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Dielectric Tensor of Weakly Relativistic Electron Distributions Separable in Momentum and Pitch Angle

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Abstract

The dielectric tensor of a weakly relativistic, magnetized plasma is derived for distributions separable in momentum and pitch angle by using an expansion in powers of the Larmor radius. The results are initially expressed in terms of an integral over the electron pitch angle distribution which is itself unrestricted apart from a single symmetry condition. These results include relativistic and finite Larmor radius effects contributed by harmonics s with $-2 \leqslant s \leqslant 2$ for all propagation angles and thus provide a useful framework for both numerical and analytical investigation of electron cyclotron phenomena (propagation and absorption of waves, maser action, current drive etc.) in a wide variety of isotropic and anisotropic plasmas. Explicit results are presented for the dielectric properties of isotropic, loss cone, anti-loss cone and hollow beam distributions, and for wave propagation perpendicular to the magnetic field. In these cases the pitch angle integrals are performed in terms of functions related to the standard plasma dispersion function.

1. Introduction

The purpose of this paper is to derive an expression for the dielectric tensor of a weakly relativistic (characteristic electron energies $\lesssim 20$ keV), magnetized electron distribution separable in momentum and pitch angle. This work is intended to provide the basis for investigation of the propagation and absorption of waves in isotropic and anisotropic plasmas, especially when the frequency is near the electron cyclotron frequency Ω_e or its second harmonic and also when the Cerenkov and/or anomalous Doppler resonances are of importance. An important aim is to obtain expressions which, not only permit straightforward computation of wave properties in a wide class of distribution functions but, enable qualitative physical insight to be gained in important limiting cases without one resorting to involved numerical computations.

Interest in electron cyclotron interactions arises from a number of sources. First, electron cyclotron maser emission from loss cone plasmas is believed to be the mechanism responsible for auroral kilometric and z-mode radiation and Jovian decametric emission (Wu and Lee 1979; Melrose *et al.* 1982; Hewitt *et al.* 1982; Hewitt *et al.* 1983), and for a number of solar and stellar microwave emissions (Melrose and Dulk 1982; Dulk 1985). Furthermore, similar instabilities may exist in loss cone plasmas in laboratory devices such as the endplugs of tandem mirrors and the hot electron rings of bumpy tori (Pritchett 1984). Second, electron cyclotron resonance heating of isotropic and anisotropic plasmas near the first and second

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harmonics is relevant in the laboratory situations mentioned above and in tokamaks; both propagation and absorption of waves are of interest in these contexts (Bornatici *et al.* 1983). Third, the dielectric properties of hot, magnetized plasmas are of interest in connection with electron cyclotron current drive and with mode coupling (Bornatici *et al.* 1983). Fourth, the Cerenkov and anomalous Doppler resonances have been discussed in connection with amplification of auroral hiss (Melrose and White 1980) and scattering of runaway electrons in solar flares and the laboratory (Liu *et al.* 1977; Kuijpers *et al.* 1981; Gandy *et al.* 1985).

In most analyses of electron cyclotron absorption the dispersion relation of the waves has been assumed to be given by cold plasma theory on the grounds that the plasma temperature is low or because the absorption is due to a small population of hot electrons in the presence of a dense, cold background plasma which determines the hermitian part of the dielectric tensor. Neither justification is valid in general—even in low temperature plasmas, intrinsically relativistic effects are important near the cyclotron frequency and its harmonics (Dnestrovskii *et al.* 1964; Shkarofsky 1966; Wu and Lee 1979); furthermore, in some laboratory and astrophysical situations few cold electrons are present and hot electrons determine both the hermitian and the antihermitian parts of the dielectric tensor (Tsai *et al.* 1981; Winglee 1983; Pritchett 1984).

A number of investigations of electron cyclotron absorption have been made which assume special forms of the (in general, anisotropic) distribution function and retain some finite Larmor radius (FLR) and relativistic effects on the dispersion. In addition to the restrictions implicit in a specific choice of distribution function, these treatments usually assume that contributions to the dielectric tensor from harmonics $s \leqslant 0$ may be approximated by their cold plasma values, or that contributions to the hermitian part of the dielectric tensor from $|s| \geqslant 2$ may be neglected entirely. The first of these assumptions implies that the Cerenkov ($s = 0$) and anomalous Doppler ($s \leqslant -1$) resonances are not important and is in general justified only if the refractive index n of the waves is less than unity. The second assumption is equivalent to neglecting FLR effects except those contributed by $s = 1$. Such a neglect is shown to be unjustified in Section 3 where we demonstrate that lowest order FLR effects from $|s| = 2$ (but *not* $|s| > 2$) can be comparable in magnitude with the zeroth order $s = 0, \pm 1$ contributions in weakly relativistic plasmas. Furthermore, near resonances in the dispersion relation (e.g. near the upper hybrid frequency), the wavevector is large and hence FLR effects are significant.

In this paper we evaluate the dielectric tensor for separable electron distributions, retaining FLR and relativistic effects for $|s| \leqslant 2$. The results, given in Section 2, are expressed in the form of an integral over the pitch angle distribution; the integrands are related to the standard plasma dispersion function (Fried and Conte 1961). In Section 3 and Appendix 2 we discuss the dielectric tensor for a number of specific cases of distribution function, propagation angle and frequency in which the pitch angle integral in Section 2 can be performed and in which the results simplify considerably. Near the first and second harmonics these specific results generalize those of Tsai *et al.* (1981) and Wong *et al.* (1982).

