Dispersion in a Relativistic Quantum Electron Gas. II*
Thermal Distributions

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Abstract

The dispersion functions which appear when the response of a non-degenerate thermal electron gas is treated using a relativistic quantum theory are shown to be the same as the functions which appear in the non-quantum limit. These functions are evaluated at two speeds v_1 and v_2 which in the non-quantum limit reduce to the phase speed $\omega/|\boldsymbol{k}|$. It is shown that the effects of partial degeneracy may be included by expanding about the non-degenerate limit, and the expansion of the response functions is given explicitly. It is also pointed out that thermal corrections to the completely degenerate limit may be included using a standard technique and the lowest order corrections are given.

1. Introduction

In an accompanying paper (Hayes and Melrose 1984, present issue p. 615; hereafter denoted as HM) we showed that dispersion in a relativistic quantum electron gas may be described in terms of three functions $S^{(0)}(k)$, $S^{(1)}(k)$ and $S^{(2)}(k)$. These involve integrals over the occupation number $\bar{n}(\varepsilon)$ which is a sum of contributions from electrons and positrons. In HM we evaluated these integrals for completely degenerate electrons, and our results reproduced those of Jancovici (1962). In the present paper we evaluate these integrals for thermal distributions of electrons and positrons.

A strictly thermal distribution is a Fermi–Dirac distribution, with the chemical potentials for electrons and positrons being equal and opposite. In the limit of small temperatures, such a distribution reduces to the completely degenerate distribution. Our main emphasis in this paper is on the opposite (non-degenerate) limit of a Boltzmann gas, i.e. $\bar{n}(\varepsilon) \propto \exp(-\varepsilon/T)$.

In Section 2 we evaluate the $S^{(n)}(k)$ for a Boltzmann gas in terms of the relativistic plasma dispersion function $T(v, \rho)$ of Godfrey *et al.* (1975) (cf. also Melrose 1982). In evaluating the integrals an important intermediate step is the definition of a function $I(t, \rho)$ whose properties are discussed in Appendix 1. In Section 3 it is shown that with the known analytic properties of $T(v, \rho)$ our formulas imply dissipation identical to that implied by the formulas for the degenerate gas (cf. Section 4 of HM) for both LD and PC, and that the $S^{(n)}(k)$ are real in the dissipation-free region $|\boldsymbol{k}|^2 < \omega^2 < 4m^2 + |\boldsymbol{k}|^2$. In Section 4 the non-quantum (but relativistic) limit is

derived and the explicit expressions given by Melrose (1982) are reproduced. In Section 5 the first order corrections to the non-degenerate and degenerate limits are obtained.

2. Plasma Dispersion Functions for a Boltzmann Gas

The general distribution function for a thermal electron–positron gas is given by

$$\bar{n}(\varepsilon) = \sum_{\zeta} \left[\exp\{(\varepsilon - \mu^{\zeta})/T\} + 1\right]^{-1}, \tag{1}$$

where $\zeta = 1$ for electrons, $\zeta = -1$ for positrons and the chemical potentials μ^+ and μ^- are determined by the total number density and the ratio of the number densities of electrons and positrons. In evaluating the dispersion functions, we may treat the electrons and positrons separately and sum over their contributions.

We now consider only the electron distribution, and write the electron chemical potential μ^+ as μ. A Boltzmann electron gas corresponds to the non-degenerate limit $\mu \to -\infty$, and in this limit the expansion for the electron distribution function

$$n(\varepsilon) = \sum_{r=1}^{\infty} (-1)^{r-1} \exp\{r(\mu - \varepsilon)/T\} \tag{2}$$

converges rapidly. The leading term $r = 1$ gives the distribution for a Boltzmann gas:

$$n(\varepsilon) = A \exp(-\varepsilon/T), \tag{3}$$

where $A = \exp(\mu/T)$. Substituting (3) into the expressions for the proper number density $\bar{n}_0$ [given by equation (21) of HM, i.e. HM (21)] and actual number density $\bar{n}$, where $\bar{n} = 2 \int d^3p/(2\pi)^3 n(\varepsilon)$, we find

$$A = \frac{\pi^2 \rho \bar{n}_0}{m^3 K_1(\rho)} = \frac{\pi^2 \rho \bar{n}}{m^3 K_2(\rho)}, \tag{4}$$

where $\rho = m/T$ is the inverse temperature in units of $(5 \cdot 93 \times 10^9 \text{ K})^{-1}$.

We evaluate the functions $S^{(n)}(k)$ given by HM (23) for a Boltzmann gas using the same initial steps as in the case of a completely degenerate gas (cf. Appendix 2 of HM). That is, we partially integrate and then express the integral over ε as one over t, where $\varepsilon/m = (1 + t^2)/(1 - t^2)$. The resulting expressions contain integrals of the form

$$I(t, \rho) = \int_{-1}^{+1} dt' \, \frac{\exp\{-\rho(1 + t'^2)/(1 - t'^2)\}}{t' - t}. \tag{5}$$

In terms of these integrals, HM (23) with HM (24) reduce to

$$S^{(0)}(k) = -A \sum_{i=1}^{4} \frac{I(t_i, \rho)}{\rho}, \tag{6a}$$

$$S^{(1)}(k) = A \frac{\partial}{\partial \rho} \sum_{i=1}^{4} \eta_i \frac{I(t_i, \rho)}{\rho}, \tag{6b}$$

$$S^{(2)}(k) = -A \frac{\partial^2}{\partial \rho^2} \sum_{i=1}^{4} \frac{I(t_i, \rho)}{\rho}, \tag{6c}$$

with $\eta_i = 1$ for $i = 1, 2$ and $\eta_i = -1$ for $i = 3, 4$ and where the t_i are given by HM (A26).

