to the representations of G contained in the plethysm

$$(\{\alpha\} + \{\beta\} + \dots + \{\omega\}) \otimes \{\lambda\} \tag{8}$$

(Wybourne and Butler 1969). The plethysm (8) is evaluated by firstly expressing the characters of $\bar{G}$ and G in terms of S-functions, then evaluating the plethysm to give S-functions pertaining to the characters of G, and finally expressing these latter S-functions in terms of the characters of G to give the final result.

As an example that will lead later to our general observations, consider the eight-dimensional adjoint irrep of SU_3 which is designated by the partition 21. This irrep of SU_3 may be embedded in the vector irrep 1 of SU_8 such that under $SU_8 \to SU_3$ we have $1 \to 21$ and hence, for an arbitrary irrep λ of SU_8, we have via the plethysm (8)

$$\lambda \to 21 \otimes \lambda .$$

If λ is a symmetric irrep of SU_8 then using tables of S-function plethysms (see e.g. Butler and Wybourne 1971) we have the simple decompositions given in Table 1 for $\lambda = 1$, 2, 3 and 4.

Table 1. Some $SU_8 \to SU_3$ decompositions

Dimension D_λ	SU_8	Branching to SU_3
8	1	21
36	2	$0+21+42$
120	3	$0+21+3+3^2+42+63$
330	4	$0+2.21+2.42+51+54+63+84$

Inspection of Table 1 shows that the identity irrep 0 of SU_3 occurs for $\lambda = 2$, 3 and 4. The $\lambda = 2$ and 3 scalars are clearly independent whereas the $\lambda = 4$ scalar can be taken as simply the square of the second order scalar. Thus if we have a set of eight Higgs scalars ϕ_{21} transforming under SU_3 as the 21 irrep then we can write the Higgs potential in terms of just three polynomial functions such that

$$P(\phi_{21}) = a_2 p_2(\phi_{21}) + a_3 p_3(\phi_{21}) + a_4 p_2^2(\phi_{21}), \tag{9}$$

where a_2, a_3 and a_4 are the parameters associated with the Higgs potential. The order of the polynomial invariants is given as a subscript. Thus $p_3(\phi_{21})$ is a group-invariant symmetric polynomial of order 3 in the Higgs scalars. We note that in this case the invariant $p_2(\phi_{21})$ and $p_3(\phi_{21})$ could be taken as just the second and third order Casimir invariants of SU_3.

The above example leads us to the following general result: *If we have a set of Higgs scalar fields*

$$\phi \equiv \{\phi_{\lambda_1}, \phi_{\lambda_2}, \dots, \phi_{\lambda_r}\}$$

then the number of distinct G*-invariant symmetric polynomials that can be constructed to be of order* N *in the fields is equal to the number of times the identity irrep 0 of the invariance group* G *occurs in the plethysm*

$$(\lambda_1 + \lambda_2 + \dots + \lambda_r) \otimes N. \tag{10}$$

We note that these G-invariant symmetric polynomials need not all be independent. Thus, for example, a fourth order invariant may be representable as the square of a second order invariant as in equation (9).

For the task at hand we may restrict our attention to just $N = 2$, 3 and 4. We start by first considering the properties of second order invariants, then third order invariants and finally fourth order invariants.

G-*invariant Second Order Symmetric Polynomials* $p_2(\phi)$

Let us first consider the case of a set of Higgs fields ϕ_λ belonging to a single irrep λ of a group G. Then a second order G-invariant symmetric polynomial will exist if and only if

$$\lambda \otimes 2 \supset 0, \tag{11}$$

where 0 is the identity irrep of G. This will be the case if and only if λ is equivalent to a real representation and hence is of the orthogonal type (Butler and King 1974). Thus a complex or a symplectic representation cannot produce a second order symmetric G-invariant polynomial.

The irreps of the classical Lie groups may all be labelled by partitions $\lambda \equiv (\lambda_1 \lambda_2 \ldots \lambda_r)$ of integers or half-integers. We may classify the irreps of the classical and exceptional Lie groups as orthogonal, symplectic or complex as follows (Mal'cev 1944; Butler and King 1974):

SU_n. If $n = 0, 1, 3 \pmod 4$ then all self-contragredient irreps are orthogonal whilst if $n = 2 \pmod 4$ the self-contragredient irreps are orthogonal (symplectic) if λ_1 is *even* (*odd*). All other irreps of SU_n are complex.

