Aust. J. Phys., 1986, 39, 339–46

The Large Numbers Hypothesis and a Relativistic Theory of Gravitation

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Abstract

A way to reconcile Dirac's large numbers hypothesis and Einstein's theory of gravitation was recently suggested by Lau (1985). It is characterized by the conjecture of a time-dependent cosmological term and gravitational term in Einstein's field equations. Motivated by this conjecture and the large numbers hypothesis, we formulate here a scalar–tensor theory in terms of an action principle. The cosmological term is required to be spatially dependent as well as time dependent in general. The theory developed is applied to a cosmological model compatible with the large numbers hypothesis. The time-dependent form of the cosmological term and the scalar potential are then deduced. A possible explanation of the smallness of the cosmological term is also given and the possible significance of the scalar field is speculated.

1. Introduction

In the past few decades, there have been numerous suggestions based on different arguments that the gravitational term G is indeed time dependent (Dirac 1938; Brans and Dicke 1961; Hoyle and Narlikar 1964). Dirac first explored this interesting and provocative possibility in 1937 motivated by the numerology uncovered by Weyl, Eddington and Dirac himself.

Dirac noticed that the ratio of the electrical force to the gravitational force operating between a proton of mass m_p and an electron of mass m_e, i.e. $e^2/Gm_e m_p$, is a large dimensionless number of the order of 10^{40}. Similarly, the age of the Universe t, expressed in terms of a unit constructed from the atomic constants (for instance $e^2/m_e c^2$), is roughly of the same size. This led Dirac to suggest that

$$e^2/Gm_e m_p \sim t, \tag{1}$$

where $\sim$ indicates of the same order of magnitude. Assuming that (1) holds at all times after the instant of the Big Bang and that the atomic parameters do not vary with time, equation (1) implies that $G \propto t^{-1}$.

Dirac put the above semi-quantitative argument on a formal footing through his large numbers hypothesis (LNH) which states that:

0004-9506/86/030339$02.00

> Any two of the very large numbers occurring in Nature are connected by a simple mathematical relation in which the coefficients are of the order of unity.

A time-dependent G then follows as a natural consequence of the LNH. Since the value of t in (1) varies with the epoch, the LNH requires that other large dimensionless numbers must also vary with the epoch. However, Einstein's theory of gravitation requires G to be a constant and independent of any coordinates, in contradiction with the LNH.

Since Dirac's early work, several attempts have been made to reconcile this apparent contradiction (Jordan 1947, 1959; Gamow 1948; Dirac 1979; Recami 1983). Recently, one of us (Lau 1985) proposed a resolution of the contradiction by considering Einstein's field equations with a nonzero cosmological term λ, with λ and G assuming time-dependent forms in the field equations in order to satisfy the LNH. In the present work, we attempt to construct a scalar–tensor theory which has its basis in the LNH and the tentative time-dependent λ and G conjecture.

In this paper we adopt the notations and conventions that the tensor field $g_{\mu\nu}$ is of signature -2. The speed of light c is set to unity, we take $x^0 = t$, and x^k $(k = 1,2,3)$ are spatial coordinates; following the usual practice, the subscripts $,\mu\nu$ and $:\mu\nu$ represent the repeated partial and covariant derivatives respectively.

2. Generalized Field Equations

In this section we shall briefly review the time-dependent λ and G conjecture proposed by Lau (1985). Einstein's field equations, with the presence of a cosmological term λ, take the form

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}, \tag{2}$$

where $R_{\mu\nu}$ is the contracted Riemann–Christoffel tensor, R is the curvature scalar and $T_{\mu\nu}$ is the energy–momentum tensor. According to the LNH, G is a function of time, hence the divergence of the contravariant form of (2) yields

$$-(8\pi G T^{\mu\nu})_{:\nu} = -8\pi(G T^{\mu\nu}{}_{:\nu} + T^{\mu\nu}\dot{G}),$$

where the dot denotes $\partial/\partial t$. Conservation of energy and momentum requires $T^{\mu\nu}{}_{:\nu} = 0$, which implies that

$$-(8\pi G T^{\mu\nu})_{:\nu} = -8\pi T^{\mu 0}\dot{G} \neq 0,$$

unless G is a constant. However, it is always true that

$$(R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R + \lambda g^{\mu\nu})_{:\nu} = 0$$

by the Bianci identities, assuming that λ is a constant. Hence, a time-dependent G is inadmissible with the usual interpretation of Einstein's field equations.

