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Alternative Scheme of Spontaneous Chiral Symmetry Breaking

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Abstract

We develop an alternative scheme of spontaneous chiral symmetry breaking which is characterized by four-quark condensates instead of $\langle\bar{q}q\rangle\neq 0$. In this scheme the Nambu–Goldstone bosons acquire a mass squared $\sim m_{\text{quark}}^2$, in comparison with m_{quark} in the usual scheme. The quark mass ratios and the parameters of the scheme are determined by an application to the pseudoscalar nonet spectrum (including π^0–η–η' mixing). The decays $\psi\to(\pi^0,\eta,\eta')\gamma$ and $\psi'\to\psi(\pi^0,\eta)$ are also considered. The results do not promote the alternative scheme.

1. Introduction and Motivation

Quantum chromodynamics (QCD) (Fritzsch *et al.* 1973; Marciano and Pagels 1978)—the present day theory of the strong interactions—has a global flavour (colour independent) $U(L)\times U(L)$ chiral symmetry when L of the quark mass parameters in the Lagrangian density are set equal to zero. In reality the quark mass parameters do not vanish and these chiral symmetries are only approximate (see for instance the reviews by Pagels 1975 and Christos 1984*a*).

Phenomenologically the two most interesting cases are $L = 2$ or 3, corresponding to small values of m_u (m_1), m_d (m_2) and m_s (m_3) on the scale of the strong interactions (≈ 1 GeV). Our considerations are based on chiral $U(3)\times U(3)$ symmetry, associated with the transformations (where q is both a flavour and colour 3-vector)

$$q\to\exp(\mathrm{i}\,\boldsymbol{\alpha}\,.\,\boldsymbol{\lambda})q\,,\qquad q\to\exp(\mathrm{i}\,\alpha)q\,,$$

$$q\to\exp(\mathrm{i}\,\boldsymbol{\beta}\,.\,\boldsymbol{\lambda}\gamma_5)q\,,\qquad q\to\exp(\mathrm{i}\,\beta\gamma_5)q\,,\tag{1}$$

where the $\boldsymbol{\lambda}$ are the usual Gell-Mann SU(3) matrices satisfying ($a, b = 1,\dots, 8$)

$$[\lambda^a,\lambda^b] = 2\mathrm{i}f^{abc}\lambda^c\,,\qquad \mathrm{tr}(\lambda^a\lambda^b) = 2\delta^{ab}\,.\tag{2}$$

Since the consequences of a manifest axial symmetry (i.e. approximately equal mass parity partners and near massless baryons) are not observed in Nature, the chiral $U(3)\times U(3)$ symmetry is believed to be spontaneously broken down to U(3) vector symmetry. The Nambu–Goldstone (NG) bosons associated with this spontaneous breaking are thought to be the lowest lying pseudoscalar nonet (π, K, η, η').

In the usual Gell-Mann *et al.* (1968) and Glashow and Weinberg (1968) scheme this spontaneous breaking is assumed to occur through the non-vanishing condensate (in the chiral limit)

$$\langle \bar{q}q \rangle \equiv \langle \bar{q} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} q \rangle = \mathrm{O}(1). \tag{3}$$

In this scheme the NG bosons acquire a mass squared of $\mathrm{O}(m_q)$ [see for instance equation (A1) in the Appendix, where many of the results of the usual scheme relevant for comparison in this paper are summarized].

A number of authors (Scadron and Jones 1974; Sazdjian and Stern 1975; Fuchs 1980, 1981; Scadron 1981, 1983) have proposed an alternative to this scheme of spontaneous chiral symmetry breaking (SCSB) where instead $m_{\mathrm{NG}}^2 = \mathrm{O}(m_{\mathrm{quark}}^2)$. The idea stems from the observation that the equation (where $\mathscr{F}^a_{\mu 5}$ is the axial–vector current $\bar{q}\gamma_\mu \gamma_5 \frac{1}{2}\lambda^a q$)

$$\begin{aligned} \langle 0 | \partial^\mu \mathscr{F}^a_{\mu 5}(0) | \pi^b(q) \rangle &= \langle 0 | \mathrm{i} \sum_{i,j=1}^{3} \bar{q}_i (m_i + m_j) \gamma_5 \frac{(\lambda^a)_{ij}}{2} q_j | \pi^b(q) \rangle \\ &= m_{\pi^a}^2 F_\pi \delta_{ab} \end{aligned} \tag{4}$$

can be satisfied with the pion decay constant $F_\pi = \mathrm{O}(1)$ (a necessary and seemingly sufficient condition for SCSB), if $m_{\mathrm{NG}}^2 = \mathrm{O}(m_q^2)$ and $\langle 0 | \bar{q}\gamma_5 \lambda^a q | \pi^b \rangle = \mathrm{O}(m_q)$ [cf. the usual scheme where $F_\pi = \mathrm{O}(1)$, $m_{\mathrm{NG}}^2 = \mathrm{O}(m_q)$ and $\langle 0 | \bar{q}\gamma_5 \lambda^a q | \pi^b \rangle = \mathrm{O}(1)$]. These schemes are usually characterized *in the literature* by the condensates

$$\langle \bar{u}u \rangle = c m_{\mathrm{u}}, \quad \langle \bar{d}d \rangle = c m_{\mathrm{d}}, \quad \langle \bar{s}s \rangle = c m_{\mathrm{s}}, \tag{5}$$

where $c = \mathrm{O}(1)$ is independent of the quark mass parameters to leading order.