The expressions (6) may be rewritten in terms of the relativistic plasma dispersion function

$$T(v_i, \rho) = \int_{-1}^{+1} dv' \, \frac{\exp\{-\rho(1-v'^2)^{-\frac{1}{2}}\}}{v' - v_i}, \tag{7}$$

introduced by Godfrey et al. (1975). If we relate t_i and v_i by

$$t_{i\pm} = \{1 \pm (1 - v_i^2)^{\frac{1}{2}}\}/v_i, \tag{8}$$

then, as shown in Appendix 1, we have

$$I(t_{i+}, \rho) + I(t_{i-}, \rho) = T(v_i, \rho), \tag{9a}$$

$$I(t_{i+}, \rho) - I(t_{i-}, \rho) = -\frac{1}{\gamma_i} \frac{\partial T(v_i, \rho)}{\partial \rho}, \tag{9b}$$

with $\gamma_i = (1 - v_i^2)^{-\frac{1}{2}}$ as usual.

We may relate the $t_{1\pm}$ and $t_{2\pm}$ to the t_1 to t_4 defined by HM (A26) by equating $(1 - v_i^2)^{\frac{1}{2}}$ to $(1 - t_i^2)/(1 + t_i^2)$. The relevant identifications are

$$t_1 = t_{1-}, \quad t_2 = t_{2-}, \quad t_3 = -t_{1+}, \quad t_4 = -t_{2+}. \tag{10}$$

However, by implication $(1 - v_i^2)^{\frac{1}{2}}$ $[= (1 - t_{i\pm}^2)/(1 + t_{i\pm}^2)]$ is positive in (8), whereas $(1 - t_i^2)/(1 + t_i^2)$ has the same sign as the resonant energy. We may correct for this by including the relevant sign in the factor γ_i in (9b). Let us define

$$\sigma_i = (-1)^{i+1}, \qquad \omega^2 < |\mathbf{k}|^2; \tag{11a}$$

$$= \omega/|\omega|, \qquad \omega^2 > 4m^2 + |\mathbf{k}|^2; \tag{11b}$$

in accord with HM (A26). In the intervening region $|\mathbf{k}|^2 < \omega^2 < 4m^2 + |\mathbf{k}|^2$, $(1 - v_1^2)^{\frac{1}{2}}$ and $(1 - v_2^2)^{\frac{1}{2}}$ are complex conjugates of each other, and we are free to choose $\sigma_i = 1$. With these identifications the $v_{1,2}$ are given by

$$v_{1,2} = \frac{\omega}{|\mathbf{k}|} \frac{\varepsilon_0 \pm |\mathbf{k}|^2/2\omega}{\varepsilon_0 \pm \frac{1}{2}\omega}, \tag{12}$$

with

$$\varepsilon_0 = \left(\frac{m^2}{1 - \omega^2/|\mathbf{k}|^2} + \tfrac{1}{4}|\mathbf{k}|^2 \right)^{\frac{1}{2}}. \tag{13}$$

Henceforth, the γ_i are positive by definition.

The functions (6) now become

$$S^{(0)}(k) = A \sum_{i=1}^{2} \frac{\sigma_i}{\rho\gamma_i} \frac{\partial T(v_i, \rho)}{\partial \rho}, \tag{14a}$$

$$S^{(1)}(k) = A \sum_{i=1}^{2} \frac{\partial}{\partial \rho} \left(\frac{T(v_i, \rho)}{\rho} \right), \tag{14b}$$

$$S^{(2)}(k) = A \sum_{i=1}^{2} \frac{\sigma_i}{\gamma_i} \frac{\partial^2}{\partial \rho^2} \left(\frac{1}{\rho} \frac{\partial T(v_i, \rho)}{\partial \rho} \right). \tag{14c}$$

These may be written in terms of $T(v_i, \rho)$ and $\partial T(v_i, \rho)/\partial v_i$ using equations (A8) and (A10) of Appendix 1:

$$S^{(0)}(k) = \frac{A}{\rho} \sum_{i=1}^{2} \frac{\sigma_i}{\gamma_i v_i} \left(\frac{1-v_i^2}{\rho} \frac{\partial T(v_i, \rho)}{\partial v_i} + 2K_1(\rho) \right), \tag{15a}$$

$$S^{(1)}(k) = \frac{A}{\rho} \sum_{i=1}^{2} \left\{ -\frac{1}{\rho} T(v_i, \rho) + \frac{1}{v_i} \left(\frac{1-v_i^2}{\rho} \frac{\partial T(v_i, \rho)}{\partial v_i} + 2K_1(\rho) \right) \right\}, \tag{15b}$$

$$S^{(2)}(k) = \frac{A}{\rho} \sum_{i=1}^{2} \frac{\sigma_i}{\gamma_i v_i} \left\{ \left(\frac{2}{\rho^2} + \gamma_i^2 \right) \left(\frac{1-v_i^2}{\rho} \frac{\partial T(v_i, \rho)}{\partial v_i} + 2K_1(\rho) \right) \right.$$
$$\left. - 2\gamma_i^2 v_i^2 K_1(\rho) - \frac{2}{\rho} v_i \gamma_i^2 \{ T(v_i, \rho) + 2v_i K_0(\rho) \} \right\}. \tag{15c}$$