In order to reconcile the LNH with Einstein's theory, it was postulated that the cosmological term λ in (2) be nonzero and also time dependent. The covariant divergence of (2), instead of being of zero, then becomes

$$g^{\mu 0}\dot{\lambda} = 8\pi T^{\mu 0}\dot{G}$$

and the contradiction may be resolved.

To preserve the success of Einstein's theory in explaining local gravitational phenomena, we note that $G \propto t^{-1}$ which implies $|\dot{G}/G| \sim 10^{-10}\,\mathrm{yr}^{-1}$.* Moreover, it was shown by a cosmological model compatible with the LNH that $\lambda = -\frac{1}{3}t^{-2}$ (Lau 1985), and hence that $|\dot{\lambda}/\lambda| \sim 10^{-10}\,\mathrm{yr}^{-1}$ as well. It is seen that the time variation deduced for λ and G is very small at the present epoch and will be even smaller in the future, so that both of these parameters will appear approximately constant over a sufficiently long period of time subsequent to the early epoch. In the light of this consequence, the usual (Einsteinian) interpretation of (2) would retain its essential validity. Furthermore, the value of λ is very small at present (for $t \sim 6\times 10^{17}$ s, $\lambda = -\frac{1}{3}t^{-2}$ gives $\lambda \sim 10^{-36}\,\mathrm{s}^{-2}$) and will be even smaller in the future, so that its influence is negligible on the scale of our solar system and can be ignored in the description of local gravitational phenomena. Hence, for most non-cosmological purposes, we can employ the familiar Einstein field equations

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu}.$$

The implication of a time-varying λ and G will become important only when the history and evolution of the Universe is considered, particularly in its early stages.

3. Lagrangian Formulation

As a starting point for a variational principle formulation of the theory, we consider the usual gravitational action

$$I = \int \surd(-g)(R - 2\lambda - 16\pi G L_{\mathrm{m}})\,\mathrm{d}^4x, \tag{3}$$

where L_{m} is the matter Lagrangian density including all non-gravitational fields. If λ and G are time dependent, we have to treat them as dynamical variables having the same importance as the components of the metric tensors. Hence, by definition, one must be able to vary them in the action (3). However, varying λ or G in (3) only gives $\surd(-g) = 0$ which is clearly not permissible. To incorporate λ and G indirectly into the action, the simplest way is to employ a time-dependent scalar field ψ, following the method of Jordan (1947), and to couple λ and G with it such that

$$\psi = \psi(t), \quad G = G(\psi), \quad \lambda = \lambda(\psi).$$

We assume a Lagrangian density of the form

$$\mathscr{L}_{\mathrm{I}} + \mathscr{L}_{\mathrm{G}},$$

* Hellings *et al.* (1983) gave an experimental upper limit of $(0\cdot 4\pm 0\cdot 2)\times 10^{-11}\,\mathrm{yr}^{-1}$ for $|\dot{G}/G|$ which seems at first sight incompatible with Dirac's LNH. However, due to the uncertainty of the Hubble constant, $|\dot{G}/G| \sim 10^{-10}\,\mathrm{yr}^{-1}$ can at best be regarded as an estimate rather than a precise value in Dirac's cosmology and it may well be considerably lower than this present estimate.

