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Method of Averaged Lagrangian for Precessional Rotation and Other Complementary Effects of Waves in Nonlinear Plasmas

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Abstract

The formalism of the averaged Lagrangian has been extended and developed to evaluate the intensity-dependent precessional rotation and wave number shift of an elliptically polarized electromagnetic wave in unmagnetized cold relativistic plasmas; the results are identical to those of Arons and Max (1974) and others. Moreover, the expression for two intensity-induced nonlinear modulation frequencies for both the right and left circular polarization components of the wave have been derived. The mathematical technique developed here may be useful in the study of other types of nonlinearly evolved rotational corrections for motions in fluids, and so the possibilities of broadening the scope of this formalism are discussed.

1. Introduction

Nonlinear effects in plasmas and other media are much discussed topics of research in modern physics because of the wide scope of their applicability in astrophysics and laboratory experiments. For theoretical investigations of nonlinear effects of frequency shift, wave number shift, precessional rotation, modulational instability etc. some standard mathematical methods are followed, which give solutions of higher order nonlinear differential equations. The method of Bogoliubov, Krylov and Mitropolsky (BKM) (Bogoliubov and Mitropolsky 1961) has been widely used by many authors for the study of problems of wave propagation through plasmas (see e.g. Montgomery and Tidman 1964; Tidman and Stainer 1965; Boyd 1967; Das 1968, 1970, 1971; Chakraborty 1977; Chakraborty and Chandra 1978; Chandra 1980). Another mathematical technique, called the derivatives expansion method, was adopted by Nayfeh (1965) to recover the results of Tidman and Stainer (1965) for the frequency shift of electromagnetic waves in a plasma medium. The Lindstedt method (Bellman 1964) is also very useful in solving nonlinear equations for weak nonlinear effects in plasmas (Chandra 1974, 1976, 1979). Arons and Max (1974), Khan and Chakraborty (1979), Bhattacharyya and Chakraborty (1979), Chakraborty *et al.* (1980, 1981) and others have used some of the available methods for finding precessional rotation (PR), shifts of wave parameters and other related nonlinear effects in a plasma medium.

Another interesting method also exists which has not yet been explored to the same extent as the other methods listed above. It is a special Lorentz transformation (LT) which reduces the partial differential equations for the field variables in time and the position coordinate parallel to the direction of propagation into a system of ordinary differential equations with respect to a new time variable permitted by the

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special LT. This method has been used for solving higher order nonlinear differential equations in the study of nonlinear effects in plasmas (Akhiezer and Polovin 1956; Winkles and Eldridge 1972; Clemmow 1974, 1975, 1977; Chain and Clemmow 1975; Kennal and Pellat 1976; Shih 1978; Decoster 1978; Lee and Lerche 1978, 1979*a*, 1979*b*, 1979*c*, 1980; Clemmow and Harding 1980; Paul and Chakraborty 1983).

Low (1958) applied the Lagrangian formalism to study plasma oscillations and hydromagnetic waves in the linear theory. Later, Suramlishvili (1964, 1965, 1967) used this formalism to solve some problems of nonlinear plasmas. But these have been found to be of little benefit to the development of the Lagrangian approach in studying nonlinear problems in plasma physics. A new kind of Lagrangian approach has been examined independently in fluid mechanics by Whitham (1965, 1967, 1974), who introduced the idea of an averaged Lagrangian to both linear and nonlinear dispersive waves. When the Lagrangian $\mathscr{L}$ of motion of a system is known, the nonlinear solution of the field equations for strong waves in a plasma can be studied with the help of the averaged Lagrangian $\langle \mathscr{L} \rangle$, which is obtained by averaging $\mathscr{L}$ over the time period of the rapidly varying fields. Thus $\langle \mathscr{L} \rangle$ becomes a function of the wave amplitudes and their derivatives with respect to time t and the coordinate variable z parallel to the direction of wave propagation. This method has been found to be very useful for the study of different types of problems in fluid mechanics as well as in electrodynamics. Dougherty (1970, 1974) has reviewed the averaged Lagrangian method for the solution of equations for cold plasmas and has discussed two covariant methods for varying the background. Dewar (1970) applied this formalism to investigate the interaction between hydromagnetic waves and a time dependent inhomogeneous medium. The problems of wave–wave interactions in a plasma were also solved, using this method, by Boyd and Turner (1972, 1973, 1978), Kim and Crawford (1977) and Das and Sihi (1980). Dysthe (1974) developed Whitham's method in such a form that it became adequate for the description of wave phenomena in plasmas. Subsequently, this modified method was applied by Das and Sihi (1977, 1979) and Sihi (1980) to investigate the modulational instability of an electromagnetic wave through the introduction of the Rayleigh dissipation function (Goldstein 1970). Leroy and Bel (1979) used the formalism of the averaged Lagrangian for the study of propagation of waves in an isothermal atmosphere in the presence of a magnetic field.

The averaged Lagrangian formalism was also used by Dewar (1977) to develop a general treatment of spatially dispersive waves in warm plasmas within the WKB approximation and to elucidate some fine points of distinction between the solution of microscopic and macroscopic sets of equations for electrodynamics in plasmas. Since this discussion on the formalisms can widen the scope of research on the method of the averaged Lagrangian for the PR effect and other complementary effects of waves in plasmas, we have summarized the main points briefly in Section 7*c*.

Lagrangian densities are known for many of the equations whose soliton solutions have been studied. Hence, Crawford (1980) searched for an improved perturbation technique to study the theory of plasma solitons. Such a technique was made distinguishable from the averaged Lagrangian method, by employing an averaged Hamiltonian density expressed in canonical variables whose choice is determined from Lie algebra considerations. However, this technique has also been introduced into plasma physics by Dewar and Kaufman and their coworkers (Dewar 1970, 1972, 1973, 1976; Johnston 1976; Johnston and Kaufman 1977, 1978, 1979; Kaufman 1978) in the course of their development of the oscillation centre theory and studies of turbulence. The investigations reported in the present paper further extend the scope of application of this method to nonlinear problems of waves in plasmas, and show that it is very powerful also for the study of nonlinear wave precession and related problems of self-generating rotations of plasmas due to waves.

The existing methods, which use the averaged Lagrangian $\langle \mathscr{L} \rangle$ for the determination of the characteristics of nonlinear dispersive wave properties in continuous media and their extension to waves in plasmas, do not directly lead to the solution of the problems of nonlinear evolution of rotational motion from these waves. These latter categories of problems include the nonlinearly induced effect of birefringence due to different rates of dispersion of the field variables of the left and right circular polarization (to be called in short the LCP and RCP respectively) wave components. To find this birefringence it is necessary to isolate the nonlinearly correct LCP and RCP components and their field vectors perpendicular to the direction of wave propagation by using rotating (complex) coordinates, and then apply the technique of varying the amplitudes of these wave components in the action integral in which the integrand is $\langle \mathscr{L} \rangle$.

Application of Hamilton's principle of least action on $\langle \mathscr{L} \rangle$, through the variation of the action integral S, gives rise to the equations for the nonlinear evolution of the fields in space or time. For plasmas, scope is thus provided for finding some higher order field evolution effects (e.g. some types of modulational instabilities) in a compact manner. The usual methods of studying this nonlinear evolution, using a process of successive approximation ensuring secular free solution of the first harmonic wave field, would require more involved mathematical manipulations.

Although the old method of using $\mathscr{L}$ for the study of rotation of rigid bodies and fluids has a long history and extensive usage, the scope and regions of applicability of this method and the other using $\langle \mathscr{L} \rangle$ are vastly different. Thus, the existence of the Lagrangian analysis of rotational motion of fluids and rigid bodies is not at all helpful to our problems, just as the known Lagrangian dynamics does not directly lead to that for $\langle \mathscr{L} \rangle$. Hence, our work also increases the scope of applications of this powerful method to a wider range of problems, including perhaps those of nonlinearly generated torques in fluid flow studies. To explore the possibility of further work, we include a brief discussion on some relevant topics in Section 7.

In Section 2 we start from the expression for the $\mathscr{L}$ of the electromagnetic field for electron motion in cold unmagnetized plasmas, in which only the electrons are mobile while ions provide the neutralizing static background of positive charges. Using the Lagrangian concept of fluid motion for the electrons, the Lagrangian of motion is expanded in a Taylor series in powers of the displacement vector $\mathscr{E}$ of the fluid from the equilibrium position vector $\boldsymbol{r}$. Terms up to the fourth power of the components of $\mathscr{E}$ and other field variables and their derivatives with respect to time t and z are retained in $\mathscr{L}$, which is then written as the sum $\mathscr{L}_2+\mathscr{L}_3+\mathscr{L}_4$, where $\mathscr{L}_i$ contains a sum of terms of the ith power in the components of $\mathscr{E}$ and other field variables. Then the usual expressions for the components of the vector fields are replaced by the expression for the LCP and RCP wave components. These are obtained from an appropriate complex combination of the field components perpendicular to the direction of propagation of the wave field. The field equations for the linearized approximation and the nonlinearly excited second harmonic wave field are obtained from the perturbation expansion of the field variables in the sum Lagrangian $\mathscr{L}_2+\mathscr{L}_3$ in terms of an unspecified expansion parameter μ. This parameter is retained merely for book-keeping purposes to specify the fields of all orders correctly according to their orders of smallness. Solutions for the first order field variables including the linear dispersion relation have been derived in Section 2.