SO_n. The true or tensor irreps are all orthogonal except for $n = 2 \pmod 4$ where if $\lambda_{\frac{1}{2}n} \neq 0$ the irrep is complex. The spinor irreps are orthogonal for $n = 0, 1, 7 \pmod 8$, symplectic for $n = 3, 4, 5 \pmod 8$ and complex for $n = 2, 6 \pmod 8$.

Sp_n. The irreps associated with *even* weight partitions are orthogonal while those with *odd* weight partitions are symplectic.

G_2, F_4, E_8. The irreps are all orthogonal.

E_6. The self-contragredient irreps are orthogonal with all other irreps being complex.

E_7. The irreps are orthogonal or symplectic according as the weight $(\lambda_1 + \lambda_2 + \ldots + \lambda_7)$ of the partition, which is necessarily even, is 0 (or 2) (mod 4).

With the above results in mind we see that if λ is the adjoint irrep of a simple group G then a second order symmetric polynomial can always be constructed and indeed may be taken as the usual second order Casimir invariant.

The fundamental irrep (1^6) of E_7 is symplectic and hence it is impossible to form a second order symmetric polynomial from a single set of 56 Higgs scalars transforming as the (1^6) irrep of E_7. Similar remarks hold for all other symplectic irreps. However, if we double the Higgs scalars to give $\phi \equiv \{\phi_{16};\ \phi'_{16}\}$ then we can form a second order symmetric polynomial since by the rule (4) we have

$$((1^6)+(1^6)')\otimes 2 = (1^6)\otimes 2 + (1^6)\times(1^6)' + (1^6)'\otimes 2 .$$

The terms $(1^6)\otimes 2$ and $(1^6)'\otimes 2$ cannot yield the identity irrep whereas it certainly does occur in $(1^6)\times(1^6)'$.

In the general case where the Higgs fields belong to several irreps of G, say $\lambda_1, \lambda_2, \ldots, \lambda_r$, we evaluate $(\lambda_1+\lambda_2+\ldots+\lambda_r)\otimes 2$ using the rule (4) to obtain

$$(\lambda_1+\lambda_2+\ldots+\lambda_r)\otimes 2 = \sum_{i=1}^{r} (\lambda_i\otimes 2) + \sum_{j>i}^{r} (\lambda_i \times \lambda_j). \tag{12}$$

The terms $\lambda_i \otimes 2$ will yield scalars only for those cases where λ_i is an orthogonal type irrep, while the terms $\lambda_i \times \lambda_j$ will yield scalars if and only if $\lambda_i \equiv \lambda_j^*$, where λ_j^* is the contragredient of λ_j. Thus if the Higgs fields all belong to symplectic irreps then G-invariant second order symmetric polynomials can only arise if the irreps are at least doubled. Alternatively if the Higgs fields belong to complex irreps then a G-invariant second order symmetric polynomial can only arise if the complex representation λ occurs with its contragredient complex partner λ^*. Thus theories based on Higgs fields in other than orthogonal irreps must be very restricted.

Using the above information we can readily determine for any set of scalar Higgs fields the number of distinct G-invariant second order symmetric polynomials. These polynomials are necessarily all independent and hence the number of independent second order symmetric polynomials is given by the number of times the identity irrep arises in equation (12).

G-*invariant Third Order Symmetric Polynomials* $p_3(\phi)$

All G-invariant third order symmetric polynomials $p_3(\phi)$ are necessarily independent. In general the number of symmetric polynomials associated with a set of Higgs fields $\phi \equiv \{\phi_{\lambda_1}, \phi_{\lambda_2}, \ldots, \phi_{\lambda_r}\}$ will be equal to the number of times the identity irrep arises in the plethysm

$$(\lambda_1+\lambda_2+\ldots+\lambda_r)\otimes 3$$
$$= \sum_{i=1}^{r} (\lambda_i\otimes 3) + \sum_{i\neq j}^{r} ((\lambda_i\otimes 2)\times\lambda_j) + \sum_{i<j<k} (\lambda_i\times\lambda_j\times\lambda_k). \tag{13}$$