When these are inserted in the plasma dispersion functions [cf. HM (20)]

$$\alpha^{\mathrm{L}}(k) = \frac{e^2 \bar{n}_0 \omega^2}{m|\boldsymbol{k}|^2} + \frac{e^2 m \omega^2}{2\pi |\boldsymbol{k}|^3} \{ \tfrac{1}{4}(\omega^2 - |\boldsymbol{k}|^2) S^{(0)}(k) - m\omega S^{(1)}(k) + m^2 S^{(2)}(k) \}, \tag{16a}$$

$$\alpha^{\mathrm{T}}(k) = -\frac{e^2 \bar{n}_0 (\omega^2 + |\boldsymbol{k}|^2)}{2m|\boldsymbol{k}|^2} - \frac{e^2 m (\omega^2 - |\boldsymbol{k}|^2)}{4\pi^2 |\boldsymbol{k}|^2} \{ (-\varepsilon_0^2 + \tfrac{1}{4}\omega^2 + \tfrac{1}{2}|\boldsymbol{k}|^2) S^{(0)}(k) $$
$$- m\omega S^{(1)}(k) + m^2 S^{(2)}(k) \}, \tag{16b}$$

they give expressions for the response of a relativistic quantum electron gas in terms of the functions $T(v, \rho)$ which are characteristic of dispersion in a relativistic non-quantum gas.

3. Dissipation and the Dissipation-free Region

As discussed in HM, Landau damping occurs for $\omega < |\boldsymbol{k}|$. There is no dissipation for $|\boldsymbol{k}|^2 < \omega^2 < 4m^2 + |\boldsymbol{k}|^2$, which we refer to as the dissipation-free region. In this region, ε_0 (cf. equation 13) is imaginary and hence v_1 and v_2 are complex. By implication the $S^{(n)}(k)$ should have imaginary parts (which describe dissipation) when v_1 and v_2 are real, and the $S^{(n)}(k)$ should be real when v_1 and v_2 are complex. In this section we show how these properties are implied by the known properties of $T(v, \rho)$.

Godfrey et al. (1975) pointed out that $T(v, \rho)$ satisfies

$$T(v^*, \rho) = T^*(v, \rho), \tag{17}$$

for complex v. Moreover, the definition (7) is for $\mathrm{Im}\, v > 0$, whereas we are actually concerned with its analytic continuation into the lower half of the v-plane. Godfrey et al. also gave the alternative form

$$T(v, \rho) = \exp(-\rho\gamma) \ln \left(\frac{1-v}{1+v} \right) + \int_{-1}^{+1} \mathrm{d}v' \, \frac{\exp(-\rho\gamma') - \exp(-\rho\gamma)}{v' - v}, \tag{18}$$

with $\gamma = (1-v^2)^{-\frac{1}{2}}$ and $\gamma' = (1-v'^2)^{-\frac{1}{2}}$. Hence for v infinitesimally below the real axis we have, for $|v| < 1$,

$$\text{Im } T(v,\rho) = \pi \exp(-\rho\gamma). \tag{19}$$

This result is consistent with the conventional procedure of giving ω an infinitesimal imaginary part ($v \to v + i0$ here) and using the Plemelj formula in (7).

Unlike the non-quantum case where $v = \omega/|\boldsymbol{k}|$ can be in the range $|v| > 1$ and be real, in our case v_1 and v_2 are either real with $|v_{1,2}| \leqslant 1$, or they are complex with $v_2 = v_1^*$. In the cases where $v_{1,2}$ are real the imaginary parts of $S^{(n)}(k)$ follow directly from (15) and (19). It is also instructive to derive these imaginary parts from the general formulas HM (27) by inserting the distribution (3) and carrying out the integrals. The contribution to LD from $i = 2$ involves a negative sign in (11a); recall that we artificially define γ_2 to be positive, and given by

$$\gamma_{1,2} = \varepsilon_0/m \pm \omega/2m, \tag{20}$$

and introduce σ_i to take account of the fact that $i = 4$ is actually the relevant resonance in this case.

In the dissipation-free region, $v_2 = v_1^*$ are off the real axis and the contribution (19) is not present. With σ_i chosen such that $\gamma_2 = \gamma_1^*$, the $S^{(n)}(k)$ are all real and equal to twice the real parts of their values when only the contribution from $i = 1$ is retained.

4. Non-quantum Limit and First Quantum Corrections

The non-quantum limit may be treated in a straightforward manner by inserting the distribution (3) into HM (23) and HM (37), and expressing the integrals in terms of $T(v,\rho)$. The same result may also be derived by starting from (15).