In Section 3 the field equations are used for another expansion procedure in which the independent variables t and z for the differentiation of the field amplitudes are stretched, or in other words, replaced by εt and εz where ε is another expansion parameter. But like μ, ε is also used merely for book-keeping purposes, and so is not necessarily specified in any manner. By using this procedure for finding the nonlinear effects, the expressions for $\mathscr{L}_2$, $\mathscr{L}_3$ and $\mathscr{L}_4$ are evaluated and their averages over the time period of the wave field are determined as functions of the amplitudes of the LCP and RCP waves and their partial derivatives with respect to t and z. The action integral, in which the averaged sum Lagrangian is the integrand, is then given a variation according to Hamilton's principle to obtain the differential equations for the nonlinear evolution of the LCP and RCP wave fields. From these, the nonlinear PR effect and the increment to the wave number in the spatial evolution problem, and to the frequency shift in the temporal evolution problem are evaluated in Section 4. The differential equations for the variation of the field amplitudes in Section 3 are used in Section 5 for finding the nonlinearly modulated wave field solutions. Two modulation frequencies, depending on the field intensity, are obtained for both the LCP and RCP waves. It is shown that frequency shift and precessional frequency are the two parts of these modulation frequencies. In Section 6, modulation frequencies are estimated numerically for the radiation of high intensity laser beams. Some relevant comments about the formalism of the averaged Lagrangian are then given in Section 7; a brief discussion about the force densities in nonlinear laser plasma interaction is also given. Section 8 gives a summary of the present work, while Section 9 contains brief proposals for investigating more general problems of the type considered here.

2. Lagrangian of Motion in an Electromagnetic Field and a Preliminary Analysis

We assume that the plasma is stationary, cold, homogeneous, unmagnetized and free from collisional and gravitational effects, and that only motions of the electrons become relativistic due to a powerful incident electromagnetic wave. Ion motion is much below the relativistic limit and so neglected. The plasma is moreover assumed to be below a certain threshold power limit so that self-action effects (e.g. self-focusing, self-steepening etc.) are insignificant.

The Lagrangian for the motion of electrons in plasmas can therefore be written as (Landau and Lifshitz 1975)

$$\mathscr{L} = -m_0 N_0\left(1-\frac{\dot{R}^2}{c^2}\right)^{\frac{1}{2}} - eN_0\,\phi(\boldsymbol{R}) - \frac{eN_0}{c}(\dot{\boldsymbol{R}}\,.\boldsymbol{A}) + \frac{E^2-H^2}{8\pi}, \tag{1}$$

where

$$\boldsymbol{H} = \operatorname{curl}\boldsymbol{A}, \tag{2}$$

$$\boldsymbol{E} = -\operatorname{grad}\phi - \dot{\boldsymbol{A}}/c, \tag{3}$$

$$\operatorname{div}\boldsymbol{A} = \dot{\phi}/c, \tag{4}$$

and where m_0, $-e$ and $\boldsymbol{R}$ are the rest mass, charge and position vector of an electron respectively. Also, N_0 is the electron number density, ϕ and $\boldsymbol{A}$ are the scalar and vector potentials, $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic fields, and dots denote time derivatives.

We use the Lagrangian concept of fluid motion and so choose a particular fluid particle to describe the location of a fluid element and its state in terms of the time coordinate alone. The dependent variables are then regarded as functions of the initial location $\boldsymbol{r}$ of a fluid element and of time t. So if $\boldsymbol{R}$ is the position vector of the displaced fluid from the equilibrium state, then $|\boldsymbol{r}| \gg |\mathscr{E}|$ and

$$\boldsymbol{R}(t) = \boldsymbol{r}(t) + \mathscr{E}(\boldsymbol{r}, t). \tag{5}$$

The displacement $\mathscr{E}$ vanishes initially and at the origin so that $\mathscr{E}(\boldsymbol{r}, 0) = 0$ and $\mathscr{E}(0, t) = 0$. The macroscopic fluid velocity in terms of $\mathscr{E}$ is given by

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{R}, t) &= \mathrm{d}\boldsymbol{R}/\mathrm{d}t = \boldsymbol{v}(\boldsymbol{r}+\mathscr{E}, t) \\ &= \boldsymbol{v}(\boldsymbol{r}, t) + (\mathscr{E}\,.\,\nabla)\boldsymbol{v}(\boldsymbol{r}, t) + \tfrac{1}{2}(\mathscr{E}\,.\,\nabla)^2\boldsymbol{v}(\boldsymbol{r}, t) + \ldots. \end{aligned} \tag{6}$$

Since the displacement amplitude is small, neglecting higher order terms gives

$$\boldsymbol{v}(\boldsymbol{R}, t) = \boldsymbol{v}(\boldsymbol{r}, t). \tag{7}$$

To the lowest order, $\mathscr{E}$ is regarded as given at a point, but to higher orders it is left undefined through the relation

$$\boldsymbol{v}(\boldsymbol{r}, t) = \partial\mathscr{E}(\boldsymbol{r}, t)/\partial t. \tag{8}$$

Assuming that the powerful first harmonic fundamental wave is transverse and propagates along the z-axis, we can write

$$E_z = 0, \qquad A_z = 0, \qquad \partial/\partial x = 0, \qquad \partial/\partial y = 0. \tag{9}$$

Then expanding the Lagrangian about the position vector $\boldsymbol{r}$, assuming that the higher harmonic waves are nonlinearly excited effects of higher order, and retaining terms correct up to the third harmonic wave solution, we obtain (see the Appendix)

$$\mathscr{L} = \mathscr{L}_2 + \mathscr{L}_3 + \mathscr{L}_4, \tag{10}$$

where

$$\begin{aligned} \mathscr{L}_2 = {} & \tfrac{1}{2}m_0 N_0(\dot{\mathscr{E}}_x^2 + \dot{\mathscr{E}}_y^2 + \dot{\mathscr{E}}_z^2) - eN_0\mathscr{E}_z\frac{\partial\phi}{\partial z} - \frac{eN_0}{c}(\dot{\mathscr{E}}_x A_x + \dot{\mathscr{E}}_y A_y) \\ & + \frac{1}{8\pi}\left(\frac{\partial\phi}{\partial z}\right)^2 + \frac{1}{8\pi c^2}(\dot{A}_x^2 + \dot{A}_y^2) - \frac{1}{8\pi}\left\{\left(\frac{\partial A_x}{\partial z}\right)^2 + \left(\frac{\partial A_y}{\partial z}\right)^2\right\}, \end{aligned} \tag{11}$$

$$\mathscr{L}_3 = -\frac{eN_0}{c}\mathscr{E}_z\left(\dot{\mathscr{E}}_x\frac{\partial A_x}{\partial z} + \dot{\mathscr{E}}_y\frac{\partial A_y}{\partial z}\right), \tag{12}$$

$$\mathscr{L}_4 = \frac{m_0 N_0}{8c^2}(\dot{\mathscr{E}}_x^4 + \dot{\mathscr{E}}_y^4 + 2\dot{\mathscr{E}}_x^2\dot{\mathscr{E}}_y^2), \tag{13}$$

with $\mathscr{L}_2$, $\mathscr{L}_3$, $\mathscr{L}_4$ containing the quadratic, cubic and bi-quadratic terms respectively.

(*a*) *Equations for the Motion of Electrons*

For the Lagrangian density $\mathscr{L}$ as a function of the field variables $\mathscr{E}$, ϕ and A, the action integral S is given by

$$S = \iiint_\tau \int_t \mathscr{L}(\mathscr{E}, \phi, A; \dot{\mathscr{E}}, \phi', \dot{A}, A'; \ddot{\mathscr{E}}, \phi'', \ddot{A}, A'', ...)\, \mathrm{d}\boldsymbol{r}\, \mathrm{d}t, \tag{14}$$

where d$\boldsymbol{r}$ is the volume element in the three-space, dt is the time differential, τ is the well-defined total volume and a prime denotes a derivative with respect to z; the limits of integration are taken from one given configuration τ_1 at time t_1 to another configuration τ_2 at time t_2.

Hamilton's principle of least action gives

$$\partial S = 0, \tag{15}$$

and so for the variation of $\mathscr{E}$, ϕ and A the following Lagrangian equations are obtained:

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \dot{\mathscr{E}}}\right) = \frac{\partial \mathscr{L}}{\partial \mathscr{E}}, \tag{16}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial \mathscr{L}}{\partial \phi'}\right) = \frac{\partial^2}{\partial z^2}\left(\frac{\partial \mathscr{L}}{\partial \phi''}\right), \tag{17}$$

$$\frac{\partial \mathscr{L}}{\partial A} = \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \dot{A}}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \mathscr{L}}{\partial A'}\right). \tag{18}$$

For $\mathscr{L} = \mathscr{L}_2 + \mathscr{L}_3$ these equations give

$$m_0 \ddot{\mathscr{E}}_x - \frac{e}{c}\dot{A}_x = \frac{e}{c}\frac{\partial}{\partial t}(\mathscr{E}_z A_x'), \tag{19}$$

$$m_0 \ddot{\mathscr{E}}_y - \frac{e}{c}\dot{A}_y = \frac{e}{c}\frac{\partial}{\partial t}(\mathscr{E}_z A_y'), \tag{20}$$

$$m_0 \ddot{\mathscr{E}}_z = -e\phi' - e\mathscr{E}_z \phi'' - \frac{e}{c}(\dot{\mathscr{E}}_x A_x' + \dot{\mathscr{E}}_y A_y'), \tag{21}$$

$$\phi'' = -2\pi e N_0 \frac{\partial^2}{\partial z^2}(\mathscr{E}_z^2) + 4\pi e N_0 \mathscr{E}_z', \tag{22}$$

$$A_x'' - \frac{1}{c^2}\ddot{A}_x - \frac{4\pi e N_0}{c}\dot{\mathscr{E}}_x = -\frac{4\pi e N_0}{c}\frac{\partial}{\partial z}(\mathscr{E}_z \dot{\mathscr{E}}_x), \tag{23}$$

$$A_y'' - \frac{1}{c^2}\ddot{A}_y - \frac{4\pi e N_0}{c}\dot{\mathscr{E}}_y = -\frac{4\pi e N_0}{c}\frac{\partial}{\partial z}(\mathscr{E}_z \dot{\mathscr{E}}_y). \tag{24}$$

For evaluation of the nonlinearly generated PR of an elliptically polarized first harmonic transverse wave we isolate the LCP and RCP waves with the help of rotating (complex) coordinates because their nonlinearly correct dispersion rates are different.

