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Counterpart of the Kasner Model in Brans–Dicke Theory

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Abstract

We present a simple and elegant generalization of the Kasner model in Brans–Dicke (BD) theory by solving the BD field equations corresponding to the Bianchi type I metric.

1. Introduction

In this paper we obtain vacuum solutions of the Brans–Dicke (1961) field equation corresponding to the spatially homogeneous and anisotropic Bianchi type I metric. It is shown in Section 3 that our solution is a generalization of the well-known Kasner (1921) model in BD theory. Some of the properties of model are given in Section 4.

2. BD Field Equations

We consider the Bianchi type I metric

$$ds^2 = dt^2 - A^2\,dx^2 - B^2\,dy^2 - C^2\,dz^2\,, \tag{1}$$

where A, B and C are functions of time only. The BD field equations for vacuum space ($T_{ij} = 0$) are

$$G_{ij} = -(\omega/\phi^2)(\phi_i\phi_j - \tfrac{1}{2}g_{ij}\phi_k\phi^k) - \phi^{-1}(\phi_{ij} - g_{ij}\phi^k{}_{;k})\,, \tag{2}$$

$$\phi^k{}_{;k} = 0\,, \tag{3}$$

where the symbols have their usual meaning. The BD field equations corresponding to the metric (1) are

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} = -\frac{\omega\phi_4^2}{2\phi^2} + \frac{C_4\phi_4}{C\phi}\,, \tag{4a}$$

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} = -\frac{\omega\phi_4^2}{2\phi^2} + \frac{A_4\phi_4}{A\phi}\,, \tag{4b}$$

$$\frac{C_{44}}{C} + \frac{A_{44}}{A} + \frac{C_4 A_4}{CA} = -\frac{\omega\phi_4^2}{2\phi^2} + \frac{B_4\phi_4}{B\phi}\,, \tag{4c}$$

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$$\frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{C_4 A_4}{CA} = \frac{\omega\phi_4^2}{2\phi^2} - \frac{(ABC)_4 \phi_4}{ABC\phi}, \tag{4d}$$

$$\frac{\phi_{44}}{\phi} + \frac{(ABC)_4 \phi_4}{ABC\phi} = 0. \tag{4e}$$

3. The Solution

Equations (4a)–(4c) yield

$$\frac{s_{44}}{s_4} + \frac{\phi_4}{\phi} = 0, \tag{5}$$

where

$$s = ABC. \tag{6}$$

Equations (4e) and (5) give the solution

$$\phi = (at+b)^{p_1}, \qquad s = s_0(at+b)^{1-p_1}, \tag{7a, b}$$

where a, b, s_0 and p_1 are arbitrary constants.

Now equations (4a)–(4c) along with equations (7) ultimately give the solution

$$A = A_0(at+b)^{p_2}, \qquad B = B_0(at+b)^{p_3}, \qquad C = C_0(at+b)^{p_4}, \tag{8a, b, c}$$

where the arbitrary constants p_2, p_3, p_4 and A_0, B_0, C_0 satisfy

$$p_2+p_3+p_4 = 1-p_1 \quad \text{or} \quad \sum_{i=1}^{4} p_i = 1, \tag{9a}$$

$$A_0 B_0 C_0 = s_0. \tag{9b}$$

One more restriction on the p_i may be imposed with the help of equation (4d), which along with (9a), gives

$$(\omega+1)p_1^2+p_2^2+p_3^2+p_4^2 = 1. \tag{10}$$

Thus, we get the following metric for an anisotropic empty BD universe:

$$ds^2 = dt^2 - A_0^2(at+b)^{2p_2}dx^2 - B_0^2(at+b)^{2p_3}dy^2 - C_0^2(at+b)^{2p_4}dz^2.$$

This metric can be transformed through a proper choice of coordinates to the form

$$ds^2 = dT^2 - T^{2p_2}dx^2 - T^{2p_3}dy^2 - T^{2p_4}dz^2, \tag{11}$$

with the scalar field $\phi = \phi_0 T^{1-p_1}$, ϕ_0 being constant.

4. Some Physical Properties

(1) The metric (11) is the generalization of the well-known Kasner (1921) metric in BD theory. This can be seen in the following way. As we know that BD theory goes over to relativistic theory as $\omega \to \infty$, equation (10) shows us immediately that in this limit

$$p_1 = 0,$$

which gives $\phi = \phi_0$, and then the constants p_i satisfy

$$\sum_{i=2}^{4} p_i = 1 \quad \text{and} \quad \sum_{i=2}^{4} p_i^2 = 1\,.$$

Thus, in the limit $\omega \to \infty$, metric (11) is converted into the Kasner metric.

(2) The volume element in the BD model is

$$(-g)^{\frac{1}{2}} = T^{1-p_1},$$

which shows the expansion of the universe with time.

(3) The expansion is anisotropic, occurring at the rates p_2/t, p_3/t and p_4/t along the x, y and z axes respectively.

References

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