where $\mathscr{L}_G$ represents the contribution from the gravitational fields alone, and $\mathscr{L}_I$ is the interaction Lagrangian density which includes the coupling of gravitational fields with all matter and non-gravitational fields. If we assume the field equations are at most of second differential order then, as shown by Bergmann (1968), the most general form of $\mathscr{L}_G$ in the presence of a scalar field is

$$\mathscr{L}_G = \sqrt{(-g)}\{h(\psi)R + l(\psi)g^{\mu\nu}\psi_{,\mu}\psi_{,\nu} - 2\lambda\},$$

where $h(\psi)$, $l(\psi)$ and λ are arbitrary functions of ψ. For our purposes, we choose $h(\psi) = l(\psi) = 1$, and take

$$\mathscr{L}_I = -16\pi G L_m,$$

where G is understood to be a variable and L_m is assumed to be a function of $g_{\mu\nu}$ only. We then have

$$I = \int \sqrt{(-g)}(R - 2\lambda + g^{\mu\nu}\psi_{,\mu}\psi_{,\nu} - 16\pi G L_m)\,d^4x. \tag{4}$$

Variation of (4) with respect to $g_{\mu\nu}$ yields

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} - \psi_{,\mu}\psi_{,\nu} + \tfrac{1}{2}g_{\mu\nu}g^{\rho\nu}\psi_{,\rho}\psi_{,\nu},$$

i.e.

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + g_{\mu\nu}(\lambda - \tfrac{1}{2}g^{\rho\nu}\psi_{,\rho}\psi_{,\nu}) = -8\pi G T_{\mu\nu} - \psi_{,\mu}\psi_{,\nu}, \tag{5}$$

where $T_{\mu\nu}$ is the energy–momentum tensor, given by

$$T_{\mu\nu} = \frac{2}{\sqrt{(-g)}}\frac{\partial}{\partial g^{\mu\nu}}\{\sqrt{(-g)}L_m\}.$$

At this point, we further define a generalized cosmological parameter

$$\Lambda = \lambda - \tfrac{1}{2}g^{\rho\nu}\psi_{,\rho}\psi_{,\nu}.$$

Since ψ is a function of time only, we write

$$\begin{aligned}\psi_{,\rho}\psi_{,\nu} &= \dot{\psi}^2, \qquad \rho = \nu = 0\\ &= 0, \qquad \text{otherwise}.\end{aligned}$$

Hence we have

$$\Lambda = \lambda - \tfrac{1}{2}g^{00}\dot{\psi}^2 \tag{6}$$

and (5) becomes

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} - \psi_{,\mu}\psi_{,\nu}, \tag{7}$$

which are the generalized field equations.

We note that Λ replaces λ as a cosmological term in the field equations. We also see from (6) that, contrary to our original proposal that the cosmological term is only time dependent, Λ is a function of spatial coordinates as well as time due to the presence of the term g^{00} which is in general a function of four coordinates. Furthermore, the extra term $\dot{\psi}^2$ appears in the (0, 0) component of the field equations as a result of the introduction of an extra degree of freedom (namely ψ) into these equations. The gravitational interaction is now characterized by the ten metric tensor components (employed by Einstein's theory) as well as by a scalar potential. Einstein's field equations become a special case of (7) when ψ is constant or the time variation of ψ can be considered as negligible.

Variation of (4) with respect to ψ yields

$$\Box\psi+\lambda'+8\pi G' L_{\mathrm{m}} = 0$$

where

$$\lambda' = \mathrm{d}\lambda/\mathrm{d}\psi, \qquad G' = \mathrm{d}G/\mathrm{d}\psi, \qquad \Box\psi = g^{\mu\nu}\psi_{:\mu\nu}.$$

Writing

$$\lambda' = \dot{\lambda}/\dot{\psi}, \qquad G' = \dot{G}/\dot{\psi},$$

we have

$$\dot{\psi}\Box\psi+\dot{\lambda}+8\pi \dot{G} L_{\mathrm{m}} = 0.$$

Invoking (6) we then get

$$\dot{\psi}\Box\psi+\dot{\Lambda}+\tfrac{1}{2}\dot{g}^{00}\dot{\psi}^2+g^{00}\dot{\psi}\ddot{\psi}+8\pi\dot{G}L_{\mathrm{m}} = 0, \tag{8}$$

which is the field equation for ψ. We note that Λ and G are not independent in (8) but are related by

$$g_{\mu\nu}\Lambda_{,\nu} = -8\pi\dot{G}T_{\mu 0} - (\psi_{,\mu}\psi_{,\nu})_{:\nu},$$

which is derived by taking the covariant divergence of (7). The results (7) and (8) are the fundamental equations of our theory.