Since the condensates in (5) vanish in the chiral limit it is unclear whether there is any SCSB at all. These schemes are resurrected by the observation (Crewther 1979) that (i is not summed over)

$$\frac{\partial}{\partial m_i} \langle Op(0) \rangle = -\mathrm{i} \int \mathrm{d}^4 x \, T \langle Op(0) \bar{q}_i q_i(x) \rangle^{\mathrm{connected}}. \tag{6}$$

Equation (6) follows directly in the path integral representation with the explicit symmetry breaking interaction in the Lagrangian density

$$\mathscr{L}' = -\bar{q} \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix} q. \tag{7}$$

This leads to the notion that these alternative schemes may be more precisely formulated in terms of four-quark condensates which do not vanish in the chiral limit, for example

$$\int_x T\langle \bar{q}q(x) . \bar{q}q(0)\rangle = O(1) .$$

Besides the alternative scheme mentioned above there are other possibilities corresponding to other values of $p \geqslant 1$ with

$$F_\pi = O(1), \quad m_{NG}^2 = O(m_q^p), \quad \langle 0|\bar{q}\gamma_5 \lambda^a q|\pi^b\rangle = O(m_q^{p-1}),$$
$$\int ... \int \langle(\bar{q}q)^p\rangle = O(1), \quad \langle \bar{q}q\rangle = O(m_q^{p-1}). \tag{8}$$

Our considerations here are limited to the simplest such alternative, corresponding to $p = 2$.

The motivation to consider such alternative schemes is twofold. Firstly, and most importantly, the usual scheme seems to have some difficulty in properly accounting for deviations from certain chiral limit theorems. The principal concern is with the Goldberger–Treiman relation (Pagels and Zepeda 1972; Pagels 1975; J. Stern 1982, unpublished), the πN σ-term, and π–π scattering (Gasser and Leutwyler 1984). The π^0–$\pi^\pm$ mass difference is also poorly understood in the usual scheme; corrections due to π^0–η mixing (Gross *et al.* 1979) and Li–Pagels logarithms (Li and Pagels 1971*a*, 1971*b*, 1972; Langacker and Pagels 1973*a*, 1973*b*; Pagels 1975) are of the same sign as the already large electromagnetic splitting calculated by Das *et al.* (1967) of $(m_{\pi^\pm} - m_{\pi^0})_{em} \approx 5 \cdot 3$ MeV. There is also a discrepancy between the strange to non-strange quark mass ratio as obtained from the meson and baryon sectors. The crucial question is whether the alternative scheme can better account for the experimental data. The present paper however is mainly concerned with a consistent *formulation* of the simplest alternative scheme (AS) of SCSB. Some of the above questions (e.g. π–π scattering) have recently been addressed by J. Stern (1982, unpublished) and Crewther (1984).

The alternative schemes are also important from a theoretical point of view. No matter how unlikely,* it remains a logical possibility that $\langle \bar{q}q\rangle \approx 0$. A small value of $\langle \bar{q}q\rangle$ could be protected if the vacuum had some discrete symmetry (e.g. Z_2 axial symmetry). We should also note that recent arguments (see for instance Veneziano 1980) supporting the hypothesis of SCSB in QCD do not specifically imply the usual scheme (see Section 5.4 of Christos 1984*a*). In addition, calculations in lattice gauge theories which seem to imply that $\langle \bar{q}q\rangle = O(1)$ (see for instance Barbour *et al.* 1984; Schierholz 1985) are not particularly convincing since they entail a manifest violation of chiral symmetry and involve numerous uncontrolled approximations (e.g. fermion loop expansion and quenching).

2. Four-quark Condensates

There are four 4-quark condensates which spontaneously break the axial U(3) symmetry but leave the vacuum invariant under U(3) vector symmetry. They are [where $\lambda^0 \equiv \sqrt{\frac{2}{3}}\,\mathrm{diag}(1, 1, 1)$ and where the superscript c denotes the connected

* A parameter is unlikely to be small if there is no symmetry to constrain it from acquiring a 'large' value in higher orders.

amplitude]

$$\int \mathrm{d}^4x\, T\langle \bar{q}\lambda^0 q(x) \,.\, \bar{q}\lambda^0 q(0)\rangle^{\mathrm{c}} \equiv \sigma ,$$

$$\int \mathrm{d}^4x\, T\langle \bar{q}\lambda^a q(x) \,.\, \bar{q}\lambda^b q(0)\rangle^{\mathrm{c}} \equiv \xi\delta_{ab} ,$$

$$\int \mathrm{d}^4x\, T\langle \bar{q}\gamma_5\lambda^0 q(x) \,.\, \bar{q}\gamma_5\lambda^0 q(0)\rangle^{\mathrm{c}} \equiv \sigma' ,$$

$$\int \mathrm{d}^4x\, T\langle \bar{q}\gamma_5\lambda^a q(x) \,.\, \bar{q}\gamma_5\lambda^b q(0)\rangle^{\mathrm{c}} \equiv \xi'\delta_{ab} , \tag{9}$$