Since the product of two symplectic irreps is orthogonal† and the product of an orthogonal irrep with a symplectic irrep is symplectic, it is usually not possible to construct a G-invariant third order polynomial by using Higgs fields involving only symplectic irreps. A set of Higgs fields involving both symplectic *and* orthogonal irreps may yield G-invariant third order symmetric polynomials. In these cases the invariant must be quadratic in the symplectic Higgs fields. Likewise, since the product of a spinor irrep with a tensor irrep necessarily yields spinor irreps, it is impossible to construct a G-invariant third order polynomial out of Higgs fields belonging just to spinor irreps. The only possible symmetric polynomials would have to be quadratic in the spinor fields and linear in the tensor fields.

If the Higgs fields are restricted to the adjoint irrep then a G-invariant third order symmetric polynomial will exist for SO_6 and for all SU_n ($n \geqslant 3$). No other cases exist among the classical and exceptional simple Lie groups. In the case of a single set of Higgs fields belonging to the defining irrep of a simple Lie group, a G-invariant third order symmetric polynomial is only possible for F_4 and E_6.

† In practice the orthogonal representation will be reducible and *may* contain a pair of symplectic irreps, although such a situation would appear to be quite exceptional.

Equation (13) allows us to readily enumerate the possible G-invariant third order symmetric polynomials for any group and set of Higgs fields of interest. Observing where the identity arises in equation (13) gives additional insight into the structure of the polynomials. It is worth noting that the third order invariants will be absent in the presence of a discrete symmetry $\phi \rightarrow -\phi$, analogously to the occurrence of a centre of inversion in finite groups (McLellan 1974).

G-invariant Fourth Order Symmetric Polynomials $p_4(\phi)$

The enumeration of G-invariant fourth order symmetric polynomials amounts to identifying the occurrence of the identity irrep in the plethysm

$$(\lambda_1+\lambda_2+\ldots+\lambda_r)\otimes 4 = \sum_{i=1}^{r}(\lambda_i\otimes 4) + \sum_{i\neq j}^{r}((\lambda_i\otimes 3)\times\lambda_j) + \sum_{i<j}^{r}(\lambda_i\otimes 2)\times(\lambda_j\otimes 2)$$
$$+ \sum_{i<j<k}((\lambda_i\otimes 2)\times\lambda_j\times\lambda_k) + \sum_{i<j<k<l}(\lambda_i\times\lambda_j\times\lambda_k\times\lambda_l). \quad (14)$$

The number of times the identity irrep occurs in the evaluation of the right-hand side of equation (14) will give the total number of G-invariant fourth order symmetric polynomials. Not all of these fourth order polynomials will be independent, as clearly any bilinear product of the second order polynomials will yield a fourth order polynomial.

If the Higgs fields belong to just the adjoint irrep of a simple Lie group then there can be no independent G-invariant fourth order symmetric polynomials in the exceptional groups nor for SU_2 and SU_3. Such a polynomial will arise, however, once in each of the groups SU_n ($n \geqslant 4$), SO_n ($n \geqslant 5$) and Sp_n ($n \geqslant 4$). The case SO_8 is exceptional as it yields two independent invariant fourth order symmetric polynomials.

In the case of the defining irreps of the classical groups the fourth order invariant polynomial is always representable as the square of the second order symmetric polynomial. Among the exceptional groups only E_7 has an independent G-invariant fourth order symmetric polynomial, in agreement with the results of Cvitanovic (1976).

We note that the *912*-dimensional irrep (2^7) of E_7 is symplectic and has been invoked as a Higgs multiplet for inducing symmetry breaking in E_7 unified models (Ramond 1976, 1977; Sikivie and Gürsey 1977). Such a representation by itself can only yield a fourth order symmetric polynomial. A second order symmetric polynomial is only possible if the symplectic irrep is doubled.