2. Analysis

In this section we evaluate the dielectric tensor of a weakly relativistic, magnetized, anisotropic plasma separable in momentum and pitch angle in terms of an integral over its pitch angle distribution. Frequencies are measured in units of the cyclotron frequency $\Omega_e = eB/m$ where B is the magnetic flux density. The waves are then described by their frequency ω and wavevector $\boldsymbol{k}$ (with $\boldsymbol{k}$ in units of c/Ω_e). The plasma frequency ω_p is given by $\omega_p^2 = Nm/\epsilon_0 B^2$ where N is the number density of electrons. The electrons are described by their momentum $\boldsymbol{p}$ (in units of mc) and distribution function F which is assumed to be separable in p and the pitch angle cosine μ with

$$F(p,\mu) = f(p)\,\Phi(\mu), \tag{1}$$

$$\int_{-1}^{1} \mathrm{d}\mu\ \Phi(\mu) = 2, \qquad 4\pi \int_0^\infty \mathrm{d}p\ p^2 f(p) = 1. \tag{2a, b}$$

The dielectric tensor $\boldsymbol{\epsilon}$ is a function of the frequency ω and the refractive index vector $\boldsymbol{n} = \boldsymbol{k}/\omega$ which has cartesian components

$$\boldsymbol{n} = (n_\perp, 0, n_\parallel) = (n\sin\theta,\ 0,\ n\cos\theta), \tag{3}$$

where the propagation angle θ is the angle between $\boldsymbol{n}$ and $\boldsymbol{B}$ ($\boldsymbol{B}$ is assumed to be uniform and directed along the z axis). The subscripts $\parallel$ and $\perp$ denote components parallel and perpendicular to $\boldsymbol{B}$ respectively. In terms of these variables $\boldsymbol{\epsilon}$ may be written (see e.g. Melrose 1980, p. 41; Bornatici *et al.* 1983)

$$\epsilon_{ij} = \delta_{ij} - (\omega_p^2/\omega^2)(L_{ij} + U_{ij}), \tag{4}$$

with

$$\mathbf{L} = -\int \mathrm{d}\boldsymbol{p}\ \mu\gamma^{-1} f(p)\{\mathrm{d}\Phi(\mu)/\mathrm{d}\mu\}\,\boldsymbol{bb}, \tag{5}$$

$$\boldsymbol{b} = \boldsymbol{B}/|\boldsymbol{B}|, \qquad \gamma = (1+p^2)^{\frac{1}{2}}, \tag{6a, b}$$

$$\mathbf{U} = \mathrm{i}\int_0^\infty \mathrm{d}\tau \sum_{s=-\infty}^{\infty} \int \mathrm{d}\boldsymbol{p}\ \omega p^{-1}\mathbf{A}_s \exp\{\mathrm{i}\tau(\gamma\omega - s - k_\parallel p_\parallel)\}$$
$$\times\left(\frac{\partial}{\partial p} + (n_\parallel \gamma^{-1} - \mu p^{-1})\frac{\partial}{\partial \mu}\right) f(p)\,\Phi(\mu), \tag{7}$$

$$\mathbf{A}_s = \begin{bmatrix} s^2 \mathrm{J}_s^2/k_\perp^2 & -\mathrm{i}sp_\perp \mathrm{J}_s \mathrm{J}_s'/k_\perp & sp_\parallel \mathrm{J}_s^2/k_\perp \\ \mathrm{i}sp_\perp \mathrm{J}_s \mathrm{J}_s'/k_\perp & p_\perp^2 (\mathrm{J}_s')^2 & \mathrm{i}p_\parallel p_\perp \mathrm{J}_s \mathrm{J}_s' \\ sp_\parallel \mathrm{J}_s^2/k_\perp & -\mathrm{i}p_\parallel p_\perp \mathrm{J}_s \mathrm{J}_s' & p_\parallel^2 \mathrm{J}_s^2 \end{bmatrix}, \tag{8}$$

$$\mathrm{J}_s = \mathrm{J}_s(k_\perp p_\perp), \qquad \mathrm{J}_s' = \{\mathrm{dJ}_s(z)/\mathrm{d}z\}_{z=k_\perp p_\perp}. \tag{9a, b}$$

Our main interest in the present work is to include those relativistic and FLR contributions to the dielectric tensor which remain important for weakly relativistic plasmas, rather than to account for small differences between γ and unity. Thus we set $\gamma = 1$ throughout equations (5) and (7) except in the resonant denominator $\gamma\omega - s - k_\parallel p_\parallel$ in (7) where retention of $\gamma - 1$ is of critical importance; there we assume the plasma to be weakly relativistic and approximate γ by $1+\frac{1}{2}p^2$. In this approximation $\mathbf{L}$ is then easily evaluated by using (2a, b) to give

$$\epsilon_{ij} = \delta_{ij} - (\omega_p^2/\omega^2)\Big(\tfrac{1}{2}\{2-\Phi(1)-\Phi(-1)\}\, b_i\, b_j + \pi \int_{-1}^{1} d\mu\, R_{ij}\Big) \tag{10}$$

with

$$\mathbf{R} = -2i\omega \int_0^\infty d\tau \int_0^\infty dp\, f(p) \exp\{i\omega\tau(1+\tfrac{1}{2}p^2 - n_\parallel\, p\mu)\}$$
$$\times \sum_{s=-\infty}^{\infty} \mathbf{A}_s \exp(-is\tau)[-p\Phi(\mu)\{\partial \ln f(p)/\partial p\} + (\mu - n_\parallel\, p)\Phi'(\mu)]. \tag{11}$$