This procedure gives rise to the nonlinearly induced effect of birefringence, and the rate of PR is proportional to this difference. So putting

$$A_x \pm \mathrm{i} A_y = A_\pm, \qquad \mathscr{E}_x \pm \mathrm{i} \mathscr{E}_y = \mathscr{E}_\pm \tag{25}$$

in relations (19)–(24) and (11)–(13) we obtain

$$A''_\pm - \frac{1}{c^2} \ddot{A}_\pm - \frac{4\pi e N_0}{c} \dot{\mathscr{E}}_\pm = -\frac{4\pi e N_0}{c} \frac{\partial}{\partial z} (\mathscr{E}_z \dot{\mathscr{E}}_\pm), \tag{26}$$

$$m_0 \ddot{\mathscr{E}}_\pm - \frac{e}{c} \dot{A}_\pm = \frac{e}{c} \frac{\partial}{\partial t} (\mathscr{E}_z \dot{A}_\pm), \tag{27}$$

$$m_0 \ddot{\mathscr{E}}_z = -e\phi' - e\mathscr{E}_z \phi'' - \frac{e}{2c} (\dot{\mathscr{E}}_+ A'_- + \dot{\mathscr{E}}_- A'_+), \tag{28}$$

$$\mathscr{L}_2 = \tfrac{1}{2} m_0 N_0 \dot{\mathscr{E}}_+ \dot{\mathscr{E}}_- - \frac{e N_0}{2c} (\dot{\mathscr{E}}_+ A_- + \dot{\mathscr{E}}_- A_+) + \frac{1}{8\pi c^2} \dot{A}_+ \dot{A}_-$$

$$- \frac{1}{8\pi} A'_+ A'_- + \tfrac{1}{2} m_0 N_0 \dot{\mathscr{E}}_z^2 - e N_0 \mathscr{E}_z \phi' + \frac{\phi'^2}{8\pi}, \tag{29}$$

$$\mathscr{L}_3 = -\frac{e N_0}{2c} (\dot{\mathscr{E}}_+ A'_- + \dot{\mathscr{E}}_- A'_+) \mathscr{E}_z, \tag{30}$$

$$\mathscr{L}_4 = \frac{m_0 N_0}{8c^2} \dot{\mathscr{E}}_+^2 \dot{\mathscr{E}}_-^2 . \tag{31}$$

(b) Perturbation Expansion of Field Variables and First and Second Order Equations

For a field variable P the perturbation expansion in series is

$$P = \mu P^{(1)} + \mu^2 P^{(2)} + \mu^3 P^{(3)} + \ldots, \tag{32}$$

where μ is an unspecified expansion parameter retained exclusively for book-keeping purposes. This series expansion gives rise to the following field equations: (i) for the linearized approximation

$$\ddot{\mathscr{E}}_\pm^{(1)} = \frac{e}{m_0 c} \dot{A}_\pm^{(1)}, \tag{33}$$

$$\ddot{\mathscr{E}}_z^{(1)} = -\frac{e}{m_0} \phi'^{(1)}, \tag{34}$$

$$\phi''^{(1)} = 4\pi e N_0 \mathscr{E}_z'^{(1)}, \tag{35}$$

$$A''^{(1)}_\pm - \frac{1}{c_2} \ddot{A}_\pm^{(1)} = \frac{4\pi e N_0}{c} \dot{\mathscr{E}}_\pm^{(1)}, \tag{36}$$

and (ii) for the nonlinearly excited second harmonic fields

$$m_0 \ddot{\mathscr{E}}_{\pm}^{(2)} - \frac{e}{c}\dot{A}_{\pm}^{(2)} = \frac{e}{c}\frac{\partial}{\partial t}(\mathscr{E}_z^{(1)}\dot{A}_{\pm}^{(1)}), \tag{37}$$

$$m_0 \ddot{\mathscr{E}}_z^{(2)} = -e\phi'^{(2)} - e\mathscr{E}_z^{(1)}\phi''^{(1)} - \frac{e}{2c}(\dot{\mathscr{E}}_+^{(1)}A_-'^{(1)} + \dot{\mathscr{E}}_-^{(1)}A_+'^{(1)}), \tag{38}$$

$$\phi''^{(2)} - 4\pi e N_0 \mathscr{E}_z'^{(2)} = -2\pi e N_0 \frac{\partial^2}{\partial z^2}(\mathscr{E}_z^{(1)})^2, \tag{39}$$

$$A_{\pm}''^{(2)} - \frac{1}{c^2}\ddot{A}_{\pm}^{(2)} - \frac{4\pi e N_0}{c}\dot{\mathscr{E}}_{\pm}^{(2)} = -\frac{4\pi e N_0}{c}\frac{\partial}{\partial z}(\mathscr{E}_z^{(1)}\dot{\mathscr{E}}_{\pm}^{(1)}). \tag{40}$$

(*c*) *Solutions for First Order Field Variables and Corresponding Dispersion Relation*

For a purely transverse first harmonic wave solution, propagating parallel to the z-axis, we have

$$E_{\pm} = \tfrac{1}{2}\{(a \pm b)\exp(\mathrm{i}\psi) + (\bar{a} \mp \bar{b})\exp(-\mathrm{i}\psi)\}, \tag{41}$$

where $\psi = kz - \omega t$. Obviously, for real amplitudes, these relations give the elliptically polarized wave form $(E_x, E_y) = (a\cos\psi, b\sin\psi)$.

We can also write (41) more generally as

$$E_{\pm} = \tfrac{1}{2}\{\alpha_{\pm}\exp(\mathrm{i}\psi_{\pm}) + \bar{\alpha}_{\mp}\exp(-\mathrm{i}\psi_{\mp})\}, \tag{42}$$

where $\psi_{\pm} = k_{\pm} z - \omega_{\pm} t$ and $\alpha_{\pm}$ are the amplitudes of the LCP and RCP components of the wave. The quantities $a, \bar{a}, b, \bar{b}$ or $\alpha_{\pm}$ and $\bar{\alpha}_{\pm}$ are complex field amplitudes having the same dimensions as the electric field intensity vector. For the nonlinear evolution of the field variables, $\alpha_{\pm}$ and $\bar{\alpha}_{\pm}$ are assumed to be slowly varying functions of z or t and from their variations the nonlinear effects are determined; for solutions correct up to squares of the field amplitudes the phases are equal and so we write

$$\psi_+ = \psi_- = \psi = kz - \omega t, \qquad \alpha_{\pm} = a \pm b. \tag{43a, b}$$

Relations (9) ensure that

$$\phi^{(1)} = 0, \qquad \mathscr{E}_z^{(1)} = 0. \tag{44a, b}$$

Hence, equation (3) gives

$$cE = \dot{A}, \tag{45}$$

and so

$$E_{\pm}^{(1)} = \dot{A}_{\pm}/c. \tag{46}$$

Therefore we have

$$A_{\pm}^{(1)} = -(\mathrm{i}c/2\omega)\{\alpha_{\pm}\exp(\mathrm{i}\psi) - \bar{\alpha}_{\mp}\exp(-\mathrm{i}\psi)\}, \tag{47}$$

and equation (33) yields

$$2m_0\omega^2\mathscr{E}_{\pm}^{(1)} = \alpha_{\pm}\exp(\mathrm{i}\psi) + \bar{\alpha}_{\mp}\exp(-\mathrm{i}\psi). \tag{48}$$

Now, eliminating $\mathscr{E}_{\pm}^{(1)}$ from equations (33) and (36) we obtain

$$c^2 A_{\pm}''^{(1)} - \ddot{A}_{\pm}^{(1)} - \omega_p^2 A_{\pm}^{(1)} = 0, \tag{49}$$

where $\omega_p^2 = 4\pi N_0 e^2/m_0$, with ω_p the characteristic plasma frequency (or the so-called Langmuir frequency). By using equation (43) in (49), the familiar dispersion relation, independent of field intensity, is obtained:

$$k^2 c^2 = \omega^2 - \omega_p^2. \tag{50}$$

3. Differential Equations for Nonlinearly Evolved Wave Fields using $\langle \mathscr{L} \rangle$

To consider the nonlinear Lagrangian of motion for both the temporal and spatial evolution problems simultaneously, we first assume the field amplitudes $\alpha_{\pm}$ and $\bar{\alpha}_{\pm}$ of (42) to be slowly varying functions of both z and t and their derivatives to be of the order of the smallness parameter ε. This parameter can be regarded as independent of the other smallness parameter μ of relation (32). For the potentials $A_{\pm}^{(1)}$ we write

$$A_{\pm}^{(1)} = \mathscr{A}_{\pm}^{(1)} \exp(\mathrm{i}\psi) + \overline{\mathscr{A}}_{\pm}^{(1)} \exp(-\mathrm{i}\psi), \tag{51}$$

and, similar to the amplitudes $\alpha_{\pm}$, we regard $\mathscr{A}_{\pm}^{(1)}$ and $\overline{\mathscr{A}}_{\pm}^{(1)}$ as slowly varying functions of z and t; $\overline{\mathscr{A}}_{\pm}^{(1)}$ are the complex conjugates of $\mathscr{A}_{\pm}^{(1)}$. Then using (42) in (46) we obtain

$$\begin{aligned} A_{\pm}^{(1)} &= \frac{\mathrm{i}c}{2\omega}\left(-\alpha_{\pm} - \frac{\varepsilon}{\omega}\dot{\alpha}_{\pm} + \frac{\varepsilon^2}{\omega^2}\ddot{\alpha}_{\pm}\right)\exp(\mathrm{i}\psi) \\ &\quad + \frac{\mathrm{i}c}{2\omega}\left(\bar{\alpha}_{\mp} - \frac{\varepsilon}{\omega}\dot{\bar{\alpha}}_{\mp} - \frac{\varepsilon^2}{\omega^2}\ddot{\bar{\alpha}}_{\mp}\right)\exp(-\mathrm{i}\psi). \end{aligned} \tag{52}$$