Enumeration for SU_5

Much of the foregoing can be illustrated by considering the enumeration of the terms in the Higgs potential for an SU_5 unified model where the Higgs fields are taken as

$$\phi \equiv \{\phi_a, \phi_f, \tilde{\phi}_f\}, \quad (15)$$

where ϕ_a is the adjoint irrep 21^3 of SU_5 and ϕ_f the defining irrep 1, with $\tilde{\phi}_f$ the conjugate irrep 1^4. The number of terms appearing in the Higgs potential will equal the number of times the identity irrep 0 arises in the plethysm

$$(21^3+1+1^4)\otimes(2+3+4). \quad (16)$$

Consider the second order symmetric polynomials first. They arise in

$$(21^3+1+1^4)\otimes 2 = 21^3\otimes 2 + 1\otimes 2 + 1^4\otimes 2 + 21^3\times 1 + 21^3\times 1^4 + 1\times 1^4, \quad (17)$$

where we have made use of the rule (4) to obtain the right-hand side. The identity irrep occurs only in the terms $21^3\otimes 2$ and 1×1^4 and thus there are just two SU_5 invariant second order symmetric polynomials, say $p_2(\phi_a^2)$ and $p_2(\phi_f, \tilde{\phi}_f)$.

Expansion of the term $(21^3+1+1^4)\otimes 3$ in the plethysm (16) shows that the identity irrep occurs twice, once in $21^3\otimes 3$ and once in $21^3\times 1\times 1^4$, leading to two independent SU_5 invariant third order symmetric polynomials, say $p_3(\phi_a^3)$ and $p_3(\phi_a, \phi_f, \tilde{\phi}_f)$.

Finally, expansion of the term $(21^3+1+1^4)\otimes 4$ in (16) gives the identity irrep twice in $21^3\otimes 4$, twice in $(21^3\otimes 2)\times((1+1^4)\otimes 2)$ and once in $(1+1^4)\otimes 4$. Of the two terms arising from $21^3\otimes 4$, one may be taken as $[p_2(\phi_a^2)]^2$ and the other as $p_4(\phi_a^4)$, while the two terms arising from $(21^3\otimes 2)\times((1+1^4)\otimes 2)$ may be taken as $p_2(\phi_a^2)\,p_2(\phi_f, \tilde{\phi}_f)$ and $p_4(\phi_a^2, \phi_f, \tilde{\phi}_f)$. The term in $(1+1^4)\otimes 4$ may be taken as $[p_2(\phi_f, \tilde{\phi}_f)]^2$.

Thus for SU_5 with the choice of Higgs fields (15) we obtain a set of nine SU_5 invariant symmetric polynomials in the Higgs fields:

$$p_2(\phi_a^2),\quad p_2(\phi_f, \tilde{\phi}_f),\quad p_3(\phi_a^3),\quad p_3(\phi_a, \phi_f, \tilde{\phi}_f),\quad [p_2(\phi_a^2)]^2,\quad p_4(\phi_a^4), \quad (18a)$$

$$[p_2(\phi_a^2)\,p_2(\phi_f, \tilde{\phi}_f)],\quad p_4(\phi_a^2, \phi_f, \tilde{\phi}_f),\quad [p_2(\phi_f, \tilde{\phi}_f)]^2. \quad (18b)$$

120-dimensional irrep of SO_{10}

The group SO_{10} has arisen in a number of grand unified theories, and a variety of Higgs fields have been considered including the *16*-dimensional basic spin irreps $\Delta_\pm \equiv [\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\pm\frac{1}{2}]$, the *10*-dimensional vector irrep [1], the *45*-dimensional adjoint irrep $[1^2]$ and the *120*-dimensional irrep $[1^3]$. There is no great difficulty in evaluating the relevant SO_{10} Kronecker products and plethysms using the method of difference characters (see e.g. Littlewood 1950; Wybourne and Butler 1969; Wybourne 1970). Extensive tables are also available via the interactive computer program of P. H. Butler (personal communication).