To perform the integrals in (10) and (11) we make the small Larmor radius approximation $(k_\perp p_\perp)^2 \ll 1$. Provided $k_\perp$ is of order unity (or smaller) this approximation is sufficiently general to enable distributions with characteristic electron energies up to ~20 keV to be treated (Tsai *et al.* 1981; Bornatici *et al.* 1983; Batchelor *et al.* 1984). This energy range includes a wide variety of plasmas of interest in astrophysics and the laboratory, as discussed in the Introduction and the references cited therein. Next the Bessel functions are expanded in powers of $k_\perp p_\perp$ retaining contributions from $|s| \leqslant 2$. The tensor $\mathbf{R}$ may then be expressed in terms of the following functionals of $f(p)$:

$$G_s(r) = -i\omega \int_0^\infty d\tau \int_0^\infty dp\, \exp\{i\tau(\omega - s - \omega n_\parallel\, p\mu + \tfrac{1}{2}\omega p^2)\}\, p^r f(p)$$
$$\times[-p\Phi(\mu)\{\partial \ln f(p)/\partial p\} + (\mu - n_\parallel\, p)\Phi'(\mu)], \tag{12a}$$

$$G_s^{\pm}(r) = \tfrac{1}{2}\{G_s(r) \pm G_{-s}(r)\}, \tag{12b}$$

to give

$$R_{11} = (1-\mu^2)[G_1^+(2) - l\{G_1^+(4) - G_2^+(4)\}], \tag{13a}$$

$$R_{12} = -i(1-\mu^2)[G_1^-(2) - l\{2G_1^-(4) - G_2^-(4)\}], \tag{13b}$$

$$R_{13} = k_\perp \mu(1-\mu^2)[G_1^-(3) - \tfrac{1}{2}l\{2G_1^-(5) - G_2^-(5)\}], \tag{13c}$$

$$R_{22} = (1-\mu^2)[G_1^+(2) + l\{2G_0(4) - 3G_1^+(4) + G_2^+(4)\}], \tag{13d}$$

$$R_{23} = ik_\perp\mu(1-\mu^2)[-G_0(3) + G_1^+(3) + \tfrac{1}{2}l\{3G_0(5) - 4G_1^+(5) + G_2^+(5)\}], \tag{13e}$$

$$R_{33} = 2\mu^2[G_0(2) - 2l\{G_0(4) - G_1^+(4)\} + \tfrac{1}{2}l^2\{3G_0(6) - 4G_1^+(6) + G_2^+(6)\}], \tag{13f}$$

with $l = \frac{1}{4}k_\perp^2(1-\mu^2)$.

Specific Distribution Functions

Equation (10) [with equations (13)] is valid for arbitrary weakly relativistic plasmas satisfying (1) provided $|k_\perp^2 p_\perp^2| \ll 1$ wherever $f(p)$ is non-negligible. We now consider a specific class of distribution functions $f(p)$ in order to evaluate the functionals

$G_s(r)$. The generic distribution function we employ is of the form

$$f(p) = C_{2b}\,p^{2b}\exp(-\tfrac{1}{2}\zeta p^2); \qquad C_{2b} = (\tfrac{1}{2}\zeta)^{b+\frac{3}{2}}\{2\pi\Gamma(b+\tfrac{3}{2})\}^{-1}, \quad \text{(14a, b)}$$

where $\zeta = mc^2/k_B T$ is the (dimensionless) inverse temperature and b is a non-negative integer. With appropriate choice of angular distribution function this class includes the usual isotropic thermal distribution ($b = 0$) and the commonly encountered DGH distribution (Dory *et al.* 1965) as special cases. If we assume that the angular distribution function $\Phi(\mu)$ satisfies $\Phi(\mu) = \Phi(-\mu)$, then the parity of the μ integrals involving $G_s(r)$ which appear in (10) [with equations (13)] enables us to replace $\exp(-i\omega\tau n_{\|}\,p\mu)$ there by

$$\tfrac{1}{2}\{\exp(-i\omega\tau n_{\|}\,p\mu) + (-1)^r\exp(i\omega\tau n_{\|}\,p\mu)\}$$

and to restrict the range of μ to positive values without affecting the resulting expression for ϵ_{ij}. For the distribution (14) we may then replace $G_s(r)$ in (12a) by the following symmetrized form:

$$G_s(r) = \zeta\Phi(\mu)I_s(r+2b+2) - n_{\|}\,\Phi'(\mu)I_s(r+2b+1)$$
$$+\{\mu\Phi'(\mu) - 2b\Phi(\mu)\}\,I_s(r+2b), \quad (15)$$

with

$$I_s(r) = -\tfrac{1}{2}i\zeta C_{2b}\int_0^\infty dt\,\exp(iz_s t)\int_0^\infty dp\,p^r\exp(-\tfrac{1}{2}\xi p^2)$$
$$\times\{\exp(-iqp) + (-1)^r\exp(iqp)\}, \quad (16)$$

$$z_s = \zeta - \zeta s/\omega, \qquad t = \omega\tau/\zeta, \qquad \xi = \zeta(1-it), \qquad q = n_{\|}\,\mu\zeta t, \quad \text{(17a–d)}$$

and hence (Gradshteyn and Ryzhik 1980; p. 496, equations 3.952.9, 10)

$$I_s(r) = -i\zeta C_{2b}\,\pi^{\frac{1}{2}}(2\zeta)^{-\frac{1}{2}r-\frac{1}{2}}\exp(-\tfrac{1}{2}i\pi r)\int_0^\infty dt\,(1-it)^{-\frac{1}{2}r-\frac{1}{2}}$$
$$\times\exp(iz_s t - \tfrac{1}{2}q^2/\xi)\,H_r\{q/(2\xi)^{\frac{1}{2}}\}, \quad (18)$$

where H_r is the rth Hermite polynomial (Abramowitz and Stegun 1970; p. 775, equation 22.3). We thus find

$$I_s(r) = (-1)^r(\tfrac{1}{2}\pi)^{\frac{1}{2}}\,r!\,\zeta^{\frac{1}{2}-\frac{1}{2}r}C_{2b}$$
$$\times\sum_{m=0}^{[\frac{1}{2}r]}\frac{H^{r-2m}}{2^m m!\,(r-2m)!}\,\mathscr{F}_{r-m+\frac{1}{2},\,r-2m}(z_s,\tfrac{1}{2}H^2), \quad (19)$$