or in terms of the u, d and s quark fields

$$\begin{aligned}
\int_x T\langle \bar{u}u(x) \,.\, \bar{u}u(0)\rangle^{\mathrm{c}} &= \int_x T\langle \bar{d}d(x) \,.\, \bar{d}d(0)\rangle^{\mathrm{c}} \\
&= \int_x T\langle \bar{s}s(x) \,.\, \bar{s}s(0)\rangle^{\mathrm{c}} \equiv \rho = \tfrac{1}{6}(\sigma+2\xi) , \\
\int_x T\langle \bar{u}u(x) \,.\, \bar{d}d(0)\rangle^{\mathrm{c}} &= \int_x T\langle \bar{u}u(x) \,.\, \bar{s}s(0)\rangle^{\mathrm{c}} \\
&= \int_x T\langle \bar{d}d(x) \,.\, \bar{s}s(0)\rangle^{\mathrm{c}} \equiv \gamma = \tfrac{1}{6}(\sigma-\xi) , \\
\int_x T\langle \bar{u}d(x) \,.\, \bar{d}u(0)\rangle^{\mathrm{c}} &= \int_x T\langle \bar{u}s(x) \,.\, \bar{s}u(0)\rangle^{\mathrm{c}} \\
&= \int_x T\langle \bar{d}s(x) \,.\, \bar{s}d(0)\rangle^{\mathrm{c}} = \rho-\gamma = \tfrac{1}{2}\xi , \\
\int_x T\langle \bar{u}\gamma_5 u(x) \,.\, \bar{u}\gamma_5 u(0)\rangle^{\mathrm{c}} &= \int_x T\langle \bar{d}\gamma_5 d(x) \,.\, \bar{d}\gamma_5 d(0)\rangle^{\mathrm{c}} \\
&= \int_x T\langle \bar{s}\gamma_5 s(x) \,.\, \bar{s}\gamma_5 s(0)\rangle^{\mathrm{c}} \equiv \rho' = \tfrac{1}{6}(\sigma'+2\xi') , \\
\int_x T\langle \bar{u}\gamma_5 u(x) \,.\, \bar{d}\gamma_5 d(0)\rangle^{\mathrm{c}} &= \int_x T\langle \bar{u}\gamma_5 u(x) \,.\, \bar{s}\gamma_5 s(0)\rangle^{\mathrm{c}} \\
&= \int_x T\langle \bar{d}\gamma_5 d(x) \,.\, \bar{s}\gamma_5 s(0)\rangle^{\mathrm{c}} \equiv \gamma' = \tfrac{1}{6}(\sigma'-\xi') , \\
\int_x T\langle \bar{u}\gamma_5 d(x) \,.\, \bar{d}\gamma_5 u(0)\rangle^{\mathrm{c}} &= \int_x T\langle \bar{u}\gamma_5 s(x) \,.\, \bar{s}\gamma_5 u(0)\rangle^{\mathrm{c}} \\
&= \int_x T\langle \bar{d}\gamma_5 s(x) \,.\, \bar{s}\gamma_5 d(0)\rangle^{\mathrm{c}} = \rho'-\gamma' = \tfrac{1}{2}\xi' .
\end{aligned} \tag{10}$$

If the vacuum was U(3)×U(3) invariant, all of the condensates (9) would be equal (up to a sign); for example, if $Q_5^3|0\rangle = 0$, where Q_5^3 is the U(1) axial charge generator, then

$$\int_x T\langle\overline{q}\lambda^0 q(x)\,.\,\overline{q}\lambda^0 q(0)\rangle^{\mathrm{c}} = -\int_x T\langle\overline{q}\lambda^0 q(x)[Q_5^3,\overline{q}\gamma_5\lambda^0 q(0)]\rangle^{\mathrm{c}}$$

$$= \int_x T\langle[Q_5^3,\overline{q}\lambda^0 q(x)]\overline{q}\gamma_5\lambda^0 q(0)\rangle^{\mathrm{c}} = -\int_x T\langle\overline{q}\gamma_5\lambda^0 q(x)\,.\,\overline{q}\gamma_5\lambda^0 q(0)\rangle^{\mathrm{c}}.$$

Consequently, there are only three relevant symmetry breaking parameters. One combination of the condensates, namely

$$\int_x T\langle\mathrm{tr}(\mathcal{M}(x)\,.\,\mathcal{M}^\dagger(0))\rangle^{\mathrm{c}} = 9(\rho-\rho')-6(\gamma-\gamma'), \tag{11}$$

where $\mathcal{M}_{ij} \equiv \overline{q}_j(1-\gamma_5)q_i$, is the vacuum expectation value of a chiral singlet operator.

It also proves convenient to use Stern's (dimensionless) variables defined by

$$A \equiv \frac{2\mathrm{i}}{F_\pi^2}(\rho+\rho'-\gamma-\gamma'),$$

$$Z_{\mathrm{s}} \equiv \frac{2\mathrm{i}}{F_\pi^2}\gamma, \qquad Z_{\mathrm{p}} \equiv -\frac{2\mathrm{i}}{F_\pi^2}\gamma'. \tag{12}$$

Note that $Z_{\mathrm{s}}(\gamma)$ and $Z_{\mathrm{p}}(\gamma')$ entail violations of Zweig's rule in the 0^+ and 0^- channels respectively. Consequently one expects Z_{s} to be smaller than A, although Z_{p} may be somewhat larger because of the U(1) axial anomaly.