In taking the non-quantum limit we exclude the region $\omega^2 > 4m^2 + |\boldsymbol{k}|^2$ where the intrinsically quantum effect of pair creation occurs. In the region $\omega < |\boldsymbol{k}|$, the $v_{1,2}$ are real and it is straightforward to expand in powers of $\hbar$. The natural expansion parameter is $\omega/2m\gamma$, with $\gamma = (1 - \omega^2/|\boldsymbol{k}|^2)^{-\frac{1}{2}}$ and $v = \omega/|\boldsymbol{k}|$:

$$v_{1,2} = v\left\{\pm \frac{\omega}{2m\gamma}\frac{1}{\gamma^2 v^2} - \left(\frac{\omega}{2m\gamma}\right)^2 \frac{1}{\gamma^2 v^2} \mp \left(\frac{\omega}{2m\gamma}\right)^3 \frac{1-2v^2}{2\gamma^2 v^4} + ...\right\}. \tag{21}$$

The expansion of (15) then involves a Taylor series expansion using (21). After re-expressing $\partial^2 T(v,\rho)/\partial v^2$ in terms of $T(v,\rho)$ and its first derivative using (A12), we obtain

$$S^{(0)}(k) = \frac{2\omega A}{mv}\left(K_0(\rho) + \tfrac{1}{2}vT(v,\rho) + \frac{\omega^2}{48m^2 v^2 \gamma^4}\{\rho^2 v^3 \gamma^4 T(v,\rho) + 3\,\partial T(v,\rho)/\partial v\right.$$

$$\left. + 2\rho^2 v^2 \gamma^4 K_0(\rho) + 2\rho\gamma^2 K_1(\rho)\} + ...\right), \tag{22a}$$

$$S^{(1)}(k) = \frac{2A}{v}\left(\frac{2K_1(\rho)}{\rho} - \frac{1}{\rho^2}\{vT(v,\rho) - (1-v^2)\,\partial T(v,\rho)/\partial v\}\right.$$

$$\left. + \frac{\omega^2}{8m^2 v^2 \gamma^2}\{(1+v^2)\gamma^2 T(v,\rho) + v\,\partial T(v,\rho)/\partial v + 4v\gamma^2 K_0(\rho)\} + ...\right), \tag{22b}$$

$$S^{(2)}(k) = \frac{2\omega A}{mv}\left(K_2(\rho) + \frac{v}{2(1-v^2)}\{2vK_0(\rho) + T(v,\rho)\}\right.$$

$$+ \frac{\omega^2}{48m^2 v^2 \gamma^4}\{2v\gamma^4(3+v^2+\rho^2 v^2\gamma^2)T(v,\rho) + \gamma^2(3+4v^2)\,\partial T(v,\rho)/\partial v$$

$$+ 4\gamma^4(1+2v^2)K_0(\rho) + \gamma^6 v^2(1+v^2)\rho^2 K_0(\rho) + \tfrac{3}{2}\gamma^2\rho K_1(\rho)$$

$$\left. + \gamma^2(\rho^2 v^2\gamma^2 - 2)K_2(\rho) + \tfrac{1}{2}\gamma^2\rho K_3(\rho)\} + \dots\right). \tag{22c}$$

The leading terms in (22) have been written in a form which, when inserted in equations (16) reproduce the known non-quantum results (cf. Melrose 1982). The explicit quantum terms in (16) must also be omitted; this involves replacing the coefficients of $S^{(0)}(k)$ by zero and $m^2|\boldsymbol{k}|^2/(\omega^2 - |\boldsymbol{k}|^2)$ respectively.

The result (22) has been derived for $\omega^2 < |\boldsymbol{k}|^2$. For $\omega^2 > |\boldsymbol{k}|^2$, in the dissipation-free region, the parameter ε_0 becomes imaginary. Nevertheless, an expansion of the form (22) applies, and on repeating the derivation of (22) one obtains the same final result. As v^2 (>1) increases, the expansion parameter $\omega(v^2-1)^{\frac{1}{2}}/2m$ also increases, and the expansion ceases to be justified as the threshold for pair production is approached.

Note that there are no quantum corrections of first order in $\hbar$ to the real parts of $\alpha^{\mathrm{L}}(k)$ and $\alpha^{\mathrm{T}}(k)$, but there are first order corrections to the imaginary parts.

5. Nearly Non-degenerate and Nearly Degenerate Limits

Nearly Non-degenerate Limit

The nearly non-degenerate limit corresponds to the electron chemical potential μ large and negative, so we may use the expansion (2) for the distribution function, i.e.

$$n(\varepsilon) = \sum_{r=1}^{\infty} (-1)^{r-1} A^r \exp(-r\varepsilon/T), \tag{23}$$

where $A = \exp(\mu/T)$.