Here we shall confine our attention to a single set of Higgs fields spanning the *120*-dimensional $[1^3]$ irrep of SO_{10}. We have in terms of S-functions (see Wybourne 1970)

$$[1^3]\otimes(2+3+4) \equiv \{1^3\}\otimes(\{2\}+\{3\}+\{4\}).$$

We now note that (Butler and Wybourne 1971)

$$\{1^3\}\otimes\{2\} = \{21^4\}+\{2^3\},$$

$$\{1^3\}\otimes\{3\} = \{2^31^3\}+\{31^6\}+\{32^21^2\}+\{3^3\},$$

$$\{1^3\}\otimes\{4\} = \{2^6\}+\{32^21^5\}+\{32^31^3\}+\{3^22^21^2\}+\{3^31^3\}+\{41^8\}$$
$$+\{42^21^4\}+\{42^4\}+\{43^21^2\}+\{4^3\}.$$

These S-functions must now be expressed in terms of SO_{10} characters, and the number of times the identity irrep occurs in each plethysm must be determined. However, we can shortcut most of the work by noting that an S-function reduced to characters of SO_n will only yield the identity irrep if the partition is comprised solely of *even* parts (see Wybourne 1970, equation 69) and then only once. Thus we immediately have the desired results

$$[1^3]\otimes 2 \supset [0], \qquad [1^3]\otimes 3 \not\supset [0], \qquad [1^3]\otimes 4 \supset 3[0].$$

From the above we see that, starting with a set of 120 Higgs scalars transforming as the $[1^3]$ irrep of SO_{10}, it is possible to construct *one* SO_{10} invariant second-order symmetric polynomial and *three* fourth-order symmetric polynomials. One of the latter may be taken as the square of the second order polynomial, leaving *two* other independent fourth order polynomials. Such a Higgs potential would require to be represented in terms of four parameters.

A multiplicity of independent fourth order symmetric polynomials can arise even when the Higgs fields are restricted to a single irrep.

Table 2. Some $SO_8 \to SU_3$ decompositions

Dimension D_λ	SO_8	Branching to SU_3
8	[1]	21
35	[2]	21+42
112	[3]	$0+3+3^2+42+63$
294	[4]	21+42+51+54+63+84

Higher Symmetry Higgs Potentials

In some cases the Higgs potential $P(\phi)$ will admit of a higher symmetry group $\bar{G} \supset G$ (see Weinberg 1972). As an example consider the eight-dimensional adjoint irrep 21 of SU_3. This irrep is orthogonal and hence can be embedded in the vector irrep of SO_8. The $SO_8 \to SU_3$ branching rules for the symmetric irreps of SO_8 can be readily reduced (Wybourne 1970) to give the results shown in Table 2. Inspection of this table shows immediately that the second and fourth order symmetric polynomials appearing in the SU_3 Higgs potential are necessarily invariant with respect to SO_8 whereas the third-order symmetric polynomial, while invariant with respect to SU_3, is not invariant under SO_8.

Consider a set of n Higgs fields which belong to a representation λ (not necessarily irreducible) of a group G. If λ is orthogonal (symplectic) then λ may be embedded in the vector irrep [1] of SO_n (Sp_n). In the case when λ is orthogonal the Higgs potential will be invariant with respect to $SO_n \supset G$ if and only if the identity irrep [0] of G does *not* occur in the reduction of the plethysm

$$\lambda \otimes ([2]+[3]+[4]), \tag{19}$$

where [2], [3] and [4] are characters of SO_n. If G contains a discrete reflection symmetry such that $\phi \to -\phi$ then we may eliminate the term $\lambda\otimes[3]$ from (19).

If λ is symplectic then, while it is possible to make the embedding $Sp_n \supset G$, it is impossible to construct any Sp_n invariant symmetric polynomials for the vector irrep of Sp_n for any order in the Higgs fields, and hence Sp_n can never be a higher symmetry group. Thus, while we can readily make the embedding $\langle 1\rangle \supset (1^6)$ for $Sp_{56} \supset E_7$, we find $\langle 4\rangle \supset (0)$ and hence Sp_{56} cannot be a higher symmetry.

By way of contrast the Higgs potentials constructed from the adjoint irreps of the exceptional groups all exhibit higher symmetry of the orthogonal type (Harvey 1980).

Concluding Remarks

We have seen here how S-function methods may be used to enumerate the terms that must appear in a G-invariant symmetric quartic polynomial constructed from a set of Higgs fields. Emphasis has been placed on the importance of classifying the Higgs fields as to their orthogonal, symplectic or complex representations and to the peculiar difficulties associated with symplectic irreps.

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