with

$$H = n_{\|}\,\mu\zeta^{\frac{1}{2}}, \qquad [\tfrac{1}{2}r] = \tfrac{1}{2}r, \qquad r\ \text{even}, \quad \text{(20a, b)}$$
$$= \tfrac{1}{2}r - \tfrac{1}{2}, \qquad r\ \text{odd}, \quad \text{(20c)}$$

and where the generalized Shkarofsky functions $\mathscr{F}_{q,r}$ are defined as

$$\mathscr{F}_{q,r}(z, a) = -\mathrm{i}\int_0^\infty \mathrm{d}t\,(\mathrm{i}t)^r(1-\mathrm{i}t)^{-q}\exp\{\mathrm{i}zt - at^2/(1-\mathrm{i}t)\}\,. \tag{21}$$

Krivenski and Orefice (1983) have shown that $\mathscr{F}_{q,r}(z, a)$ can be re-expressed in terms of the standard plasma dispersion function (Fried and Conte 1961). This expression and other important properties of the functions $\mathscr{F}_{q,r}$ are summarized in Appendix 1.

The results of this section enable the dielectric tensor of the distribution (14) to be written in the form

$$\epsilon_{ij} = \delta_{ij} - (\omega_{\mathrm{p}}^2/\omega^2)\Big(\{1-\Phi(1)\}\,b_i\,b_j + 2\pi\int_0^1 \mathrm{d}\mu\; R_{ij}\Big) \tag{22}$$

with $\Phi(\mu) = \Phi(-\mu)$ and where R_{ij} is given by (13) and the functionals G_s and I_s by (15), (18) and (19). Since the evaluation of the Shkarofsky functions in (19) is reasonably straightforward, equation (22) is well suited to numerical calculations involving arbitrary $\Phi(\mu)$ satisfying $\Phi(\mu) = \Phi(-\mu)$.

3. Special Cases

In this section we consider a number of special cases of the results in Section 2 in which the form of the dielectric tensor simplifies significantly. We shall assume that the momentum distribution is of the form (14), that the angular distribution satisfies $\Phi(\mu) = \Phi(-\mu)$, and hence that the dielectric tensor is of the form (22). We note that considerable additional simplifications to the form of ϵ occur in some regimes of frequency and propagation angle where the functions $\mathscr{F}_{q,r}$ assume one of their several limiting forms. These limiting forms have been extensively discussed by Shkarofsky (1966), Airoldi and Orefice (1982) and Krivenski and Orefice (1983) and are given in Appendix 1.

(a) Generalized DGH Distribution

An important distribution for which the threefold μ, p and t integral in (22) can be evaluated in closed form is that given by (14) and (23) with b and j integers—a 'generalized' DGH distribution

$$\Phi(\mu) = \frac{2\Gamma(b+\frac{3}{2})}{\Gamma(b-j+\frac{1}{2})\,\Gamma(j+1)}(1-\mu^2)^j\mu^{2b-2j}\,. \tag{23}$$

If $b = j = 0$ this distribution reproduces an isotropic Maxwellian distribution, whereas it reproduces the widely used, loss cone-like DGH distribution (Dory *et al.* 1965) if $b = j \neq 0$. [A DGH distribution was used by Tsai *et al.* (1981) whose analysis considered the lowest order (in λ) contribution to ϵ for each harmonic.] Furthermore, it can be used to represent a pair of counterstreaming beams ($j = 0$, $b > 0$) or counterstreaming 'hollow beams' ($b > j > 0$), each similar to the hollow distribution considered by Freund *et al.* (1983). Fig. 1 illustrates these four types of distribution.

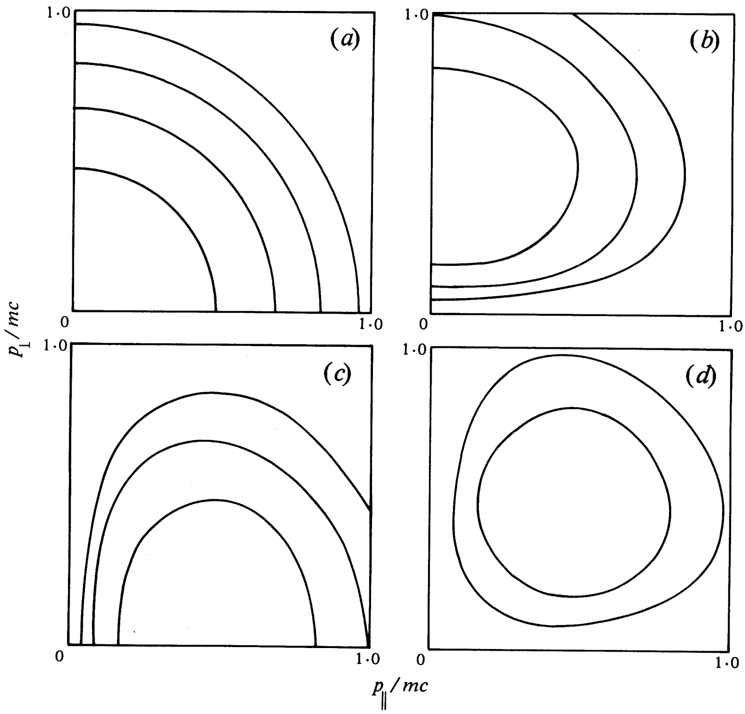

Fig. 1. Contours of generalized DGH distribution [equations (14) and (23)] in $p_\parallel$–$p_\perp$ space for $\zeta = 20$ and (*a*) an isotropic Maxwellian distribution with $b = j = 0$; (*b*) a DGH distribution with $b = j = 2$; (*c*) counterstreaming beams with $b = 2$, $j = 0$; (*d*) hollow beams with $b = 4$, $j = 2$. Each distribution has reflection symmetry across the $p_\perp$ axis. Contours are spaced apart by a factor of 10 with the peak of each distribution normalized to unity.