The lowest order term in this expansion reduces to a Boltzmann distribution and higher order terms involve integrals which are essentially the same as the ones we evaluated in Section 2. Substituting (23) into the expressions for the proper and actual number densities, and the plasma dispersion functions $S^{(n)}(k)$ given by HM (23), we find

$$(\bar{n}_0, \bar{n}) = \frac{m^3}{\pi^2} \sum_{r=1}^{\infty} (-1)^{r-1} \frac{\exp(r\mu/T)}{r\rho}(K_1(r\rho), K_2(r\rho)), \tag{24}$$

$$S^{(0)}(k) = \sum_{r=1}^{\infty} \frac{A^r}{r\rho} \sum_{i=1}^{2} \frac{\sigma_i}{\gamma_i v_i}\left(\frac{1-v_i^2}{r\rho}\frac{\partial T(v_i, r\rho)}{\partial v_i} + 2K_1(r\rho)\right), \tag{25a}$$

$$S^{(1)}(k) = \sum_{r=1}^{\infty} \frac{A^r}{r\rho} \sum_{i=1}^{2} \left\{-\frac{1}{r\rho}T(v_i, r\rho) + \frac{1}{v_i}\left(\frac{1-v_i^2}{r\rho}\frac{\partial T(v_i, r\rho)}{\partial v_i} + 2K_1(r\rho)\right)\right\}, \tag{25b}$$

$$S^{(2)}(k) = \sum_{r=1}^{\infty} \frac{A^r}{r\rho} \sum_{i=1}^{2} \frac{\sigma_i}{\gamma_i v_i} \Biggl\{ \left(\left(\frac{2}{r^2\rho^2} + \gamma_i^2 \right) \left(\frac{1-v_i^2}{r\rho} \frac{\partial T(v_i, r\rho)}{\partial v_i} + 2K_1(r\rho) \right) \right.$$

$$\left. -2\gamma_i^2 v_i^2 K_1(r\rho) - \frac{2}{r\rho} v_i \gamma_i^2 \{ T(v_i, r\rho) + 2v_i K_0(r\rho) \} \Biggr\}. \quad (25c)$$

Nearly Degenerate Limit

In obtaining the nearly degenerate limit we make use of a standard expansion (see e.g. Landau and Lifshitz 1959; §57), which enables one to approximate the integral of a differentiable function $F(\varepsilon)$ multiplied by the electron thermal distribution function $[\exp\{(\varepsilon - \mu)/T\} + 1]^{-1}$ by

$$\int_m^{\infty} d\varepsilon \, F(\varepsilon) [\exp\{(\varepsilon - \mu)/T\} + 1]^{-1} = \int_m^{\mu} d\varepsilon \, F(\varepsilon) + \tfrac{1}{6}\pi^2 T^2 F'(\mu) + \dots, \quad (26)$$

where a prime denotes a derivative with respect to ε. This expansion converges rapidly for $T \ll \mu$.

In the completely degenerate limit we have $T = 0$ and $\mu = \varepsilon_F$, so the lowest order term in (26) reproduces the completely degenerate expressions given by HM (41). Subsequent terms give thermal corrections to these expressions. To lowest order these thermal corrections are

$$\delta S^{(0)}(k) = \frac{\pi^2 T^2 4\varepsilon_F(\omega^2 - |k|^2)}{3mp_F}$$

$$\times \left(\frac{p_F + \tfrac{1}{2}|k|}{4\varepsilon_F^2 \omega^2 - (\omega^2 - |k|^2 - 2p_F|k|)^2} - \frac{p_F - \tfrac{1}{2}|k|}{4\varepsilon_F^2 \omega^2 - (\omega^2 - |k|^2 + 2p_F|k|)^2} \right),$$

$$(27a)$$

$$\delta S^{(1)}(k) = \frac{\pi^2 T^2}{6m^2} \Biggl\{ \ln \Lambda_{2F} + \frac{8\varepsilon_F}{p_F}$$

$$\times \left(\frac{(\varepsilon_F\omega + p_F|k|)(\varepsilon_F|k| + p_F\omega)}{4(\varepsilon_F\omega + p_F|k|)^2 - (\omega^2 - |k|^2)^2} - \frac{(\varepsilon_F\omega - p_F|k|)(\varepsilon_F|k| + p_F\omega)}{4(\varepsilon_F\omega - p_F|k|)^2 - (\omega^2 - |k|^2)^2} \right) \Biggr\},$$

$$(27b)$$

$$\delta S^{(2)}(k) = \frac{\pi^2 T^2}{6m^3} \Biggl\{ 2\varepsilon_F \ln \Lambda_{1F} + \frac{8\varepsilon_F^3(\omega^2 - |k|^2)}{mp_F}$$

$$\times \left(\frac{p_F + \tfrac{1}{2}|k|}{4\varepsilon_F^2 \omega^2 - (\omega^2 - |k|^2 - 2p_F|k|)^2} - \frac{p_F - \tfrac{1}{2}|k|}{4\varepsilon_F^2 \omega^2 - (\omega^2 - |k|^2 + 2p_F|k|)^2} \right) \Biggr\}.$$

$$(27c)$$

6. Discussion

A particularly notable feature of the results we derive in this paper is that dispersion in a non-degenerate or partially degenerate relativistic electron gas may be expressed in terms of functions which have already been introduced in connection

with the non-quantum limit. We choose to use the function $T(v, \rho)$ defined by Godfrey *et al.* (1975); the relation of this function to several others has been discussed by Melrose (1982). One would anticipate that known properties of $T(v, \rho)$ could be used to treat a variety of different limiting cases. However, we have encountered difficulties in treating two important limiting cases: the long-wavelength and the ultra-relativistic limits. We now comment on these two cases.