From equations (10) and (6) it follows that

$$\langle\overline{u}u\rangle = -\mathrm{i}m_1\,\rho - \mathrm{i}m_2\,\gamma - \mathrm{i}m_3\,\gamma + \mathrm{O}(m_q^2),$$

$$\langle\overline{d}d\rangle = -\mathrm{i}m_1\,\gamma - \mathrm{i}m_2\,\rho - \mathrm{i}m_3\,\gamma + \mathrm{O}(m_q^2),$$

$$\langle\overline{s}s\rangle = -\mathrm{i}m_1\,\gamma - \mathrm{i}m_2\,\gamma - \mathrm{i}m_3\,\rho + \mathrm{O}(m_q^2). \tag{13}$$

These relations reduce to (5) only if $\gamma = 0$. This would be the case in the large N_{c} (number of colours) limit ('t Hooft 1974; Witten 1980).

3. Pseudoscalar Masses

We consider the two-point function

$$\int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}q.x}\,T\langle\phi_\pi(x)\,\phi_\pi(0)\rangle,$$

where ϕ_π is some operator which can create (and annihilate) a NG pseudoscalar meson (denoted generically by π) with a non-vanishing amplitude in the chiral limit. [For the present purposes the form of ϕ_π does not need to be known, but it is worth mentioning at this stage that the usual two-quark interpolating field $\sim\overline{q}\gamma_5\lambda q$ does not satisfy this condition in the AS of SCSB (see equation 8). In this case ϕ_π will need to be some four-quark pseudoscalar operator.] In the limit $q \to 0$ the dominant contribution to the two-point function comes from the 'pion' (NG) pole, since $m_\pi^2 \to 0$ as $m_q \to 0$. Therefore, we get

$$\int \mathrm{d}^4 x\, \mathrm{e}^{\mathrm{i}q.x}\, T\langle \phi_\pi(x)\, \phi_\pi(0)\rangle \approx |\langle 0|\phi_\pi|\pi\rangle|^2 \frac{\mathrm{i}}{q^2 - m_\pi^2} \{1 + \mathrm{O}(m_q)\}\,. \tag{14}$$

An alternative expression for the two-point function can be obtained by performing a chiral expansion about the U(3)×U(3) limit and saturating the resulting Green's functions with massless 'pion' poles:

$$\begin{aligned}\int \mathrm{d}^4 x\, \mathrm{e}^{\mathrm{i}q.x}\, T\langle \phi_\pi(x)\, \phi_\pi(0)\rangle_{\mathscr{L}} &= \int \mathrm{d}^4 x\, \mathrm{e}^{\mathrm{i}q.x}\, T\langle \phi_\pi(x)\, \phi_\pi(0)\rangle_{\mathscr{L}_0} \\ &+ \int \mathrm{d}^4 x\, \mathrm{e}^{\mathrm{i}q.x} \int \mathrm{d}^4 y_1\; T\langle \phi_\pi(x)\, \phi_\pi(0)\, \mathrm{i}\mathscr{L}'(y_1)\rangle^{\mathrm{c}}_{\mathscr{L}_0} \\ &+ \frac{1}{2!} \int \mathrm{d}^4 x\, \mathrm{e}^{\mathrm{i}q.x} \int \mathrm{d}^4 y_1 \int \mathrm{d}^4 y_2\; T\langle \phi_\pi(x)\, \phi_\pi(0)\, \mathrm{i}\mathscr{L}'(y_1)\, \mathrm{i}\mathscr{L}'(y_2)\rangle^{\mathrm{c}}_{\mathscr{L}_0} \\ &+ \dots,\end{aligned} \tag{15}$$

where $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}'$, $\mathscr{L}_0$ is U(3)×U(3) invariant and $\mathscr{L}'$ is the explicit symmetry breaking quark mass term (7). The subscripts $\mathscr{L}$ and $\mathscr{L}_0$ specify whether the amplitudes are to be calculated in the full non-chiral theory or the exact symmetry limit respectively. [Equation (15) can easily be derived in the path integral representation by expanding the exponential $\exp\{\mathrm{i} \int \mathrm{d}^4 y\, \mathscr{L}'(y)\}$ into a power series in $\mathscr{L}'$.]

In the usual scheme the dominant contribution (as $q \to 0$) to the terms on the right-hand side of (15) come from the diagrams in which the operators are connected by massless 'pions' (see Fig. 1). Although each term is progressively more infrared singular, the sum is infrared finite. Formally summing this series of diagrams gives

$$\begin{aligned}\int \mathrm{d}^4 x\, \mathrm{e}^{\mathrm{i}q.x}\, T\langle \phi_\pi(x)\, \phi_\pi(0)\rangle_{\mathscr{L}} &= |\langle 0|\phi_\pi|\pi\rangle_{\mathscr{L}_0}|^2 \frac{\mathrm{i}}{q^2 + \langle \pi|\mathscr{L}'|\pi\rangle_{\mathscr{L}_0}} \\ &\quad \times \{1 + \mathrm{O}(m_q)\}\,.\end{aligned} \tag{16}$$

Equating this with (14) gives an expression for the masses of the NG bosons ($a, b = 1, \dots, 8, 9$):*

$$m_{ab}^2 = -\langle \pi^a|\mathscr{L}'|\pi^b\rangle_{\mathscr{L}_0} + \mathrm{O}(m_q^2)\,. \tag{17}$$

On using soft 'pion' theorems this reduces to the usual formulae for the NG boson masses (see equation A1).