Inspection of the integrals involved in evaluating (22) for the generalized DGH distribution shows that each involves a sum of terms of the form

$$Q = 2\pi \int_0^1 \mathrm{d}\mu\, \mu^{r-2g}(1-\mu^2)^g\, G_s(r)\,, \tag{24}$$

with g an integer and

$$r-2g = 0, \quad i,j \neq 3\,, \tag{25a}$$

$$= 1, \quad i = 3 \text{ or } j = 3,\ i \neq j\,, \tag{25b}$$

$$= 2, \quad i = j = 3\,. \tag{25c}$$

These integrals can be expressed in terms of

$$\int_0^1 \mathrm{d}\mu\, \mu^{2\delta-1}(1-\mu^2)^{\beta-1} \exp(-\psi^2\mu^2)\, H_{2\delta+2\beta-1}(\psi\mu)$$
$$= \Gamma(\beta)\, 2^{2\beta-1}(-1)^\beta \exp(-\psi^2)\, H_{2\delta-1}(\psi)\,, \tag{26}$$

where β and 2δ are positive integers and $\psi = n_\parallel\, \zeta t/(2\xi)^{\frac{1}{2}}$ is independent of μ. The remaining t integrals can then be performed, leading to an expression for ϵ in terms of generalized Shkarofsky functions $\mathcal{F}_{q,r}$. The results of the above steps are given in Section 3b for $b = j = 0$ and the μ integral (24) is performed in Appendix 2 for arbitrary positive integers b and j.

(b) Maxwellian Distribution

The most important special case of the generalized DGH distribution is the isotropic Maxwellian distribution ($b = j = 0$) which serves to illustrate a number of features of the more general result. The dielectric tensor for this distribution is

$$\epsilon_{ij} = \delta_{ij} - (\omega_p^2 \zeta/\omega^2) X_{ij}, \tag{27}$$

with

$$X_{11} = \mathcal{F}^{+}_{5/2} - \lambda(\mathcal{F}^{+}_{7/2} - \mathcal{F}^{2+}_{7/2}), \qquad X_{22} = X_{11} + 2\lambda(\mathcal{F}_{7/2} - \mathcal{F}^{+}_{7/2}), \tag{28a, b}$$

$$X_{12} = -\mathrm{i}\{\mathcal{F}^{-}_{5/2} - \lambda(2\mathcal{F}^{-}_{7/2} - \mathcal{F}^{2-}_{7/2})\}, \tag{28c}$$

$$X_{13} = n\ln_\perp \omega\{\mathcal{F}^{-}_{7/2,1} - \tfrac{1}{2}\lambda(2\mathcal{F}^{-}_{9/2,1} - \mathcal{F}^{2-}_{9/2,1})\}, \tag{28d}$$

$$X_{23} = \mathrm{i}n\ln_\perp \omega\{(\mathcal{F}_{7/2,1} - \mathcal{F}^{+}_{7/2,1}) - \tfrac{1}{4}\lambda(3\mathcal{F}_{9/2,1} - 4\mathcal{F}^{+}_{9/2,1} + \mathcal{F}^{2+}_{9/2,1})\}, \tag{28e}$$

$$X_{33} = \mathcal{F}_{5/2} + n_\|^2 \zeta\mathcal{F}_{7/2,2} - \lambda\{(\mathcal{F}_{7/2} - \mathcal{F}^{+}_{7/2}) + n_\|^2 \zeta(\mathcal{F}_{9/2,2} - \mathcal{F}^{+}_{9/2,2})\}$$
$$+ \tfrac{1}{4}\lambda^2 n_\|^2 \zeta(3\mathcal{F}_{11/2,2} - 4\mathcal{F}^{+}_{11/2,2} + \mathcal{F}^{2+}_{11/2,2}), \tag{28f}$$

$$\mathcal{F}^{s\pm}_{q,r} = \tfrac{1}{2}\{\mathcal{F}_{q,r}(z_s, \tfrac{1}{2}n_\|^2 \zeta) \pm \mathcal{F}_{q,r}(z_{-s}, \tfrac{1}{2}n_\|^2 \zeta)\}; \quad \lambda = k_\perp^2/\zeta. \tag{29a, b}$$

To simplify our notation we have written

$$\mathcal{F}^{s\pm}_{q} = \mathcal{F}^{s\pm}_{q,0}, \quad \mathcal{F}_{q,r} = \mathcal{F}^{0+}_{q,r}, \quad \mathcal{F}^{\pm}_{q,r} = \mathcal{F}^{1\pm}_{q,r}. \tag{29c–e}$$

Equations (27) and (28) reproduce the corresponding results of Shkarofsky (1966) for $|s| \leqslant 2$ apart from his omission of some terms proportional to λ and λ^2 which are included here.

Important Features of the Dielectric Tensor. We note the following important features of equations (27) and (28) which are shared with the corresponding results for the generalized DGH distribution (see also Airoldi and Orefice 1982):

(i) The largest terms omitted in a given dielectric tensor element consist of terms arising from small differences between γ and unity (other than in the resonance condition) and terms involving higher powers of λ than those retained. If we neglect the former class of terms and recall the relations $\zeta \gg 1$ and $\lambda = k_\perp^2/\zeta$, it is seen that the largest FLR terms omitted in a given element are negligible compared with the smallest retained, provided $k_\perp^2 \ll \zeta$. In particular, the contribution of higher harmonic ($|s| \geqslant 3$) FLR terms to the hermitian part of the dielectric tensor may be neglected because these terms are of at least one higher order in λ than those retained. If required to treat absorption at higher harmonics, these terms may be included in the antihermitian part via the usual formalism for weakly damped waves.