In the non-quantum case the long-wavelength limit may be treated by expanding $T(v, \rho)$ in a power series for v^{-1}. The relevant expansion is (Godfrey *et al.* 1975)

$$T(v, \rho) = \sum_{s=1}^{\infty} a_s(\rho)/v^{2s+1}, \tag{28}$$

with

$$a_0(\rho) = -2\mathrm{Ki}_2(\rho), \qquad a_1(\rho) = -\tfrac{1}{3}\{K_0(\rho) - \rho K_1(\rho)\} + \tfrac{1}{6}(3 - \rho^2)a_0(\rho),$$

$$a_2(\rho) = -\tfrac{1}{10}\rho K_1(\rho) - \tfrac{3}{10}a_0(\rho) + \tfrac{1}{20}(27 - \rho^2)a_1(\rho),$$

$$a_n(\rho) = [\{3(2n-1)^2 - \rho^2\}a_{n-1}(\rho) - 6(n-1)(2n-3)a_{n-2}(\rho)$$

$$+ (2n-3)(2n-5)a_{n-3}(\rho)]/2n(2n+1), \quad n > 2. \tag{29}$$

Although this expansion requires $v = \omega/|\boldsymbol{k}| > 1$, the results obtained using (28) seem to be valid in larger parameter regions than expected.

Now consider the quantum case. In the dissipation-free region, v_1 and v_2 are complex conjugates of each other. However, as the threshold for pair production ($\omega^2 = 4m^2 + |\boldsymbol{k}|^2$) is approached, v_1 and v_2 approach $|\boldsymbol{k}|/\omega$, so the condition $|v_{1,2}| > 1$ in (28) does not necessarily make this a small $|\boldsymbol{k}|$ expansion. In the LD and PC regimes we have $|v_{1,2}| < 1$, so use of the expansion (28) is not justified. Godfrey *et al.* also gave an expansion of $T(v, \rho)$ for $|v_{1,2}| < 1$, but this condition is satisfied for all values of $|\boldsymbol{k}|$, so this does not correspond to a small $|\boldsymbol{k}|$ expansion either. Hence, there is no obvious way to use the known properties of $T(v, \rho)$ to obtain the long-wavelength expansion.

However, we can use the small $|\boldsymbol{k}|$ expansions given by HM(28) and HM(29) for any isotropic particle distribution, and in the case of a Boltzmann distribution the regions of validity of the expansions are given by HM(34), where the maximum particle energy ε_m is of the order of the thermal energy.

In the ultra-relativistic limit $\rho \to 0$, it would seem relatively straightforward to evaluate the functions $T(v, \rho)$ [or $I(t, \rho)$] by expanding the exponentials in (7) (or equation 5) in a power series. The leading term gives a logarithm. However, the next order term contains divergent integrals, and hence this procedure is unacceptable. Godfrey *et al.* (1975) mentioned the difficulty of expanding $T(v, \rho)$ for $\rho \to 0$ and did not give any expansion. Tsytovich (1961) presented a specific result in the limit of small ρ and we comment on this result in Appendix 2.

In view of these difficulties, further progress requires a more detailed examination of the properties of the plasma dispersion function $T(v, \rho)$.

Acknowledgments

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Appendix 1

The function $I(t, \rho)$, defined by (5), has the following properties:

$$I(-t, \rho) = -I(t, \rho),\tag{A1}$$

$$\frac{\partial I(t, \rho)}{\partial t} = -\frac{4t}{(1-t^2)^2}\rho I(t, \rho) - \frac{2}{1-t^2}\rho D_2(\rho) + \frac{2(1+t^2)}{(1-t^2)^2}\rho D_1(\rho),\tag{A2}$$

$$\frac{\partial I(t, \rho)}{\partial \rho} = -\frac{1+t^2}{1-t^2}I(t, \rho) + \frac{2t}{1-t^2}D_1(\rho),\tag{A3}$$

with

$$D_n(\rho) = \int_{-1}^{+1} dt\, \frac{\exp\{-\rho(1+t^2)/(1-t^2)\}}{(t-1)^n}\tag{A4}$$

given by

$$D_1(\rho) = -K_0(\rho), \quad D_2(\rho) = K_1(\rho), \quad D_3(\rho) = -\tfrac{1}{2}\{K_2(\rho) + K_1(\rho)\},$$

$$D_{n+1}(\rho) = \frac{\partial D_n(\rho)}{\partial \rho} - D_n(\rho) + \tfrac{1}{2}\left(\frac{\partial D_{n-1}(\rho)}{\partial \rho} - D_{n-1}(\rho)\right).\tag{A5}$$

The function $T(v, \rho)$ defined by (7), when rewritten in terms of t and t', through

$$v = 2t/(1+t^2), \quad v' = 2t'/(1+t'^2),\tag{A6a, b}$$

becomes

$$T(v, \rho) = \frac{1+t^2}{1-t^2}\int_{-1}^{+1} dt'\, \frac{1-t'^2}{1+t'^2}\left(\frac{1}{t'-t} - \frac{1}{t'-t^{-1}}\right)$$

$$\times \exp\{-\rho(1+t'^2)/(1-t'^2)\}.\tag{A7}$$

On differentiating with respect to ρ and using

$$\frac{\partial T(v, \rho)}{\partial \rho} = \frac{1}{v}\left(\frac{1-v^2}{\rho}\frac{\partial T(v, \rho)}{\partial v} + 2K_1(\rho)\right),\tag{A8}$$

one finds

$$I(t, \rho) - I(t^{-1}, \rho) = -\frac{(1-v^2)^{\frac{1}{2}}}{v}\left(\frac{1-v^2}{\rho}\frac{\partial T(v, \rho)}{\partial v} + 2K_1(\rho)\right).\tag{A9}$$