Putting this expression for $A_{\pm}^{(1)}$ into (49) and keeping terms up to ε^2 we find that

$$\dot{\alpha}_{\pm} + v_g \alpha'_{\pm} = O(\varepsilon), \tag{53}$$

where v_g is the group velocity $\partial\omega/\partial k$; for equation (50)

$$v_g = kc^2/\omega. \tag{54}$$

Now, using (33) and (44) in the second order equations (38) and (39) we obtain

$$\ddot{\mathscr{E}}_z^{(2)} + \omega_p^2 \mathscr{E}_z^{(2)} = -\tfrac{1}{2}\left(\frac{e}{m_0 c}\right)^2 \frac{\partial}{\partial z}(A_+^{(1)} A_-^{(1)}), \tag{55}$$

$$\phi^{(2)} = 4\pi e N_0 \int \mathscr{E}_z^{(2)}\, \mathrm{d}\psi. \tag{56}$$

Therefore, with the help of (52), equation (55) yields

$$\mathscr{E}_z^{(2)} = -\frac{\mathrm{i}ke^2\{\alpha_+\alpha_- \exp(2\mathrm{i}\psi) - \bar{\alpha}_+\bar{\alpha}_- \exp(-2\mathrm{i}\psi)\}}{4m_0^2\omega^2(4\omega^2 - \omega_p^2)}. \tag{57}$$

So, from (56) and (57) we get

$$\phi^{(2)} = -\frac{e^2\omega_p^2}{8m_0^2\omega^2(4\omega^2-\omega_p^2)}\{\alpha_+\alpha_-\exp(2i\psi)+\bar{\alpha}_+\bar{\alpha}_-\exp(-2i\psi)\}. \tag{58}$$

As the second order transverse electric field components are zero, i.e. $E_\pm^{(2)} = 0$, equation (45) gives

$$A_\pm^{(2)} = 0. \tag{59}$$

Therefore, (37) yields

$$\mathscr{E}_\pm^{(2)} = 0. \tag{60}$$

Taking averages of the terms of (29), (30) and (31) over ψ between 0 and 2π, correct up to terms in μ^4, $\varepsilon^2\mu^2$ and μ^2, the non-vanishing terms become

$$\langle\dot{\mathscr{E}}_z^2\rangle = \mu^4\langle\dot{\mathscr{E}}_z^{(2)2}\rangle = \frac{\mu^4k^2e^4\alpha_+\bar{\alpha}_+\alpha_-\bar{\alpha}_-}{2m_0^4\omega^2(4\omega^2-\omega_p^2)^2}, \tag{61}$$

$$\langle\mathscr{E}_z\partial\phi/\partial z\rangle = \frac{\mu^4k^2e^3\omega_p^2\alpha_+\bar{\alpha}_+\alpha_-\bar{\alpha}_-}{8m_0^3\omega^4(4\omega^2-\omega_p^2)^2}, \tag{62}$$

$$\langle(\partial\phi/\partial z)^2\rangle = \frac{\mu^4k^2e^2\omega_p^2\alpha_+\bar{\alpha}_+\alpha_-\bar{\alpha}_-}{8m_0^2\omega^4(4\omega^2-\omega_p^2)^2}, \tag{63}$$

$$\langle\dot{\mathscr{E}}_+\dot{\mathscr{E}}_-\rangle = \frac{\mu^2e^2}{4m_0^2\omega^2}\left(\alpha_+\bar{\alpha}_+ +\alpha_-\bar{\alpha}_- + \frac{i\varepsilon}{\omega}(\alpha_+\dot{\bar{\alpha}}_+ -\bar{\alpha}_+\dot{\alpha}_+ -\bar{\alpha}_-\dot{\alpha}_- +\alpha_-\dot{\bar{\alpha}}_-)\right), \tag{64}$$

$$\langle\dot{\mathscr{E}}_+A_- +\dot{\mathscr{E}}_-A_+\rangle = \frac{\mu^2ec}{2m_0\omega^2}\left(\alpha_+\bar{\alpha}_+ +\alpha_-\bar{\alpha}_- + \frac{i\varepsilon}{\omega}(\alpha_+\dot{\bar{\alpha}}_+ -\bar{\alpha}_+\dot{\alpha}_+ -\bar{\alpha}_-\dot{\alpha}_- +\alpha_-\dot{\bar{\alpha}}_-)\right), \tag{65}$$

$$\langle(\dot{\mathscr{E}}_+A'_- +\dot{\mathscr{E}}_-A'_+)\mathscr{E}_z\rangle = \frac{\mu^4k^2e^3c\alpha_+\bar{\alpha}_+\alpha_-\bar{\alpha}_-}{4m_0^3\omega^4(4\omega^2-\omega_p^2)}, \tag{66}$$

$$\langle\dot{A}_+\dot{A}_-\rangle = \tfrac{1}{4}\mu^2c^2(\alpha_+\bar{\alpha}_+ +\alpha_-\bar{\alpha}_-), \tag{67}$$

$$\langle A'_+A'_-\rangle = \frac{\mu^2k^2c^2}{4\omega^2}\left(\alpha_+\bar{\alpha}_+ +\alpha_-\bar{\alpha}_- + \frac{i\varepsilon}{\omega}(\alpha_+\dot{\bar{\alpha}}_+ -\bar{\alpha}_+\dot{\alpha}_+ -\bar{\alpha}_-\dot{\alpha}_- +\alpha_-\dot{\bar{\alpha}}_-) + \frac{i\varepsilon}{k}(\alpha_+\bar{\alpha}'_+ -\bar{\alpha}_+\alpha'_+ -\bar{\alpha}_-\alpha'_- +\alpha_-\bar{\alpha}'_-)\right), \tag{68}$$

$$\langle\dot{\mathscr{E}}_+^2\dot{\mathscr{E}}_-^2\rangle = \frac{\mu^4e^4}{16m_0^4\omega^4}(\alpha_+^2\bar{\alpha}_+^2 +\alpha_-^2\bar{\alpha}_-^2 +4\alpha_+\bar{\alpha}_+\alpha_-\bar{\alpha}_-). \tag{69}$$

In relations (64)–(69), only those terms are retained which contain only one first order derivative of the amplitudes $\alpha_{\pm}$ and $\bar{\alpha}_{\pm}$ with respect to t or z, because the still higher order derivatives are higher order nonlinear terms which have been neglected from other terms. Using equations (61)–(69) in (29)–(31) we find that

$$\begin{aligned}\langle \mathscr{L} \rangle &= \langle \mathscr{L}_2 + \mathscr{L}_3 + \mathscr{L}_4 + \ldots \rangle \\ &= -\frac{\mu^4 k^2 e^2 \omega_p^2 \alpha_+ \bar{\alpha}_+ \alpha_- \bar{\alpha}_-}{64\pi m_0^2 \omega^4 (4\omega^2 - \omega_p^2)} \\ &\quad + \frac{\mu^4 \omega_p^2 e^2}{512\pi m_0^2 c^2 \omega^4}(\alpha_+^2 \bar{\alpha}_+^2 + \alpha_-^2 \bar{\alpha}_-^2 + 4\alpha_+ \bar{\alpha}_+ \alpha_- \bar{\alpha}_-) \\ &\quad - \frac{\mathrm{i}\mu^2 \varepsilon k c^2}{32\pi\omega^2}(\alpha_+ \bar{\alpha}'_+ - \bar{\alpha}_+ \alpha'_+ - \bar{\alpha}_- \alpha'_- + \alpha_- \bar{\alpha}'_-) \\ &\quad - \frac{\mathrm{i}\mu^2 \varepsilon}{32\pi\omega}(\alpha_+ \dot{\bar{\alpha}}_+ - \bar{\alpha}_+ \dot{\alpha}_+ - \bar{\alpha}_- \dot{\alpha}_- + \alpha_- \dot{\bar{\alpha}}_-). \end{aligned} \tag{70}$$

The action integral for the averaged Lagrangian $\langle \mathscr{L} \rangle$ is given a variation with respect to the amplitude $\bar{\alpha}_+$:

$$\partial S = \iiint_\tau \int_t \{\langle \mathscr{L}(\bar{\alpha}_+ + \partial\bar{\alpha}_+, \bar{\alpha}'_+ + \partial\bar{\alpha}'_+, \dot{\bar{\alpha}}_+ + \partial\dot{\bar{\alpha}}_+)\rangle - \langle \mathscr{L}(\bar{\alpha}_+, \bar{\alpha}'_+, \dot{\bar{\alpha}}_+)\rangle\}\, \mathrm{d}\boldsymbol{r}\, \mathrm{d}t = 0, \tag{71}$$

or similarly

$$\partial S = 0 = \iiint_\tau \int_t \left\{\frac{\partial\langle \mathscr{L}\rangle}{\partial\bar{\alpha}_+}\partial\bar{\alpha}_+ + \partial z\left(\frac{\partial\langle \mathscr{L}\rangle}{\partial\bar{\alpha}'_+}\partial\bar{\alpha}_+\right) - \frac{\partial^2\langle \mathscr{L}\rangle}{\partial z\,\partial\bar{\alpha}'_+}\partial\bar{\alpha}_+ + \frac{\partial}{\partial t}\left(\frac{\partial\langle \mathscr{L}\rangle}{\partial\dot{\bar{\alpha}}_+}\partial\bar{\alpha}_+\right) - \frac{\partial^2\langle \mathscr{L}\rangle}{\partial t\,\partial\dot{\bar{\alpha}}_+}\partial\bar{\alpha}_+\right\} \mathrm{d}\boldsymbol{r}\, \mathrm{d}t. \tag{72}$$