4. Application to Cosmology

It was shown by Dirac (1938, 1979) that, if the Universe is assumed to be spatially homogeneous and isotropic, the simplest cosmological model compatible with the LNH is characterized by the metric

$$\mathrm{d}s^2 = (\mathrm{d}x^0)^2 - R^2(t)\{(\mathrm{d}x^1)^2+(\mathrm{d}x^2)^2+(\mathrm{d}x^3)^2\}, \tag{9a}$$

with

$$R^2(t) = \beta t^{2/3}, \tag{9b}$$

where β is a proportional constant, x^1, x^2, x^3 are co-moving coordinates and $R(t)$ is the scale factor governing the rate of expansion of the Universe. Since the cosmological pressure is negligibly small, the energy–momentum tensor can be taken as that of a pressure-free perfect fluid, that is

$$T_{\mu\nu} = \rho V_\mu V_\nu,$$

where ρ is the proper matter density of the Universe and V_μ is a velocity four-vector which has components $(1,0,0,0)$ in a co-moving frame.

From (9a) and (9b), we have

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -\beta t^{2/3}, \qquad g_{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu .$$

It then follows that

$$g^{00} = 1, \quad g^{11} = g^{22} = g^{33} = -\beta^{-1} t^{-2/3}, \quad g^{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu ,$$

and that the non-vanishing Christoffel symbols are given by

$$\Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = \tfrac{1}{3}\beta t^{-1/3},$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \tfrac{1}{3} t^{-1},$$

which, in turn, implies that

$$R_{00} \equiv R = -\tfrac{2}{3} t^{-2} .$$

The $(0,0)$ component of (7) is then

$$-\tfrac{1}{3} t^{-2} + \Lambda = -8\pi G T_{00} - \dot{\psi}^2 . \tag{10}$$

In a moving frame in respect to which matter is at rest we have $T_{00} = \rho$. Equation (10) then becomes

$$\Lambda = \tfrac{1}{3} t^{-2} - 8\pi G\rho - \dot{\psi}^2 . \tag{11}$$

Also, we have from the deductions of the LNH (Dirac 1938, 1979) that

$$G = \beta_1 t^{-1}, \qquad \rho = \beta_2 t^{-1}, \tag{12a, b}$$

where β_1 and β_2 are constants, and hence

$$\Lambda = \tfrac{1}{3} t^{-2} - 8\pi\beta_1 \beta_2 t^{-2} - \dot{\psi}^2 . \tag{13}$$

Furthermore, in the context of our cosmological model, (8) now becomes

$$2\dot{\psi}\ddot{\psi} + \dot{\Lambda} + 8\pi \dot{G}\rho = 0 ,$$

where $L_m = \rho$ in this case. Thus, with the help of (12a) and (12b), we have

$$\dot{\Lambda} = 8\pi\beta_1 \beta_2 t^{-3} - \mathrm{d}\dot{\psi}^2/\mathrm{d}t$$

and, as a result,

$$\Lambda = -4\pi\beta_1 \beta_2 t^{-2} - \dot{\psi}^2 + f , \tag{14}$$

where f is an arbitrary function of the spatial coordinates.

Comparing (11) and (14), we find $f = 0$ and $\beta_1 \beta_2 = 1/12\pi$. It then follows that

$$\Lambda = -\tfrac{1}{3}t^{-2} - \dot{\psi}^2 . \tag{15}$$

Compared with the previously derived equation $\Lambda = -\frac{1}{3}t^{-2}$ (Lau 1985), the time-dependent form of Λ is now modified by the presence of the term $-\dot{\psi}^2$ due to the introduction of a scalar potential ψ into our theory.