In the AS of SCSB, the term $\langle \pi|\bar{q}q|\pi\rangle_{\mathscr{L}_0}$ vanishes. [This can best be seen through the Feynman–Hellmann theorem (Pauli 1933; Hellmann 1937; Feynman 1939) which relates $\langle \pi^a|\bar{q}q|\pi^b\rangle$ to $(\partial/\partial m_q)(m_{\pi^a}^2)\delta_{ab} = \mathrm{O}(m_q)$.] Consequently, only the first diagram in Fig. 1 makes a contribution. There is however another set of infrared singular diagrams which now make a leading contribution; these are given

* In deriving (17) we have ignored possible gluonic contributions to the flavour singlet pseudoscalar through the U(1) axial anomaly. Electromagnetic interactions also make a separate contribution.

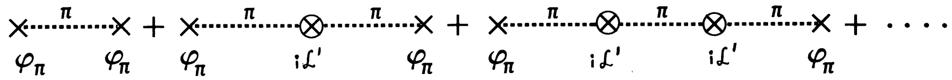

Fig. 1. Leading order 'pion' pole contributions to the right-hand side of equation (15) in the usual scheme of SCSB.

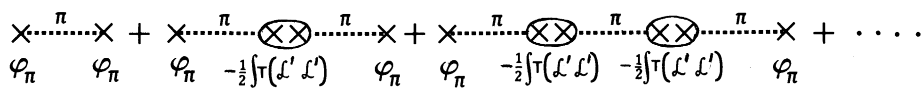

Fig. 2. Leading order 'pion' pole contributions to the right-hand side of equation (15) in the alternative scheme of SCSB.

in Fig. 2. Formally summing this series gives

$$\int \mathrm{d}^4 x\, \mathrm{e}^{\mathrm{i}q.x}\, T\langle \phi_\pi(x)\, \phi_\pi(0)\rangle_{\mathscr{L}} = |\langle 0|\phi_\pi|\pi\rangle_{\mathscr{L}_0}|^2$$

$$\times \left\{ \mathrm{i} \Big/ \left(q^2 + \tfrac{1}{2}\mathrm{i}\langle\pi| \int_x T\{\mathscr{L}'(x)\,\mathscr{L}'(0)\}\,|\pi\rangle^{\mathrm{c}}_{\mathscr{L}_0} \right) \right\}$$

$$\times \{1 + \mathrm{O}(m_q)\}\,. \tag{18}$$

Equating this with (14), we can derive an expression for the masses of the NG bosons (minus anomaly and electromagnetic effects):

$$m^2_{ab} = -\tfrac{1}{2}\mathrm{i}\langle\pi^a| \int_x T\{\mathscr{L}'(x)\,\mathscr{L}'(0)\}\,|\pi^b\rangle^{\mathrm{c}}_{\mathscr{L}_0} + \mathrm{O}(m_q^3)\,. \tag{19}$$

On using the soft 'pion' theorems* and evaluating the commutators one finds that

$$m^2_{ab} \approx \frac{\mathrm{i}}{2F^2_\pi} \int_x \langle [F^a_5, [F^b_5, T\{\mathscr{L}'(x)\,\mathscr{L}'(0)\}]]\rangle^{\mathrm{c}}_{\mathscr{L}_0}$$

$$= \frac{\mathrm{i}}{F^2_\pi} \int_x T\langle \bar{q}\left[\frac{\lambda_a}{2}, \left[\frac{\lambda_b}{2}, \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix}\right]\right] q(x)\,.\,\bar{q}\begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix} q(0)$$

$$+\bar{q}\gamma_5\left[\frac{\lambda_a}{2}, \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix}\right] q(x)\,.\,\bar{q}\gamma_5\left[\frac{\lambda_b}{2}, \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix}\right] q(0)\rangle^{\mathrm{c}}_{\mathscr{L}_0}\,. \tag{20}$$

* The validity and range of applicability of soft 'pion' theorems in the AS of SCSB will be considered in Section 4.

Evaluating the anticommutators and using (10) and (12) leads to the formulae

$$m_{\pi^\pm}^2-(m_{\pi^\pm}^2)_{\rm em} = \tfrac{1}{4}(m_1+m_2)^2A+\tfrac{1}{2}(m_1+m_2)(m_1+m_2+m_3)Z_{\rm s}\,, \tag{21a}$$

$$m_{\rm K^\pm}^2-(m_{\rm K^\pm}^2)_{\rm em} = \tfrac{1}{4}(m_1+m_3)^2A+\tfrac{1}{2}(m_1+m_3)(m_1+m_2+m_3)Z_{\rm s}\,, \tag{21b}$$

$$m_{\rm K^0,\bar K^0}^2 = \tfrac{1}{4}(m_2+m_3)^2A+\tfrac{1}{2}(m_2+m_3)(m_1+m_2+m_3)Z_{\rm s}\,, \tag{21c}$$

$$m_{33}^2 = \tfrac{1}{2}(m_1^2+m_2^2)A+\tfrac{1}{2}(m_1+m_2)(m_1+m_2+m_3)Z_{\rm s} -\tfrac{1}{2}(m_1-m_2)^2Z_{\rm p}\,, \tag{21d}$$