On differentiating (A9) with respect to ρ and using (A8) with

$$\frac{\partial^2 T(v,\rho)}{\partial \rho^2} = \frac{1}{1-v^2}\{T(v,\rho)+2vK_0(\rho)\}, \tag{A10}$$

and (A3) with (A5), one finds

$$I(t,\rho)+I(t^{-1},\rho) = T(v,\rho). \tag{A11}$$

Now writing $t = t_+$ and $t^{-1} = t_-$, in accord with (8), (A11) and (A9) give (9a) and (9b) respectively.

A result we require in Section 4 involves using (Godfrey *et al.* 1975)

$$\frac{\partial^2 T(v,\rho)}{\partial v^2} = (\gamma^2/v)(1+2v^2)\frac{\partial T(v,\rho)}{\partial v} + \rho^2 v^2 \gamma^6 T(v,\rho)$$

$$+ 2v\gamma^6 \rho^2 K_0(\rho) + 2(\gamma^4/v)\rho K_1(\rho) \tag{A12}$$

to derive

$$\frac{\partial}{\partial v}\left\{\frac{1}{\gamma v}\left(\frac{1-v^2}{\rho}\frac{\partial T(v,\rho)}{\partial v}+2K_1(\rho)\right)\right\} = \rho\gamma^3\{vT(v,\rho)+2K_0(\rho)\}. \tag{A13}$$

We also use

$$\partial^2 K_0(\rho)/\partial \rho^2 = \tfrac{1}{2}\{K_2(\rho)+K_1(\rho)\}, \tag{A14}$$

which follows from standard properties for the modified Bessel functions.

Appendix 2

Tsytovich (1961) quoted an expression for the longitudinal part of the dielectric tensor in the ultra-relativistic limit. Using the identities

$$\frac{\omega^2}{(\omega^2-|\boldsymbol{k}|^2)(\varepsilon_1-\varepsilon_2)}\frac{m^2+\varepsilon_{1,2}(|\boldsymbol{k}|^2-\omega^2)/2\omega}{\varepsilon_{1,2}^2-m^2} = -\frac{\omega}{2|\boldsymbol{k}|}, \tag{A15}$$

Tsytovich's expression (44) simplifies in our notation to

$$\alpha^{\mathrm{L}}(k) = \frac{e^2\rho\bar{n}\omega^2}{m|\boldsymbol{k}|^2}\left(1-\frac{\omega}{2|\boldsymbol{k}|}\sum_{i=1}^{2}\ln\left|\frac{\varepsilon_i+(\varepsilon_i^2-m^2)^{\frac{1}{2}}}{m}\right|\right), \tag{A16}$$

where the ε_i are given by HM (A27) and $\bar{n}$ is the actual number density. We have attempted to derive (A16) in three ways, two of which reproduce (A16); the third method highlights difficulties with the limit $\rho \to 0$.

Method 1

We wish to evaluate (16a) with equations (6) in the limit $\rho \to 0$. Setting $\gamma = (1+t^2)/(1-t^2)$, the relevant integral (5) becomes

$$I(t,\rho) = (\gamma^2-1)^{\frac{1}{2}}\int_1^{\infty} \mathrm{d}\gamma' \frac{\exp(-\rho\gamma')}{(\gamma'^2-1)^{\frac{1}{2}}(\gamma'-\gamma)}. \tag{A17}$$

After expanding $(\gamma'^2 - 1)^{-\frac{1}{2}}$ in powers of γ'^{-2} we find

$$I(t, \rho) = (\gamma^2 - 1)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{n!\sqrt{\pi}} \gamma^{-2n-1}$$

$$\times \int_1^{\infty} d\gamma' \exp(-\rho\gamma') \left(\frac{1}{\gamma'-\gamma} - \sum_{k=1}^{2n+1} \frac{\gamma^{k-1}}{\gamma'^k} \right), \tag{A18}$$

which may be evaluated in terms of the exponential integral Ei(x) (Gradsteyn and Ryzhik 1980; §8.2).