To further determine the functional form of Λ and ψ, we see from experimental results (Ohanian 1976) that $|\Lambda| < 10^{-35}\,\mathrm{s}^{-2}$ which is very small. The reciprocal of Λ, expressed in atomic units, then becomes a large dimensionless number which varies with the epoch. Thus, according to the LNH, we must have $\Lambda \propto t^n$ for some (negative) real number n; this implies

$$-\tfrac{1}{3}t^{-2} - \dot{\psi}^2 \propto t^n .$$

Assuming that ψ has a simple time-dependent form, it must then follow that

$$\Lambda \propto t^{-2}, \tag{16}$$

and that one of the following cases holds: (i) $\dot{\psi} = 0$, (ii) $\dot{\psi} =$ constant or (iii) $\dot{\psi}^2 \propto t^{-2}$. Case (i) is immediately ruled out because ψ is time dependent. For case (ii), we have $\dot{\psi} = \alpha$ for some constant α. If $\alpha = 0$, then case (ii) is identical to case (i). Let us consider $\alpha \neq 0$. We note that Einstein's theory of gravitation is required to be a limiting case of our theory when the Universe is sufficiently old, i.e. we require $\dot{\psi} \to 0$ as the age of the Universe increases indefinitely. This is clearly inconsistent with $\dot{\psi} = \alpha \neq 0$. We are thus left with $\dot{\psi}^2 \propto t^{-2}$. This implies $\psi \propto \ln |t|$ since a time close enough to the Big Bang, where $|t|$ is the magnitude of the epoch.

Also (15) and (16) imply

$$\Lambda = -\beta_3 t^{-2}, \tag{17}$$

where β_3 is a positive constant.

The reason(s) why Λ, if nonzero, is so small has been a puzzling question ever since Einstein first introduced it into his field equations. A possible explanation of this problem follows as an interesting consequence of our present work. In the argument above, we deduced that although Λ is, in general, a function of spatial coordinates and time, in a Dirac-type homogeneous, isotropic and expanding Universe with zero space curvature, Λ is time dependent only. From (17) we also note that Λ is a decreasing function of the epoch and is always negative. As a result, we are led to speculate that Λ is very small at the present epoch because the Universe is sufficiently old (for example, for $t \sim 6\times10^{17}$ s at present and $\beta_3 \sim 1$ as required by the LNH, we have $\lambda \sim 10^{-35}\,\mathrm{s}^{-2}$). Furthermore, the magnitude of Λ will be decreasing in the future as the age of the Universe increases.

5. Concluding Remarks: A Possible Physical Significance of the Scalar Field

We conclude this paper with a highly speculative remark on the physical meaning of the scalar field. Let us consider the case of a gravitational field in empty space so that $T_{\mu\nu} = 0$ and (7) can then be rewritten as

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu} = -(\Lambda g_{\mu\nu} + \psi_{,\mu}\psi_{,\nu}).$$

In terms of modern quantum electrodynamics, an empty space is no longer considered as an inert region but rather as a place giving rise continually to quantum fluctuations. Thus the vacuum itself also has energy and momentum content. The term $-(\Lambda g_{\mu\nu} + \psi_{,\mu}\psi_{,\nu})$ might then be interpreted as the energy–momentum tensor of the vacuum in the presence of a gravitational field. It is well known that Λ may be considered to represent a measure of the constant background energy density and pressure of the vacuum. But we ask what is the connection of ψ with the vacuum?

We observe that $\psi_{,\mu}\psi_{,\nu} \neq 0$ only when $\mu = \nu = 0$. Hence, in a co-moving frame, ψ is related only to the energy density of the vacuum and possibly represents the average strength of the deviations of energy density from that of the constant background. From quantum electrodynamics, we know that the vacuum can be distorted (or polarized) by the presence of an electric field which causes an alternation in the energy of the vacuum. Such an effect, in principle, could also be produced by a gravitational field. So, it is possible that ψ characterizes in some way the distortion of the vacuum by a gravitational field. This effect, if it exists, would have manifested itself significantly in the early stages of the Universe when matter was extremely compressed and the gravitational field was immensely intense. Like the cosmological term, the scalar field might then also be a clue in linking up macrophysics and microphysics.

Acknowledgments

One of us (Y.K.L.) would like to thank Professor P. G. Bergmann for his helpful suggestions. We are also grateful to Mr J. R. Shepanski for reading the manuscript.

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Manuscript received 22 October 1985, accepted 30 January 1986