The Lagrangian equation thus becomes

$$\frac{\partial\langle \mathscr{L}\rangle}{\partial\bar{\alpha}_+} = \frac{\partial^2\langle \mathscr{L}\rangle}{\partial z\,\partial\bar{\alpha}'_+} + \frac{\partial^2\langle \mathscr{L}\rangle}{\partial t\,\partial\dot{\bar{\alpha}}_+}. \tag{73}$$

Similarly, for the variation of α_+, $\bar{\alpha}_-$, α_- we obtain

$$\frac{\partial\langle \mathscr{L}\rangle}{\partial\alpha_+} = \frac{\partial^2\langle \mathscr{L}\rangle}{\partial z\,\partial\alpha'_+} + \frac{\partial^2\langle \mathscr{L}\rangle}{\partial t\,\partial\dot{\alpha}_+}, \tag{74}$$

$$\frac{\partial\langle \mathscr{L}\rangle}{\partial\bar{\alpha}_-} = \frac{\partial^2\langle \mathscr{L}\rangle}{\partial z\,\partial\bar{\alpha}'_-} + \frac{\partial^2\langle \mathscr{L}\rangle}{\partial t\,\partial\dot{\bar{\alpha}}_-}, \tag{75}$$

$$\frac{\partial\langle \mathscr{L}\rangle}{\partial\alpha_-} = \frac{\partial^2\langle \mathscr{L}\rangle}{\partial z\,\partial\alpha'_-} + \frac{\partial^2\langle \mathscr{L}\rangle}{\partial t\,\partial\dot{\alpha}_-}. \tag{76}$$

Using (70) in equations (73)–(76) we obtain the following relations:

$$\frac{\mathrm{i}\,n^2}{k}\alpha'_+ + \frac{\mathrm{i}}{\omega}\dot{\alpha}_+ = \frac{e^2\alpha_+ X}{4m_0^2c^2\omega^2}\left(\frac{n^2}{4-X}\alpha_-\bar{\alpha}_- - \tfrac{1}{4}(\alpha_+\bar{\alpha}_+ + 2\alpha_-\bar{\alpha}_-)\right), \tag{77}$$

$$\frac{\mathrm{i}\,n^2}{k}\alpha'_- + \frac{\mathrm{i}}{\omega}\dot{\alpha}_- = \frac{e^2\alpha_- X}{4m_0^2c^2\omega^2}\left(\frac{n^2}{4-X}\alpha_+\bar{\alpha}_+ - \tfrac{1}{4}(\alpha_-\bar{\alpha}_- + 2\alpha_+\bar{\alpha}_+)\right), \tag{78}$$

where we have put $\varepsilon = 1$, $\mu = 1$, $n = kc/\omega$, $X = \omega_p^2/\omega^2$ and $n^2 = 1-X$. Equations (77) and (78) are the differential equations for the nonlinearly evolved wave field. By substituting the variations of α_+ and α_- with respect to time and space into (77) and (78), the nonlinear dispersion relation of the elliptically polarized electromagnetic wave can be derived.

4. Precessional Rotation and Shifts of Wave Parameters

It is known that the frequency or wave number of an electromagnetic wave is changed due to nonlinear interactions in plasmas (Montgomery and Tidman 1964; Sluijter and Montgomery 1965; Tidman and Stainer 1965; Boyd 1967; Das 1968, 1971; Chandra 1974, 1979). For an elliptically polarized wave Arons and Max (1974) noticed that the nonlinear effects give rise to precession of the polarization ellipse without effecting the ellipticity. Katz *et al.* (1975) and Lie and Wonnacott (1976) verified this result in the long wavelength limit for an elliptically polarized wave passing through a cold plasma, using the Lagrangian and Hamiltonian formalisms of classical mechanics. Several groups have developed the theory of precessional rotation and showed that it becomes significant in laser–plasma interactions (see e.g. Chakraborty 1977; Chakraborty and Chandra 1978; Khan and Chakraborty 1979; Bhattacharyya and Chakraborty 1979, 1982; Chandra 1980; Chakraborty *et al.* 1980, 1981, 1982; Bhattacharyya 1981, 1983). In this section we derive the expressions for wave number shift or frequency shift and the complementary effect of precessional rotation of an elliptically polarized wave from the nonlinearly evolved birefringence effect (Paul and Chakraborty 1983; Chakraborty and Paul 1983) in cold unmagnetized plasmas.

(*a*) *Wave Number Shift and Precessional Rotation*

In spatial evolution problems, only the derivatives of $\alpha_\pm$ with respect to z are necessary; these are independent of the time coordinate t. Therefore, in these problems we assume $\alpha_\pm = \alpha_\pm^0 \exp(\mathrm{i}\,\delta k_\pm z)$ in equations (77) and (78) and $\alpha_\pm^0$ to be real quantities. The expressions for the wave number shift and precessional rotation derived can be represented as

$$\frac{\partial k_+ + \partial k_-}{2k} = \frac{e^2X}{4n^2m_0^2c^2\omega^2}\left(\frac{3}{4} - \frac{1-X}{4-X}\right)(a^2+b^2), \tag{79}$$

$$\frac{\partial k_- - \partial k_+}{2k} = \frac{e^2X}{2n^2m_0^2c^2\omega^2}\left(\frac{1}{4} - \frac{1-X}{4-X}\right)ab, \tag{80}$$

where we have replaced $\alpha_\pm$ by $a \pm b$.

The precessional angle ρ_S and the precessional number ρ_N are given by

$$\rho_S = \tfrac{1}{2}z(\partial k_- - \partial k_+)\ \text{rad}, \qquad \rho_N = \tfrac{1}{2}(\partial k_- - \partial k_+). \tag{81a, b}$$

(b) Frequency Shift and Precessional Rotation

In temporal evolution problems the derivatives of $\alpha_\pm$ with respect to time t are considered and $\alpha_\pm$ are assumed to be independent of the space coordinate z. By assuming that $\alpha_\pm = \alpha_\pm^0 \exp(\mathrm{i}\,\partial\omega_\pm t)$, where $\alpha_\pm^0$ are real quantities, the expressions for the frequency shift and precessional rotation of the elliptically polarized wave are

$$\frac{\partial\omega_- + \partial\omega_+}{2\omega} = \frac{e^2 X}{4m_0^2 c^2 \omega^2}\left(\frac{3}{4} - \frac{1-X}{4-X}\right)(a^2+b^2), \tag{82}$$

$$\frac{\partial\omega_- - \partial\omega_+}{2\omega} = \frac{e^2 X}{2m_0^2 c^2 \omega^2}\left(\frac{1}{4} - \frac{1-X}{4-X}\right)ab. \tag{83}$$

The precessional angle ρ_t and the precessional frequency $\dot{\rho}$ are given by

$$\rho_t = \tfrac{1}{2}t(\partial\omega_- - \partial\omega_+)\ \text{rad}, \qquad \dot{\rho} = \tfrac{1}{2}(\partial\omega_- - \partial\omega_+). \tag{84a, b}$$

5. Self-induced Wave Modulation

Nonlinear interactions in plasmas also create modulation instabilities for acoustic and electromagnetic waves during their propagation through the medium; as a result, the frequencies of the waves are modulated. In this section, we obtain two nonlinearly evolved modulation frequencies for a stable elliptically polarized wave in cold plasmas and notice that these frequencies are different in nature compared with the frequencies obtained by previous authors (see e.g. Taniuty and Yajima 1969; Shimizu and Ichikawa 1972; Ichikawa *et al.* 1972; Kakutani and Sugimoto 1974; Watanabe 1977; Das and Sihi 1977, 1979; Sharma *et al.* 1978; Sihi 1980; Murtaza and Salahuddin 1982).

Equations (77) and (78) can be written as

$$D\alpha_+ = -\frac{\mathrm{i}\,e^2\alpha_+ X}{4m_0^2 c^2\omega}\left(\frac{n^2}{4-X}\alpha_-\bar{\alpha}_- - \tfrac{1}{4}(\alpha_+\bar{\alpha}_+ + 2\alpha_-\bar{\alpha}_-)\right), \tag{85}$$

$$D\alpha_- = -\frac{\mathrm{i}\,e^2\alpha_- X}{4m_0^2 c^2\omega}\left(\frac{n^2}{4-X}\alpha_+\bar{\alpha}_+ - \tfrac{1}{4}(\alpha_-\bar{\alpha}_- + 2\alpha_+\bar{\alpha}_+)\right), \tag{86}$$

where

$$D = \frac{\partial}{\partial t} + v_g\frac{\partial}{\partial z} = \frac{\partial}{\partial s}\quad (\text{say}). \tag{87}$$

For the evaluation of intensity-induced modulation in the wave we now write

$$\alpha_+ = \alpha_+^0(1+\gamma_+)\exp(-\mathrm{i}\lambda_+ s), \tag{88}$$

$$\alpha_- = \alpha_-^0(1+\gamma_-)\exp(-\mathrm{i}\lambda_- s). \tag{89}$$

Then putting

$$\lambda_+ = \frac{e^2X}{4m_0^2c^2\omega}(g\alpha_-^{02}-\tfrac{1}{4}\alpha_+^{02}), \qquad \lambda_- = \frac{e^2X}{4m_0^2c^2\omega}(g\alpha_+^{02}-\tfrac{1}{4}\alpha_-^{02}), \tag{90a, b}$$

where

$$g = n^2/(4-X)-\tfrac{1}{2}, \tag{91}$$

into equations (85) and (86) we obtain two nonlinear equations for the wave modulation $\gamma_\pm$. Ignoring squares and higher powers of $\gamma_\pm$, these reduce to the following mutually coupled linear differential equations:

$$D\gamma_+ - \mathrm{i}\lambda_+\gamma_+ = -\frac{\mathrm{i}e^2X}{4m_0^2c^2\omega}\{g(\gamma_+ + \gamma_- + \bar{\gamma}_-)\alpha_-^{02} - \tfrac{1}{4}(2\gamma_+ + \bar{\gamma}_+)\alpha_+^{02}\}, \tag{92}$$

$$D\gamma_- - \mathrm{i}\lambda_-\gamma_- = -\frac{\mathrm{i}e^2X}{4m_0^2c^2\omega}\{g(\gamma_+ + \bar{\gamma}_+ + \gamma_-)\alpha_+^{02} - \tfrac{1}{4}(2\gamma_- + \bar{\gamma}_-)\alpha_-^{02}\}. \tag{93}$$