(ii) When the inequality

$$1,\ |n_{\|}^2\zeta| \ll |z_s| = |\zeta - \zeta s/\omega| \tag{30}$$

is satisfied we may replace $\mathscr{F}_q(z_s, a)$ by its asymptotic form $\mathscr{F}_q(z_s, a) \approx z_s^{-1}$ (equation A12). Hence if $|n_{\|}^2| \ll 1$ and $\lambda \ll 1$ equations (27) and (28) reproduce the cold plasma dielectric tensor except near the cyclotron resonances at $|s| \leqslant 2$. If, however, $|n_{\|}^2| \gtrsim 1$ the inequality in (30) is violated and the longitudinal Doppler effect significantly modifies the dielectric tensor for frequencies between, but well removed from, the cyclotron resonances at $|s| \leqslant 2$.

(iii) If $|\omega - s| \lesssim \zeta^{-1}$ the function $\mathscr{F}_q^{s\pm}$ is of order unity, whereas $\mathscr{F}_q^{s'\pm}$ is of order ζ^{-1} in the same frequency range if $s' \neq s$. This implies, for example, that FLR terms from the $s = 2$ resonance can be comparable in magnitude with the lowest order dielectric tensor elements in a narrow range of frequency near $\omega = 2$, regardless of the temperature. For example, in ϵ_{11} we have $\lambda\mathscr{F}_{7/2}^{2+} \sim \mathscr{F}_{5/2}^{+} \sim \zeta^{-1}$ near $\omega = 2$ if $k_\perp^2 \sim 1$.

(iv) The coefficient $n_{\|}^2\zeta$ of some terms is a large quantity except if $|n_{\|}| \lesssim \zeta^{-\frac{1}{2}}$. Hence these terms can easily dominate other terms of the same order in λ at most propagation angles for frequencies satisfying $\omega \approx 0, 1, 2$. This observation justifies retention of terms proportional to λ^2 in ϵ_{33}.

(c) Quasi-perpendicular Propagation

If $|n_{\|}^2\zeta\mu^2| \ll 1$ then the Shkarofsky functions $\mathscr{F}_q(z_s, \frac{1}{2}n_{\|}^2\zeta\mu^2)$ in (19) may be replaced by the simpler Dnestrovskii functions $F_q(z_s)$ in (A10). Under these conditions the dispersion of the waves is well approximated by that at $n_{\|} = 0$ and the functionals $G_s(r)$ and $I_s(r)$ become

$$G_s(r) = \zeta\Phi(\mu)I_s(r+2b+2) + \{\mu\Phi'(\mu) - 2b\Phi(\mu)\}\, I_s(r+2b), \tag{31a}$$

$$I_s(2j) = C_{2b}(\tfrac{1}{2}\pi)^{\frac{1}{2}}\zeta^{\frac{1}{2}-j}\frac{(2j)!}{2^j j!}F_{j+\frac{1}{2}}(z_s), \tag{31b}$$

$$I_s(2j+1) = 0, \tag{31c}$$

where j is an integer and the momentum distribution (14) has been assumed.

The regime in which $|n_{\|}^2\zeta\mu^2| \ll 1$ is satisfied wherever $\Phi(\mu)$ is non-negligible is termed that of *quasi-perpendicular propagation*; it prevails at propagation angles θ with $\cos^2\theta \ll 1$, and in regions near to cutoffs ($|n^2| \ll 1$). In the quasi-perpendicular regime the dielectric tensor elements with indices 13, 23, 31 and 32 may be neglected and the evaluation of the dielectric tensor then involves only calculation of the following six integrals:

$$\int_0^1 \mathrm{d}\mu\ \mu^{2j}\{\Phi(\mu), \mu\Phi'(\mu)\}, \quad j = 0, 1, 2. \tag{32}$$

In the case $j = 0$ these integrals are [from (2a) and $\Phi(\mu) = \Phi(-\mu)$]

$$\int_0^1 \mathrm{d}\mu\ \Phi(\mu) = 1, \qquad \int_0^1 \mathrm{d}\mu\ \mu\Phi'(\mu) = \Phi(1) - 1. \tag{33a, b}$$

(d) Loss Cone and Anti-loss Cone Distributions

A specific distribution function widely considered in the theory of electron cyclotron phenomena is the sharp-edged, two-sided loss cone distribution represented by

$$\begin{aligned}\Phi(\mu) &= \mu_c^{-1}, \qquad &|\mu| \leqslant \mu_c,\\ &= 0, &\mu_c < |\mu| \leqslant 1,\end{aligned} \tag{34a}$$

with $f(p)$ given by (14a) with $b = 0$. A closely related angular distribution is the two-sided anti-loss cone, represented by

$$\begin{aligned}\Phi(\mu) &= 0, \qquad &|\mu| < \mu_c\\ &= (1-\mu_c)^{-1}, &\mu_c \leqslant |\mu| \leqslant 1,\end{aligned} \tag{34b}$$

which approximates a pair of counterstreaming beams, each of which fills a cone in momentum space. It is possible to evaluate the dielectric tensor explicitly for the distribution (34a) (and hence for 34b). Here we shall restrict our attention to the cases of quasi-perpendicular propagation in which the effect of the loss cone on the dielectric properties of the plasma is most readily apparent.