$$m_{38}^2 = \tfrac{1}{2}\surd\tfrac{1}{3}\,(m_1-m_2)\{(m_1+m_2)A+(m_1+m_2+m_3)Z_{\rm s} -(m_1+m_2-2m_3)Z_{\rm p}\}\,, \tag{21e}$$

$$m_{39}^2 = \surd\tfrac{1}{6}\,\alpha(m_1-m_2)\{(m_1+m_2)A+(m_1+m_2+m_3)(Z_{\rm s}-Z_{\rm p})\}\,, \tag{21f}$$

$$m_{88}^2 = \tfrac{1}{6}(m_1^2+m_2^2+4m_3^2)A+\tfrac{1}{6}(m_1+m_2+m_3)(m_1+m_2+4m_3)Z_{\rm s} -\tfrac{1}{6}(m_1+m_2-2m_3)^2Z_{\rm p}\,, \tag{21g}$$

$$m_{99}^2 = \tfrac{1}{3}\alpha^2\{(m_1^2+m_2^2+m_3^2)A +(m_1+m_2+m_3)^2(Z_{\rm s}-Z_{\rm p})\}+\chi^2/N_{\rm c}\,, \tag{21h}$$

$$m_{89}^2 = \tfrac{1}{3}\surd\tfrac{1}{2}\,\alpha\{(m_1^2+m_2^2-2m_3^2)A +(m_1+m_2-2m_3)(m_1+m_2+m_3)(Z_{\rm s}-Z_{\rm p})\}\,, \tag{21i}$$

where in equations (21a) and (21b) we have allowed for an electromagnetic (em) contribution to $m_{\pi^\pm}^2$ and $m_{\rm K^\pm}^2$, while in equations (21f), (21h) and (21i) we have allowed for the fact that the singlet decay constant $F_{\rm s} \equiv F_{\pi^9}$ need not be equal to F_π in the leading order of chiral symmetry breaking (Christos 1984*b*). We have defined $\alpha = F_\pi/F_{\rm s}$. In other words, U(3) vector symmetry does *not* imply that $\alpha = 1$. In addition, we have allowed for a possible 'anomaly' contribution to the singlet–singlet matrix element (for more details see Section 6). The π^0, η and η' are obtained on diagonalizing the π^3–π^8–π^9 mixing matrix, which is also considered in Section 6.

4. Soft 'Pion' Theorems, 'Pion' Pole Dominance and Chiral Ward Identities

We consider the Ward identity (in the chiral limit)

$$\int {\rm d}^4x\,\partial_x^\mu\, T\langle\mathscr{F}_{\mu5}^a(x)\,P(0)\rangle = \langle[F_5^a,\,P(0)]\rangle\,, \tag{22}$$

where $\mathscr{F}_{\mu5}^a$ is the axial–vector current $\bar q\gamma_\mu\gamma_5\,\tfrac{1}{2}\lambda^a q$, F_5^a is the corresponding charge generator $\int \mathrm{d}^3x \mathscr{F}^a_{05}(x)$, and $P(0)$ is some operator such that the vacuum expectation value of the commutator on the right-hand side does not vanish. [In the simplest AS $P(0)$ will have to consist of at least four quark fields.] Saturating the left-hand side of (22) with a 'pion' pole, we derive the soft 'pion' theorem

$$\lim_{q\to 0} \langle 0|P(0)|\pi^a(q)\rangle = -\frac{\mathrm{i}}{F_\pi}\langle [F^a_5, P(0)]\rangle. \tag{23}$$

Goldstone's theorem (i.e. the existence of a massless pseudoscalar meson for every spontaneously broken axial generator) and the fact that $F_\pi = \mathrm{O}(1) \neq 0$ also follow in the course of deriving (23).

The extension of (23) to the non-chiral limit (i.e. nonzero quark masses) is straightforward so long as the vacuum expectation value of the commutator does not vanish as $m_q \to 0$. When this is not the case, 'pion' pole dominance does not hold and there is no soft 'pion' theorem. To illustrate this consider the Ward identity

$$\int \mathrm{d}^4x\, T\langle \partial^\mu \mathscr{F}^3_{\mu 5}(x) . \partial^\nu \mathscr{F}^3_{\nu 5}(0)\rangle = \mathrm{i}\langle \overline{q} \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & 0 \end{bmatrix} q\rangle . \tag{24}$$

The one-particle intermediate state insertion contribution to the left-hand side of (24) is given by

$$|\langle 0|\partial^\mu \mathscr{F}^3_{\mu 5}|\pi^0\rangle|^2 \frac{(-\mathrm{i})}{m^2_\pi} + \sum_{n\neq\pi^0} |\langle 0|\partial^\mu \mathscr{F}^3_{\mu 5}|n\rangle|^2 \frac{(-\mathrm{i})}{m^2_n}$$
$$= -\mathrm{i}m^2_\pi F^2_\pi - \mathrm{i}\sum_{n\neq\pi^0} m^2_n F^2_n, \tag{25}$$

where we have defined $\langle 0|\partial^\mu \mathscr{F}^3_{\mu 5}|n\rangle = m^2_n F_n$. In the usual scheme of SCSB, the first term in (25) is $\mathrm{O}(m_q)$, while the second is $\mathrm{O}(m^2_q)$ [recall that $m^2_\pi = \mathrm{O}(m_q)$, $F_\pi = \mathrm{O}(1)$, $m^2_n = \mathrm{O}(1)$ and $F_n = \mathrm{O}(m_q)$]. Consequently, pion pole dominance holds in the usual scheme. In the AS, however, both terms are $\mathrm{O}(m^2_q)$ [recall that $m^2_\pi = \mathrm{O}(m^2_q)$, $F_\pi = \mathrm{O}(1)$, $m^2_n = \mathrm{O}(1)$ and $F_n = \mathrm{O}(m_q)$] and pion pole dominance does *not* hold. The absence of 'pion' pole dominance seems to be a general feature of the alternative schemes of SCSB for amplitudes that vanish in the chiral limit.