Hence, we get

$$I(t, \rho) = -\exp(-\rho\gamma) \, \text{Ei}(\rho\gamma - \rho) + \text{Ei}(-\rho) - (\gamma^2 - 1)^{\frac{1}{2}}$$

$$\times \left(\exp(-\rho) \sum_{s=0}^{\infty} (-\rho)^s \sum_{n=\text{int}(\frac{1}{2}s)+1}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{n!\sqrt{\pi}} \gamma^{-2n-1} \sum_{k=s+2}^{2n+1} \frac{\gamma^{k-1}}{(k-1)\dots(k-1-s)} \right.$$

$$\left. + \text{Ei}(-\rho) \sum_{k=1}^{\infty} (-1)^k \frac{(\rho\gamma)^{k-1}}{(k-1)!} \sum_{n=\text{int}(\frac{1}{2}k)}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{n!\sqrt{\pi}} \gamma^{-2n-1} \right), \tag{A19}$$

where int$(\frac{1}{2}n)$ is the largest integer less than $\frac{1}{2}n$. We now insert (A19) in equations (6) and use (4) with $K_2(\rho) \to 2/\rho^2$ as $\rho \to 0$. The lowest order terms in an expansion in ρ are then of order ρ^{-2} and we retain only terms of this order in approximating the transcendental functions. We find, after some lengthy algebra,

$$S^{(0)}(k) = 0, \tag{A20a}$$

$$S^{(1)}(k) = \frac{\pi^2 \rho \bar{n}}{2m^2} \sum_{i=1}^{4} \eta_i \ln \left| \frac{1+t_i}{1-t_i} \right|, \tag{A20b}$$

$$S^{(2)}(k) = -\frac{\pi^2 \rho \bar{n}}{2m^3} \sum_{i=1}^{4} \left(-\frac{2}{\rho} \ln \left| \frac{1+t_i}{1-t_i} \right| - \frac{2t_i}{1-t_i^2} \right), \tag{A20c}$$

where we write $(\gamma_i^2 - 1)^{\frac{1}{2}} = 2t_i/(1-t_i^2)$ and $\gamma_i = (1+t_i^2)/(1-t_i^2)$. The sums follow from HM (A23) and HM (34a):

$$\sum_{i=1}^{4} \eta_i \ln \left| \frac{1+t_i}{1-t_i} \right| = 2 \sum_{i=1}^{2} \ln \left| \frac{\varepsilon_i + (\varepsilon_i^2 - m^2)^{\frac{1}{2}}}{m} \right|, \tag{A21a}$$

$$\sum_{i=1}^{4} \ln \left| \frac{1+t_i}{1-t_i} \right| = 0, \qquad \sum_{i=1}^{4} \frac{2t_i}{1-t_i^2} = \frac{2|k|}{m}. \tag{A21b, c}$$

Then substituting equations (A20) into (16a) reproduces (A16).

Method 2

Alternatively we may evaluate (5) for small ρ by the following expansion:

$$I(t, \rho) = \int_{-1}^{1} \frac{dt'}{t'-t} \left(1 + \rho \frac{1+t'^2}{1-t'^2} + \dots \right)^{-1}$$

$$= \frac{1}{1+\rho} \int_{-1}^{1} dt' \frac{1-t'^2}{(t'-t)(1-t'^2 a^2)} + O(\rho^2), \tag{A22}$$

with $a^2 = (1-\rho)/(1+\rho)$. The integral is elementary and gives

$$I(t,\rho) = \frac{1}{1+\rho}\left(\frac{1-t^2}{1-a^2t^2}\ln\left|\frac{1-t}{1+t}\right| + \frac{(1-a^2)t}{a(1-a^2t^2)}\ln\left|\frac{1-a}{1+a}\right|\right) + O(\rho^2).\qquad \text{(A23)}$$

Now setting $t = t_i$, the zero and first order terms in an expansion in ρ are

$$I(t_i,\rho) \approx \left(1 - \rho\frac{1+t_i^2}{1-t_i^2}\right)\ln\left|\frac{1-t_i}{1+t_i}\right| + \rho\frac{2t_i}{1-t_i^2}\ln\left|\tfrac{1}{2}\rho\right|.\qquad \text{(A24)}$$

Then using equations (6) we rederive (A20) and the result (A16) follows.

Method 3

This method highlights the formal difficulties we encounter in evaluating $I(t,\rho)/\rho$ and its derivatives in the limit $\rho \to 0$. Bateman (1931), in introducing the functions $k_n(x)$ which bear his name, noted the generating function

$$\exp\left(-\rho\frac{1+t^2}{1-t^2}\right) = \sum_{n=0}^{\infty} k_{2n}(\rho)t^{2n}.\qquad \text{(A25)}$$

Using (A25), equation (5) gives

$$I(t,\rho) = \sum_{n=0}^{\infty} k_{2n}(\rho)I_{2n}(t),\qquad \text{(A26a)}$$

with

$$I_{2n}(t) = \int_{-1}^{1} dt'\,\frac{t'^{2n}}{t'-t}.\qquad \text{(A26b)}$$

One may evaluate the first and second derivatives of $I(t,\rho)/\rho$ as they appear in (6b) and (6c), either directly using (A26a) or by using the differential equation (A3) to evaluate $\partial I(t,\rho)/\partial\rho$ and hence obtaining $(\partial/\partial\rho)\{I(t,\rho)/\rho\}$. The results are not the same for the second derivative of $I(t,\rho)/\rho$, and the reason is that the Bateman functions $k_{2n}(\rho)$ have discontinuous first derivatives and singular second derivatives at $\rho = 0$. The function $I(t,\rho)$ itself is defined only for positive ρ and the limit $\rho \to 0$ is obviously not well defined.

We are unable to comment on the possible range of validity of Tsytovich's approximation (A16). We note that Godfrey et al. (1975) mentioned the difficulties with this limit and declined to present approximate expressions for small ρ. The limit $\rho \to 0$ needs to be treated with considerably more care than we have exercised here.