Quasi-perpendicular Propagation. For the distribution (34a) the condition defining the quasi-perpendicular regime is

$$|n_{\|}^2\, \zeta\mu_c^2| \ll 1 \tag{35}$$

which, if satisfied, implies that ϵ is given by (27) with

$$X_{11} = Y^+, \quad X_{22} = X_{11} - \tfrac{1}{4}\lambda\{CF_{7/2}^+ - 3(1-\rho)^2 F_{5/2}^+\}, \tag{36a, b}$$

$$X_{12} = -X_{21} = -\mathrm{i}[Y^- - \tfrac{1}{8}\lambda\{CF_{7/2}^- - 3(1-\rho)^2 F_{5/2}^-\}], \tag{36c}$$

$$X_{33} = \zeta^{-1} + \rho[F_{5/2} - F_{3/2} + \tfrac{1}{2}\lambda\{(5-3\rho)F_{7/2}^+ - 3(1-\rho)F_{5/2}^+\}], \tag{36d}$$

$$\begin{aligned}Y^{\pm} = \tfrac{1}{2}[&(3-\rho)F_{5/2}^{\pm} - (1-\rho)F_{3/2}^{\pm}\\ &- \tfrac{1}{4}\lambda\{C(F_{7/2}^{\pm} - F_{7/2}^{2\pm}) - 3(1-\rho)^2(F_{5/2}^{\pm} - F_{5/2}^{2\pm})\}],\end{aligned} \tag{37a}$$

$$\rho = \mu_c^2, \qquad C = 15 - 10\rho + 3\rho^2. \tag{37b, c}$$

The other X_{ij} are zero and the functions $F_{q,r}$ are defined by (29) with $n_{\|} = 0$. Equations (36) reproduce (28) in the limit $\mu_c = 1$ apart from the term $\zeta^{-1} - F_{3/2}$ which, however, is non-resonant and of order $\zeta^{-2} \ll 1$.

Quasi-two-dimensional Distribution. Trubnikov and Yakubov (1963) discussed the dielectric tensor of a strictly two-dimensional Maxwellian distribution in a perpendicularly oriented magnetic field. Such a distribution may be approximated by the present distribution if $\mu_c \to 0$ (resulting in a singular angular distribution function perpendicular to $\boldsymbol{B}$). In this case it is found that equations (35) and (36) apply at all propagation angles and reproduce the corresponding results of Trubnikov and Yakubov (1963) except that the intrinsically three-dimensional nature of the present momentum space leads to the appearance of the indices $q = 3/2, 5/2, \ldots$ on the functions F_q in place of the values $q = 1, 2, \ldots$ which result if Shkarofsky's (1966) approach is applied to a strictly two-dimensional Maxwellian distribution.

4. Summary

We have derived the dielectric tensor of a weakly relativistic, anisotropic, magnetized plasma whose distribution function can be separated into the product of a momentum distribution $f(p)$ and an angular distribution $\Phi(\mu)$; the result is expressed in terms of an integral over $\Phi(\mu)$. In addition we have considered a number of special cases of distribution function, frequency and propagation angle where simpler versions of our results apply. A summary of these results is as follows:

(i) Equations (10), (12) and (13) present the dielectric tensor of a weakly relativistic, magnetized plasma in the most general form considered in this paper—in terms of the functionals $G_s(r)$— while $f(p)$ and $\Phi(\mu)$ are otherwise unrestricted. The FLR and relativistic effects are retained throughout for harmonics $|s| \leqslant 2$, thereby avoiding a number of restrictive assumptions made in existing treatments.

(ii) Equations (15), (18) and (19) for the functionals G_s and I_s are valid for the broad class of $f(p)$ represented by equation (14). These results enable the dielectric tensor to be written in the form (22) which is suitable for numerical calculations involving arbitrary pitch angle distributions satisfying $\Phi(\mu) = \Phi(-\mu)$.

(iii) The dielectric tensor is discussed explicitly in Section 3 in the following special cases of distribution function and/or wavevector:

(*a*) isotropic Maxwellian distribution,

(*b*) quasi-perpendicular propagation,

(*c*) sharp-edged loss cone distribution for quasi-perpendicular propagation.

The integrals required to evaluate the dielectric tensor of a generalized DGH distribution are performed in Appendix 2.

(iv) Appendix 1 summarizes the properties of the functions $\mathscr{F}_q$ in terms of which the dielectric tensor is expressed, including the relationship of these functions to the usual plasma dispersion function.

The results obtained in this paper may be used in the investigation of a variety of electron cyclotron phenomena (e.g. propagation and absorption of waves, maser action, current drive etc.) in isotropic and anisotropic plasmas in both laboratory and astrophysical contexts. In forthcoming papers we shall use these results to discuss relativistic effects on accessibility conditions for cyclotron waves in hot, anisotropic plasmas and to investigate modifications to cold plasma modes due to thermal effects and anisotropy.

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Appendix 1. Special Functions

We list a number of the most important properties of the special functions used in this paper.

Definitions, Notation

The generalized Shkarofsky functions appropriate to the present analysis are defined thus if $\mathrm{Im}(z-a) > 0$ (Krivenski and Orefice 1983):

$$\mathscr{F}_{q,r}(z,a) = -\mathrm{i}\int_0^\infty \mathrm{d}t\,(\mathrm{i}t)^r(1-\mathrm{i}t)^{-q}\exp\{\mathrm{i}zt - at^2/(1-\mathrm{i}t)\}\,. \tag{A1}$$

Analytic continuation is used to extend this definition to $\mathrm{Im}(z-a) \leqslant 0$. The corresponding Dnestrovskii functions (Dnestrovskii *et al.* 1964) are defined by

$$F_{q,r}(z) = \mathscr{F}_{q,r}(z,0)\,. \tag{A2}$$

We make the following identifications for notational convenience:

$$\mathscr{F}_q(z,a) = \mathscr{F}_{q,0}(z,a); \qquad F_q(z) = F_{q,0}(z)\,. \tag{A3}$$