Now we can assume $\gamma_\pm$ to be real and write $D\gamma_\pm = \mathrm{i}\sigma\gamma_\pm$; then the quadratic for the modulation frequency σ is

$$\left(\sigma + \frac{e^2\alpha_+^{02}X}{8m_0^2c^2\omega}\right)\left(\sigma + \frac{e^2\alpha_-^{02}X}{8m_0^2c^2\omega}\right) = \frac{e^4X^2g^2}{4m_0^4c^4\omega^2}(\alpha_+^0\alpha_-^0)^2. \tag{94}$$

The solution of this equation is

$$\sigma = -\frac{e^2X}{16m_0^2c^2\omega}(\alpha_+^{02}+\alpha_-^{02}) \pm \frac{e^2X}{2m_0^2c^2\omega}\{g^2\alpha_+^{02}\alpha_-^{02} + \tfrac{1}{64}(\alpha_+^{02}-\alpha_-^{02})^2\}^{\frac{1}{2}}, \tag{95}$$

where the two roots are real and different, and the RCP and LCP parts have two modulation frequencies. Thus, the strong elliptically polarized wave nonlinearly develops intensity dependent splitting due to two modulation frequencies in addition to the birefringence of the wave (shift in wave number or frequency, and precessional rotation etc.).

From equation (95) it is observed that:

(i) when $\alpha_+^0 = y\alpha_-^0$, y being any positive integer,

$$\sigma = -\frac{e^2(1+y^2)\alpha_-^{02}X}{16m_0^2c^2\omega} \pm \frac{e^2\alpha_-^{02}X}{2m_0^2c^2\omega}\{g^2y^2 + \tfrac{1}{64}(y^2-1)^2\}^{\frac{1}{2}}; \tag{96}$$

(ii) when $\alpha_+^0 \approx \alpha_-^0$,

$$\sigma = -\frac{e^2\alpha_+^{02}X}{2m_0^2c^2\omega}\left(\frac{3}{4}-\frac{1-X}{4-X}\right) \quad \text{and} \quad \frac{e^2\alpha_+^{02}X}{2m_0^2c^2\omega}\left(\frac{1}{4}-\frac{1-X}{4-X}\right); \tag{97}$$

(iii) when $\alpha_+^0 = 0$, or $\alpha_-^0 = 0$,

$$\sigma = 0 \quad \text{and} \quad -e^2\alpha_{+,-}^{02}X/8m_0^2c^2\omega. \tag{98}$$

Writing equation (95) in terms of the amplitudes a and b we find more interesting results. The modulation frequency thus obtained can be written as

$$\sigma = -f_1 - 2\Delta\omega \quad \text{and} \quad -(f_2 + \dot{\rho}_1), \tag{99}$$

where $\Delta\omega$ is the frequency shift and

$$\dot{\rho}_1 = \dot{\rho}(a^2+b^2)/ab\,,$$

$$f_1 = \frac{e^2b^2X}{2m_0^2c^2\omega}\left(\frac{X+2}{X-4}+\frac{a^2(X-4)}{4(X+2)(a^2-b^2)}\right),$$

$$f_2 = \frac{e^2b^2X}{4m_0^2c^2\omega}\left(1+\frac{a^2(X+4)}{2(X+2)(a^2-b^2)}\right).$$

Therefore, frequency shift and precessional rotation may be considered as the two parts of the modulation frequencies. It is observed that one of the modulation frequencies turns out to be zero when the frequency f_1 is equal to twice the frequency shift of the wave. For a circularly polarized wave, if we put $b = 0$, then $f_1 = f_2 = 0$. Therefore, the modulation frequencies have the values

$$\frac{e^2a^2X}{2m_0^2c^2\omega}\left(\frac{1-X}{4-X}-\frac{3}{4}\right),\qquad \frac{e^2a^2X}{2m_0^2c^2\omega}\left(\frac{1-X}{4-X}-\frac{1}{4}\right).$$

It is to be noted that when the wave frequency is very close to the plasma frequency, both the frequency shift and the PR become large and so the modulation frequencies become most significant. For a high frequency wave, if $\omega \gg \omega_p$ (i.e. $X \ll 1$), the PR almost vanishes even at higher intensities. So, in this case, the modulation frequencies are

$$\sigma = -(f_0+2\Delta\omega_0) \quad\text{and}\quad -f_0\,, \tag{100}$$

where

$$\Delta\omega_0 = \frac{e^2(a^2+b^2)X}{4m_0^2c^2\omega}\,,\qquad f_0 = \frac{e^2b^2X}{4m_0^2c^2\omega}(1+\eta^{-2})\,,$$

and η is the eccentricity of the polarization ellipse of the wave. Furthermore, we see that the modulation frequencies in equation (100) are always negative.

6. Numerical Estimation

We consider a Nd–glass laser beam with wavelength $1\cdot06\,\mu$m and frequency $1\cdot78\times10^{15}$ Hz, propagating through an over-dense plasma ($N_0 \approx 5\times10^{20}\,\text{cm}^{-3}$). Let the power of the laser beam be $10^{16}\,\text{W cm}^{-2}$. Then we obtain (i) a wave number shift of $1\cdot05\times10^2\,\text{cm}^{-1}$, (ii) a precessional rotation of 1°, for a distance of $5\cdot5\times10^{-3}$ cm, and (iii) modulation frequencies of $14\cdot5\times10^{12}$ Hz and $16\cdot7\times10^{12}$ Hz.

Let us now consider a CO_2 laser beam with wavelength $10\cdot6\,\mu$m, frequency $1\cdot78\times10^{14}$ Hz, passing through a dense plasma ($N_0 \approx 10^{18}\,\text{cm}^{-3}$), with power $10^{12}\,\text{W cm}^{-2}$. We see that (i) the wave number shift is $1\cdot2\,\text{cm}^{-1}$, (ii) the precessional rotation is $3\cdot4°$ for a distance of 1 cm, and (iii) the modulation frequencies are $1\cdot83\times10^{10}$ Hz and $3\cdot42\times10^8$ Hz.

7. Some Relevant Comments

(*a*) *Consequences of Wave-precession Effect*

The wave parameter shift effects were first discovered at about the time of the advent of masers and lasers. Being easier to evaluate, these effects have been much studied and are comparatively better known. But the effect of PR is of more recent origin and, because of complications due to rotation and other reasons, has been very much neglected until now. Some theoretical results have been published, but no published experimental results have been cited, although the consequences of such results should be important and definitely more interesting than those of wave parameter shift. The PR effect and all its consequences can be enhanced by some physically possible conditions of resonance or phase matching, or possibly by other artificial means. The most important consequences are induced magnetization and synchrotron radiation. If the effects of PR are experimentally detected in the near future, these would provide useful information on the influence of strong fields of certain frequency ranges on various states of matter, including plasmas.

The theory of the nonlinear interaction of electromagnetic waves with a plasma was initially developed in the microwave range and subsequently extended to the optical range of frequencies. The theory of nonlinear optics has been rapidly expanded into a new domain with the help of tunable dye lasers and ultrashort pulsed lasers, and the activity has been extended to a wide range of media including plasmas. Laser fields develop significant nonlinear characteristics, including the PR effect, in plasmas and other media.

The Faraday rotation (FR) effect, which is much used for plasma diagnostic purposes, is considerably modified for strong electromagnetic waves by the PR effect. Experimental results reported recently (Lax 1982) indicate the observation in the ISX-B tokamak of this modified FR effect at the sub-millimetre wavelengths used to measure the poloidal magnetic field produced by the driving current. Thus, an evaluation of the nonlinearly correct FR effect by the method explained in the present work would be useful.

Another promising consequence, pointed out by Chakraborty and Chandra (1978), is the nonlinearly induced rotation of the plane of polarization of a strong plane polarized electromagnetic wave by internal noise fields, even in the absence of any biasing magnetic field. Similar to FR, the PR of an electromagnetic wave can therefore be used for plasma diagnostic purposes in unmagnetized plasmas.

(*b*) *Discussion on Forces of Self-action Effects*

The possibility of the occurrence of nonlinearly induced self-action effects such as self-focusing, self-trapping, and self-precession (due to birefringence) in a material medium was suggested by Askar'yan (1962), Akhmanov *et al.* (1967), Chiao and Godine (1969) and others. In plasmas, self-focusing and self-phase modulation were initially reported in experiments on the laser breakdown of gases by Korobkin and Alcoc (1968). However, Johnson and Chu (1974) have pointed out that the observed self-focusing is actually due to a plasma gradient. Max *et al.* (1974) have considered the effect of self-focusing and self-phase modulation of light in a relativistic plasma. The self-focusing mechanism requires a threshold power depending mainly on the plasma density, pulse duration and laser frequency, below which it is insignificant. For the Nd–glass laser ($\lambda = 1\cdot06\ \mu$m, $\omega = 1\cdot78\times10^{15}$ Hz) in an over-dense plasma $N_0 = 10^{21}\ \mathrm{cm^{-3}}$), the threshold power, according to estimates by Kaw (1969) and Kaw and Dawson (1970), is about $10^{19}\ \mathrm{W\,cm^{-2}}$.