We consider now the Ward identity

$$\int \mathrm{d}^4x\, T\langle \partial^\mu \mathscr{F}^3_{\mu 5}(x) . \overline{u}\gamma_5 u(0)\rangle = \mathrm{i}m_1 \int \mathrm{d}^4x\, T\langle \overline{u}\gamma_5 u(x) . \overline{u}\gamma_5 u(0)\rangle$$
$$-\mathrm{i}m_2 \int \mathrm{d}^4x\, T\langle \overline{d}\gamma_5 d(x) . \overline{u}\gamma_5 u(0)\rangle = \langle \overline{u}u\rangle. \tag{26}$$

Since this equation relates the pseudoscalar–pseudoscalar amplitudes $\int T\langle \overline{u}\gamma_5 u . \overline{u}\gamma_5 u\rangle$ and $\int T\langle \overline{d}\gamma_5 d . \overline{u}\gamma_5 u\rangle$ to $\langle \overline{u}u\rangle$, which in turn is related to (see equations 13) $\int T\langle \overline{u}u . \overline{u}u\rangle$ and $\int T\langle \overline{d}d . \overline{u}u\rangle$, one hopes to be able to use it to express ρ' and γ' in terms of ρ and γ.

If we set (these equations will have to be re-examined below)

$$\int T\langle \bar{u}\gamma_5\, u\, .\, \bar{u}\gamma_5\, u\rangle = \rho' + \mathrm{O}(m_q),$$

$$\int T\langle \bar{u}\gamma_5\, u\, .\, \bar{d}\gamma_5\, d\rangle = \gamma' + \mathrm{O}(m_q) \tag{27}$$

in (26) we obtain

$$\mathrm{i} m_1\, \rho' - \mathrm{i} m_2\, \gamma' + \mathrm{O}(m_q^2) = \langle \bar{u} u\rangle. \tag{28}$$

It follows from this equation that

$$\rho = \int T\langle \bar{u}u\, .\, \bar{u}u\rangle = \mathrm{i}\frac{\partial}{\partial m_1}\langle \bar{u}u\rangle = -\rho' + \mathrm{O}(m_q),$$

$$\gamma = \int T\langle \bar{u}u\, .\, \bar{d}d\rangle = \mathrm{i}\frac{\partial}{\partial m_2}\langle \bar{u}u\rangle = \gamma' + \mathrm{O}(m_q). \tag{29}$$

In addition, the Ward identity

$$\int \mathrm{d}^4x\, T\langle \partial^\mu \mathscr{F}^3_{\mu 5}(x)\, .\, \bar{s}\gamma_5 s(0)\rangle = \mathrm{i} m_1 \int \mathrm{d}^4 x\, T\langle \bar{u}\gamma_5\, u(x)\, .\, \bar{s}\gamma_5\, s(0)\rangle$$

$$- \mathrm{i} m_2 \int \mathrm{d}^4x\, T\langle \bar{d}\gamma_5\, d(x)\, .\, \bar{s}\gamma_5\, s(0)\rangle = 0 \tag{30}$$

seems to imply that (since it must be true for all values of m_1 and m_2)

$$\gamma' = 0. \tag{31}$$

If equations (29) and (31) were true the Stern condensate parameters (12) would vanish,

$$A = Z_s = Z_p = 0, \tag{32}$$

and there would be no SCSB. This dilemma is resolved by noticing that the pseudoscalar–pseudoscalar amplitudes receive an additional contribution not included in (27) which is O(1). For simplicity we illustrate what is happening by considering chiral U(2)×U(2) symmetry. [The chiral U(2)×U(2) theory is formulated as in Sections 2 and 3 except that all of the condensates in (10) which involve the strange quark (e.g. $\int T\langle \bar{u}u\, .\, \bar{s}s\rangle$) now vanish. As a consequence of this one also finds slight changes to the formulae for $m^2_{\pi^\pm} - (m^2_{\pi^\pm})_{\mathrm{em}}$ and $m^2_{\pi^0}$ ($\approx m^2_{33}$) given by (21). In this case, we get

$$m^2_{\pi^\pm} - (m^2_{\pi^\pm})_{\mathrm{em}} \approx \tfrac{1}{4}(m_1 + m_2)^2 (A + Z_s), \tag{33a}$$

$$m^2_{\pi^0} \approx \tfrac{1}{2}(m_1^2 + m_2^2)A + \tfrac{1}{2}(m_1+m_2)^2 Z_s - \tfrac{1}{2}(m_1 - m_2)^2 Z_p$$

$$= (\mathrm{i}/F^2_\pi)\{(m_1^2 + m_2^2)(\rho + \rho') + 2m_1\, m_2(\gamma - \gamma')\}. \tag{33b}$$

Equations (26)–(29) are unchanged.] Chiral U(3)×U(3) will be reconsidered later.