Interrelations

Equation (A1) immediately implies that the generalized Shkarofsky functions $\mathscr{F}_{q,r}$ may be re-expressed in terms of the usual Shkarofsky functions ($r = 0$), with

$$\mathscr{F}_{q,r}(z,a) = \sum_{j=0}^{r} \frac{(-1)^j r!}{j!\,(r-j)!} \mathscr{F}_{q-j}(z,a). \tag{A4}$$

Krivenski and Orefice (1983) derived the following recursion relation for the functions $\mathscr{F}_q$:

$$a\mathscr{F}_{q+2}(z,a) = 1 + (a-z)\mathscr{F}_q(z,a) - q\mathscr{F}_{q+1}(z,a); \quad q \geqslant \tfrac{1}{2}. \tag{A5}$$

If $a = 0$, (A5) contains the better known recursion relation for the Dnestrovskii functions as a special case, namely

$$qF_{q+1}(z) = 1 - zF_q(z). \tag{A6}$$

Connection with the Plasma Dispersion Function

The well-known plasma dispersion function $Z(u)$ (Fried and Conte 1961) may be defined for arbitrary complex u by

$$Z(u) = \mathrm{i}\pi^{\frac{1}{2}} \exp(-u^2)\{1+\mathrm{erf}(\mathrm{i}u)\}. \tag{A7}$$

Krivenski and Orefice (1983) derived the following expressions for $\mathscr{F}_{1/2}$ and $\mathscr{F}_{3/2}$ in terms of Z:

$$\mathscr{F}_{1/2}(z,a) = -\mathrm{i}Z^+/(z-a)^{\frac{1}{2}}, \qquad \mathscr{F}_{3/2}(z,a) = -Z^-/a^{\frac{1}{2}} \tag{A8a, b}$$

with

$$Z^{\pm} = \tfrac{1}{2}[Z\{a^{\frac{1}{2}} + \mathrm{i}(z-a)^{\frac{1}{2}}\} \pm Z\{-a^{\frac{1}{2}} + \mathrm{i}(z-a)^{\frac{1}{2}}\}]. \tag{A9}$$

In practice only one root $a^{\frac{1}{2}}$ is relevant in (A8) and (A9) since the other root leads to identical expressions for $\mathscr{F}_{1/2}(z,a)$ and $\mathscr{F}_{3/2}(z,a)$. In general, however, both roots $(z-a)^{\frac{1}{2}}$ must be considered, implying that $\mathscr{F}_q(z,a)$ is double-valued with a branch point at $z = a$ [or, equivalently, that $\mathscr{F}_q(z,a)$ is defined on the same two-sheeted Riemann surface as $(z-a)^{\frac{1}{2}}$]. Equations (A5), (A8) and (A9) together enable the functions $\mathscr{F}_q$ of half-odd-integer index to be re-expressed in terms of the plasma dispersion function.

Limiting Forms

Equations (A5), (A7) and (A8) imply the following limiting forms for the functions $\mathscr{F}_q$ (Krivenski and Orefice 1983):

$$\mathscr{F}_q(z,a) = F_q(z), \qquad |a| \ll 1, \tag{A10}$$

$$= -\tfrac{1}{2}a^{-\frac{1}{2}} Z(\tfrac{1}{2}za^{-\frac{1}{2}}), \qquad 1,\ |z| \ll |a|,\ -\tfrac{5}{4}\pi < \arg(a^{\frac{1}{2}}) < \tfrac{1}{4}\pi, \tag{A11}$$

$$= z^{-1}, \qquad 1,\ |a| \ll |z|,\ |\arg(z)| < \tfrac{3}{2}\pi. \tag{A12}$$

Appendix 2. Evaluation of Integrals

Here we evaluate the integrals required to obtain the dielectric tensor of the generalized DGH distribution given by (14) and (23). These integrals are of the form (24) (with 25). By using equations (15) and (18) for the functionals G_s and I_s and the recursion relation for the Hermite polynomials

$$H_{r+1}(z) = 2zH_r(z) - 2rH_{r-1}(z), \tag{A13}$$

the integral Q in (24) may be expressed in terms of a linear combination of integrals of the form (26). Straightforward manipulation then yields the result

$$Q = -\tfrac{1}{2}\mathrm{i}\pi^{\frac{1}{2}}\zeta^{2-\frac{1}{2}r}(-\mathrm{i})^L \frac{2^{\frac{1}{2}r-L}\Gamma(g+j)}{\Gamma(b-j+\frac{1}{2})\Gamma(j+1)} \int_0^\infty \mathrm{d}t \frac{\exp(\mathrm{i}z_s t - \alpha^2)}{(1-\mathrm{i}t)^{\frac{1}{2}(r+3)+b}}$$

$$\times[\{g+\mathrm{i}jt(1-n_{\|}^2)\}H_L(\alpha) - (\mathrm{i}n_{\|}/\zeta)(2\xi)^{\frac{1}{2}}(2gb-jr)H_{L-1}(\alpha)] \tag{A14}$$

with

$$L = r-g+2(b-j), \qquad \alpha = n_{\|}\zeta^{\frac{1}{2}}t/2^{\frac{1}{2}}(1-\mathrm{i}t)^{\frac{1}{2}}. \tag{A15a, b}$$

Equation (21) may then be used to obtain an explicit expression for ϵ_{ij} for the generalized DGH distribution.

Important special cases in which (A14) simplifies significantly include

(i) DGH distribution: $b = j$, $L = r-2g = 0, 1, 2$.

(ii) Maxwellian distribution: $b = j = 0$, $L = r-2g = 0, 1, 2$.

(iii) Quasi-perpendicular propagation: $n_{\|} \approx 0$, $\alpha \approx 0$. In this case the only values of i and j of interest are those for which $r-2g$ is even; this implies that $H_{L-1}(\alpha) \approx H_{L-1}(0) = 0$.

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