In general, all self-action effects occur simultaneously in laser induced plasmas, but which one dominates depends on the experimental conditions (e.g. long pulse or short pulse excitation) and the power of the laser beam. Below the threshold power the intrinsic nonlinear instabilities due to stimulated Raman scattering (SRS), driven by a strong laser beam, can be minimized. The threshold power for self-focusing is larger for picosecond pulses than for nanosecond pulses. Also, if the medium is isotropic, homogeneous and cold (i.e. a thermal velocity much less than c), then other self-action effects (e.g. SRS, self-focusing etc.) arising from the ponderomotive forces and thermal instabilities are negligible. Therefore, below a certain threshold power the manifestation of the self-precession mechanism is possible. By increasing the field intensity, the refractive index becomes non-uniform and, as a result, bends the rays towards a focus. This results in the destruction of the PR effect and therefore by this means it seems possible to deposit some amount of wave energy into the plasma.

The analysis of the expressions for force densities is important in the generation of nonlinear self-action effects of wave propagation in plasmas, especially for forces at the laser–plasma interaction. Many peculiarities are observed in the interaction of focused laser beams with materials, and the discussion on the role of the effective part of the forces is not yet over. For instance, the ith component of the electromagnetic force density $\boldsymbol{F}$ of a plasma is given by

$$F_i = \frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial g_i}{\partial t}, \tag{101}$$

where $\boldsymbol{g}$ is the electromagnetic momentum $-(\varepsilon\mu/4\pi c)(\boldsymbol{E} \times \boldsymbol{H})$, and

$$\tau_{ij} = -p_{ij} + (4\pi)^{-1}(\varepsilon E_i E_j + \mu H_i H_j) - (8\pi)^{-1}(\varepsilon E^2 + \mu H^2)\delta_{ij} \tag{102}$$

(δ_{ij} is the Kronecker delta) is the ijth component of the three-dimensional stress tensor $\boldsymbol{\tau}$. Hora (1969) has defined equation (101) as the equation of motion and regarded it as the ponderomotive force description of Lorentz. The alternative force description in the two-fluid model of the macroscopic theory, neglecting gravitation, used by Hora (1981, Sect. 6.1) is

$$\boldsymbol{F} = -\nabla p + \frac{1}{c}(\boldsymbol{J} \times \boldsymbol{H}) + \boldsymbol{J}.\nabla\left(\frac{1}{\omega^2}\frac{\partial \boldsymbol{E}}{\partial t}\right), \tag{103}$$

where ω is the wave frequency, $\boldsymbol{J}$ is the current density and p represents the total gas dynamical pressure in the plasma. These force equations have been extensively considered by Hora and his group to find the effects of laser–plasma interactions, even at the local regions where the beam intensity increases very much. The relativistic generalizations of some aspects of the force densities involved have also been provided by Jones *et al.* (1982). These investigations are relevant to the proper assessment of the role of wave precession in laser–plasma interactions and we briefly discuss some of these points below.

The problem of ponderomotive forces of strong electromagnetic waves, giving rise to self-action effects in plasmas, is of considerable interest in laser–plasma interactions, as well as in magnetic confinement devices, r.f. confinement and microwave heating. The reason for this interest is that the particles of a plasma are weakly interacting and so may be adequately described by the Vlasov equations, or some fluid approximation to it, which give a self-consistent model.

Interaction of laser radiation with a plasma gives rise to ion and molecular scattering of light, light mixing and self-focusing, in addition to precessional rotation of the radiation below a threshold limit. Study of plasma heating due to absorption of laser light by means of the usual process is useful in programs of controlled thermonuclear fusion. Macroscopic motion of a plasma, generated by the interaction of strong electromagnetic fields in the region of the focused laser beam, was found by Hora (1969) to be directed against the laser light, and the high values of ion energies measured could be theoretically verified under some assumptions. Jones *et al.* (1982) have considered the mechanisms for high intensity laser beam propagation in a plasma with the help of the three-dimensional development of the plasma density. The detailed workings of the counteracting mechanism and the limiting conditions of the relativistic self-focusing for the production of extremely high intensities have been given by these authors.

When very high intensity laser beams are focused to diameters of about 30 wavelengths at a target in a vacuum, a relativistic quivering velocity of the electrons of the target is produced. This gives rise to a relativistic self-focusing of the propagated laser beam within the target plasma immediately produced. Thus the beam rapidly shrinks down to diameters of about a wavelength. This focusing generates a very high laser intensity and consequently accelerates the charged particles by the ponderomotive forces to very high energies. According to Hora (1981, Sects 13.3 and 13.4) the beam shrinks within a propagation length equal to the initial beam diameter at electron densities close to the cut-off value, and even at intensities 10–100 times less than the relativistic threshold intensity.

A relativistic inertial current, in combination with the nonlinear force proportional to $\boldsymbol{v} \times \boldsymbol{H}$, is an important source of relativistic self-focusing in a homogeneous plasma. The nonlinear radial force due to the radial gradient of the laser field, $-(\partial/\partial r)(E^2+H^2/8\pi)$, which expels the plasma from the beam centre, is another type of self-focusing mechanism. While the relativistic self-focusing occurs almost instantaneously, in a time of the order of the optical oscillation, the ponderomotive self-focusing has a delay due to the motion of the plasma from the beam centre. Due to this delay of the ponderomotive self-focusing, the non-ponderomotive parts of the general expression for the total nonlinear force can be neglected when compared with the nonlinear self-focusing force including thermal gradients for these types of self-action effects.

(*c*) *Energy–Momentum Tensor and Lagrangian Formalism*

A general microscopic derivation of the energy–momentum tensor for non-dispersive electromagnetic waves in material media is difficult to obtain, but a macroscopic solution in arbitrarily moving media, using the method of virtual power, has been given by Penfield and Haus (1967). They showed that the ponderomotive force density acting on the medium is that expected from the Abraham form of the energy–momentum tensor, plus a part described macroscopically as electrostrictive and magnetostrictive effects. Calculations of the electrostrictive and magnetostrictive coefficients from first principles by using an analytical microscopic treatment are difficult to carry out. Only for collisionless plasmas can these coefficients be evaluated microscopically. The reason is that particles of such a plasma are weakly interacting and so can be described using the self-consistent Vlasov equations or even the equations for the fluid mixture approximation. The ponderomotive force in a cold plasma can be obtained from the Abraham tensor with the electrostrictive correction. Results are available for the total perturbed energy–momentum tensor connected with a one-dimensional wave packet, calculated from the ponderomotive force expression combined with the electromagnetic energy–momentum tensor both in the non-dispersive and cold plasma cases.

A ponderomotive force description based on the momentum–flux-density tensor by Hora (1969) showed that the resulting forces are positive for Maxwell's theory of electrodynamics and negative for the Lorentz theory. This difference is noticeable for oblique incidence and vanishes for perpendicular incidence of light on a plane inhomogeneous plasma. Moreover, at high electron densities in inhomogeneous and collision dominated plasmas a confining force was determined, which acted in the direction of the light, producing the radiation pressure due to the nonlinear ponderomotive force. From the expression for the total momentum transferred by the collisionless de-confining acceleration of the plasma (the acceleration of the plasma against the direction of laser propagation), the range of the electron density was determined for which this acceleration is found to dominate over the thermal absorption process.

Hamilton's principle of least action and the method or relativistic field theory led to a canonical procedure which, from a Hamiltonian viewpoint, is also very natural. Just as the canonical momentum for a particle in general differs from its physical momentum, so does the canonical energy–momentum tensor for a subsystem differ from its physical energy–momentum tensor. This distinction is different from that between the canonical and the symmetrized energy–momentum tensor for the system as a whole. The canonical and physical split-up procedures could be applied to either the canonical or symmetrized tensor, although according to Dewar (1977), the physical split-up of the canonical energy–momentum tensor is not useful.

Hamilton's principle is open to the objection that it requires one to postulate the form of the Lagrangian density. But, as pointed out by Dewar (1977), any macroscopic theory involves a number of postulates and Hamilton's principle may be deeper than many of these. However, the arbitrariness of the Lagrangian density disappears due to the fact that (i) the macroscopic Lagrangian is an average of the microscopic density, which is known; (ii) 'Lorentz invariance' is imposed by the macroscopic Lagrangian; and (iii) the Maxwell equations, which are actually the Euler–Lagrangian equations of the system, are known *a priori*. Thus a definite form of the total Lagrangian density without any ambiguity is obtained. Penfield and Haus (1967) have discussed the rules for forming Lagrangian densities, and remarked that the systematic book-keeping and standardized set of rules for applying the variational principle allow the derivation of the equation of motion without errors.

8. Summary

In the present paper, we have extended the method of the averaged Lagrangian to obtain the PR and shift of a wave parameter of an electromagnetic wave. Our results are identical to equations (24) and (25) of Arons and Max (1974), if their notation is followed. We have derived the results for the PR by analytically finding the nonlinearly developed birefringence in a cold, relativistic, unmagnetized, collisionless plasma. Two modulation frequencies have been obtained as a consequence of the nonlinear interaction of an electromagnetic wave in the plasma, and each of these is shown to be a sum of the frequency shift and the PR. Some aspects of the physics of strong waves in plasmas and other media have been briefly discussed to widen the scope of applying the formalism of the averaged Lagrangian.

9. Concluding Remarks

In this section we suggest further investigations involving the present work.

(i) In the cases where the effects of collisions, gravitation, kinetic temperature, static magnetic field etc. are included, the evaluation of nonlinear effects such as precessional rotation will be more fascinating and should give some important results in the study of nonlinear Faraday rotation, inverse Faraday effect etc. The method of the averaged Lagrangian should be further extended to investigate these types of problems. Expressions for modulation frequencies in these cases should show some interesting features of nonlinear interactions in plasmas.

(ii) An interesting modification of the formalism presented here is proposed in the application of the transformation to the space-independent frame, developed by Paul and Chakraborty (1983), for the evaluation of the nonlinearly induced wave precession and other complementary effects.

(iii) Due to several difficulties, the collisional effect between particles is avoided in the calculation using the averaged Lagrangian principle. Recently Das and Sihi (1977, 1979) introduced the collision term through the Rayleigh dissipation function (Goldstein 1970) in a study of modulation instabilities in nonlinear plasmas. Following this procedure, the collision term could perhaps be easily introduced in a generalization of the problem considered in the present paper.

(iv) According to Whitham (1974, p. 396), the variational approach for the averaged Lagrangian can be extended to the cases of slowly varying media, particularly if the variation is small in one period. The argument that the wave action is conserved in these cases is valid only for non-conducting fluids, but this idea seems to provide some useful hints for an extension to slowly varying plasma-like media as well.

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Appendix

The Lagrangian of equation (1) has four terms. In the first term, replacing $\dot{\boldsymbol{R}}$ by $\dot{\mathscr{E}}$, expanding binomially in powers of the component of $\dot{\mathscr{E}}$ and keeping up to fourth powers of these components we find that

$$-m_0 N_0 c^2(1-\dot{R}^2/c^2)^{\frac{1}{2}} = -m_0 N_0 c^2 + \tfrac{1}{2}m_0 N_0(\dot{\mathscr{E}}_x^2+\dot{\mathscr{E}}_y^2+\dot{\mathscr{E}}_z^2)$$

$$+(m_0 N_0/8c^2)(\dot{\mathscr{E}}_x^4+\dot{\mathscr{E}}_y^4+\dot{\mathscr{E}}_z^4+2\dot{\mathscr{E}}_x^2\dot{\mathscr{E}}_y^2+2\dot{\mathscr{E}}_y^2\dot{\mathscr{E}}_z^2+2\dot{\mathscr{E}}_z^2\dot{\mathscr{E}}_x^2)+\dots \quad \text{(A1)}$$

In the second term of (1), replacing $\boldsymbol{R}$ by $\boldsymbol{r}+\mathscr{E}$ and expanding in a Taylor series about $\boldsymbol{R} = \boldsymbol{r}$ we obtain

$$-eN_0\,\phi(\boldsymbol{r}+\mathscr{E}) = -eN_0\{\phi(\boldsymbol{r})+(\mathscr{E}\,.\,\nabla)\phi(\boldsymbol{r})+(1/2!)(\mathscr{E}\,.\,\nabla)^2\phi(\boldsymbol{r})$$

$$+(1/3!)(\mathscr{E}\,.\,\nabla)^3\phi(\boldsymbol{r})+\dots\}\,. \quad \text{(A2)}$$

In the third term we use $\dot{\boldsymbol{R}} = \dot{\mathscr{E}}$ and expanding $A(\boldsymbol{r}+\mathscr{E})$ in a Taylor series we get

$$(-eN_0/c)(\dot{\boldsymbol{R}}\,.\,A) = (-eN_0/c)[\dot{\mathscr{E}}\,.\,A(\boldsymbol{r}) + (\dot{\mathscr{E}}\,.\,\nabla)A(\boldsymbol{r}) + (1/2!)\{\dot{\mathscr{E}}\,.\,(\mathscr{E}\,.\,\nabla)^2A(\boldsymbol{r})\} + \ldots]. \tag{A3}$$

Now using equations (2) and (3) in the fourth term of (1) and expanding similarly by Taylor series we find

$$\frac{1}{8\pi}\left\{\left(\nabla\phi - \frac{\dot{A}}{c}\right)^2 - (\nabla \times A)^2\right\} = \frac{1}{8\pi}\left(\{\nabla\,\phi(\boldsymbol{r})\}^2 - \frac{2}{c}(\dot{A}\,.\,\nabla)\phi(\boldsymbol{r}) + \frac{\dot{A}^2}{c^2} - \{\nabla \times A(\boldsymbol{r})\}^2\right). \tag{A4}$$

Therefore, on expansion the Lagrangian $\mathscr{L}$ of (1) becomes

$$\mathscr{L} = \mathscr{L}_2 + \mathscr{L}_3 + \mathscr{L}_4, \tag{A5}$$

where $\mathscr{L}_2$ contains only the quadratic terms, and $\mathscr{L}_3$ and $\mathscr{L}_4$ have the cubic and bi-quadratic terms respectively:

$$\mathscr{L}_2 = N_0\left\{\tfrac{1}{2}m_0\dot{\mathscr{E}}^2 + e(\mathscr{E}\,.\,\nabla)\phi(\boldsymbol{r}) - \frac{e}{c}\{\dot{\mathscr{E}} - A(\boldsymbol{r})\} + \frac{1}{8\pi N_0}\left((\nabla\phi)^2 - \frac{2}{c}(\dot{A}\,.\,\nabla)\phi + \frac{\dot{A}^2}{c^2} + (\nabla \times A)^2\right)\right\}, \tag{A6}$$

$$\mathscr{L}_3 = -N_0\left(-\tfrac{1}{2}e(\mathscr{E}\,.\,\nabla)^2\phi(\boldsymbol{r}) - \frac{e}{c}\{\dot{\mathscr{E}}(\mathscr{E}\,.\,\nabla)A(\boldsymbol{r})\}\right), \tag{A7}$$

$$\mathscr{L}_4 = N_0\left(\frac{m_0}{8c^2}(\dot{\mathscr{E}}_x^4 + \dot{\mathscr{E}}_y^4 + \dot{\mathscr{E}}_z^4 + 2\dot{\mathscr{E}}_x^2\dot{\mathscr{E}}_y^2 + 2\dot{\mathscr{E}}_y^2\dot{\mathscr{E}}_z^2 + 2\dot{\mathscr{E}}_z^2\dot{\mathscr{E}}_x^2) - \tfrac{1}{6}(\mathscr{E}\,.\,\nabla)^3\phi(\boldsymbol{r}) - \frac{e}{2c}\{\dot{\mathscr{E}}\,.\,(\mathscr{E}\,.\,\nabla)^2A(\boldsymbol{r})\}\right). \tag{A8}$$

The wave is assumed to propagate along the z-axis, so we put $E_z = 0$, $A_z = 0$, $\partial/\partial x = 0$, $\partial/\partial y = 0$ and find that equations (A6)–(A8) reduce to

$$\mathscr{L}_2 = N_0\left[\tfrac{1}{2}m_0(\dot{\mathscr{E}}_x^2 + \dot{\mathscr{E}}_y^2 + \dot{\mathscr{E}}_z^2) + e\mathscr{E}_z\frac{\partial\phi}{\partial z} - \frac{e}{c}(\dot{\mathscr{E}}_x A_x + \dot{\mathscr{E}}_y A_y) + \frac{1}{8\pi N_0}\left\{\left(\frac{\partial\phi}{\partial z}\right)^2 + \frac{1}{c^2}(\dot{A}_x^2 + \dot{A}_y^2) - \left(\frac{\partial A_x}{\partial z}\right)^2 - \left(\frac{\partial A_y}{\partial z}\right)^2\right\}\right], \tag{A9}$$

$$\mathscr{L}_3 = N_0\left\{-\tfrac{1}{2}e\mathscr{E}_z^2\frac{\partial^2\phi}{\partial z^2} - \frac{e\mathscr{E}_z}{c}\left(\dot{\mathscr{E}}_x\frac{\partial A_x}{\partial z} + \dot{\mathscr{E}}_y\frac{\partial A_y}{\partial z}\right)\right\}, \tag{A10}$$

$$\mathscr{L}_4 = N_0\left\{\frac{m_0}{8c^2}(\dot{\mathscr{E}}_x^4 + \dot{\mathscr{E}}_y^4 + \dot{\mathscr{E}}_z^4 + 2\dot{\mathscr{E}}_x^2\dot{\mathscr{E}}_y^2 + 2\dot{\mathscr{E}}_y^2\dot{\mathscr{E}}_z^2 + 2\dot{\mathscr{E}}_z^2\dot{\mathscr{E}}_x^2) - \tfrac{1}{6}\mathscr{E}_z^3\frac{\partial^2\phi}{\partial z^2} - \frac{e\mathscr{E}_z^2}{2c}\left(\dot{\mathscr{E}}_x^2\frac{\partial^2 A_x}{\partial z^2} + \dot{\mathscr{E}}_y^2\frac{\partial^2 A_y}{\partial z^2}\right)\right\}. \tag{A11}$$

Assuming that the fundamental wave of the first harmonic is transverse and the nonlinearly excited longitudinal wave appears only in the second order approximation, relations (A9)–(A11) reduce to

$$\mathscr{L}_2 = \tfrac{1}{2}m_0 N_0(\dot{\mathscr{E}}_x^2 + \dot{\mathscr{E}}_y^2 + \dot{\mathscr{E}}_z^2) - eN_0 \mathscr{E}_z \frac{\partial \phi}{\partial z} - \frac{eN_0}{c}(\dot{\mathscr{E}}_x A_x + \dot{\mathscr{E}}_y A_y)$$

$$+ \frac{1}{8\pi}\left(\frac{\partial \phi}{\partial z}\right)^2 + \frac{1}{8\pi c^2}(\dot{A}_x^2 + \dot{A}_y^2) - \frac{1}{8\pi}\left\{\left(\frac{\partial A_x}{\partial z}\right)^2 + \left(\frac{\partial A_y}{\partial z}\right)^2\right\},$$

$$\mathscr{L}_3 = -\frac{eN_0 \mathscr{E}_z}{c}\left(\dot{\mathscr{E}}_x \frac{\partial A_x}{\partial z} + \dot{\mathscr{E}}_y \frac{\partial A_y}{\partial z}\right),$$

$$\mathscr{L}_4 = \frac{m_0 N_0}{8c^2}(\dot{\mathscr{E}}_x^4 + \dot{\mathscr{E}}_y^4 + 2\dot{\mathscr{E}}_x^2 \dot{\mathscr{E}}_y^2).$$

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