We consider the chiral expansion [$\mathscr{L}_0$ is U(2)×U(2) invariant and $\mathscr{L}' = -\bar{q}\begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} q$]

$$\int T\langle \bar{u}\gamma_5\, u\,.\,\bar{u}\gamma_5\, u\rangle_{\mathscr{L}} = \int T\langle \bar{u}\gamma_5\, u\,.\,\bar{u}\gamma_5\, u\rangle_{\mathscr{L}_0} + \iint T\langle \bar{u}\gamma_5\, u\,.\,\bar{u}\gamma_5\, u\,.\,\mathrm{i}\mathscr{L}'\rangle^{\mathrm{c}}_{\mathscr{L}_0}$$
$$+\frac{1}{2!}\iiint T\langle \bar{u}\gamma_5\, u\,.\,\bar{u}\gamma_5\, u\,.\,\mathrm{i}\mathscr{L}'\,.\,\mathrm{i}\mathscr{L}'\rangle^{\mathrm{c}}_{\mathscr{L}_0} + \ldots . \qquad (34)$$

The third term on the right-hand side receives a singular contribution from the diagram (see Fig. 3) in which a (massless) pion connects two $\bar{u}\gamma_5\, u\,.\,\mathrm{i}\mathscr{L}'$ operators (note that there is no such contribution to the second term because $\langle 0|\bar{u}\gamma_5\, u|\pi\rangle_{\mathscr{L}_0} = 0$):

$$\frac{1}{2!}\iiint T\langle \bar{u}\gamma_5\, u\,.\,\bar{u}\gamma_5\, u\,.\,\mathrm{i}\mathscr{L}'\,.\,\mathrm{i}\mathscr{L}'\rangle_{\mathscr{L}_0} = \lim_{q\to 0} \left|\langle 0|\bar{u}\gamma_5\, u\,.\,\mathrm{i}\mathscr{L}'|\pi\rangle_{\mathscr{L}_0}\right|^2 \frac{\mathrm{i}}{q^2}$$
$$+\text{regular terms.} \qquad (35)$$

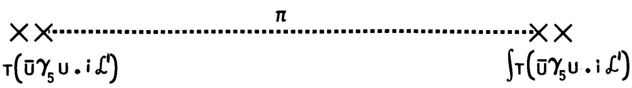

Fig. 3. Infrared singular contribution in equation (35).

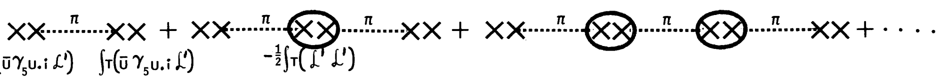

Fig. 4. Summation of the infrared singular set of diagrams contributing to the right-hand side of equation (34).

The singularity in (35) is removed by summing the set of diagrams (see Fig. 4) through which the pion becomes massive (cf. with Fig. 2). This gives

$$\int T\langle \bar{u}\gamma_5\, u\,.\,\bar{u}\gamma_5\, u\rangle_{\mathscr{L}} = \int T\langle \bar{u}\gamma_5\, u\,.\,\bar{u}\gamma_5\, u\rangle_{\mathscr{L}_0}$$
$$+\left|\langle 0|\bar{u}\gamma_5\, u\,.\,\mathrm{i}\mathscr{L}'|\pi\rangle_{\mathscr{L}_0}\right|^2 \frac{(-\mathrm{i})}{m^2_{\pi^0}} + \mathrm{O}(m_q). \qquad (36)$$

Reducing the matrix element on the right-hand side of this equation by a soft pion theorem gives

$$\int T\langle \bar{u}\gamma_5\, u\,.\,\bar{u}\gamma_5\, u\rangle_{\mathscr{L}} = \rho' - \frac{\mathrm{i}}{m^2_{\pi^0}\, F^2_\pi}(m_1\,\rho + m_2\,\gamma + m_1\,\rho' - m_2\,\gamma')^2 + \mathrm{O}(m_q). \qquad (37)$$

Similarly one can show that

$$\int T\langle \overline{u}\gamma_5\, u \,.\, \overline{d}\gamma_5\, d\rangle_{\mathscr{L}} = \gamma' - \frac{\mathrm{i}}{m^2_{\pi^0}\, F^2_{\pi}}(m_1\,\rho + m_2\,\gamma + m_1\,\rho' - m_2\,\gamma')$$

$$\times(-\,m_1\,\gamma - m_2\,\rho + m_1\,\gamma' - m_2\,\rho') + \mathrm{O}(m_q)\,. \quad (38)$$

Since $m^2_{\pi} = \mathrm{O}(m^2_q)$, the pion pole contributions are seen to be O(1). Inserting (37) and (38) into (26) and using (33) gives (in place of 28)

$$\langle \overline{u}u\rangle = -\mathrm{i}m_1\,\rho - \mathrm{i}m_2\,\gamma + \mathrm{O}(m^2_q)\,.$$

This is nothing else but the U(2)×U(2) version (i.e. with no m_3 term) of (13). As a result the chiral Ward identities do not relate the condensates ρ, γ, ρ' and γ'.

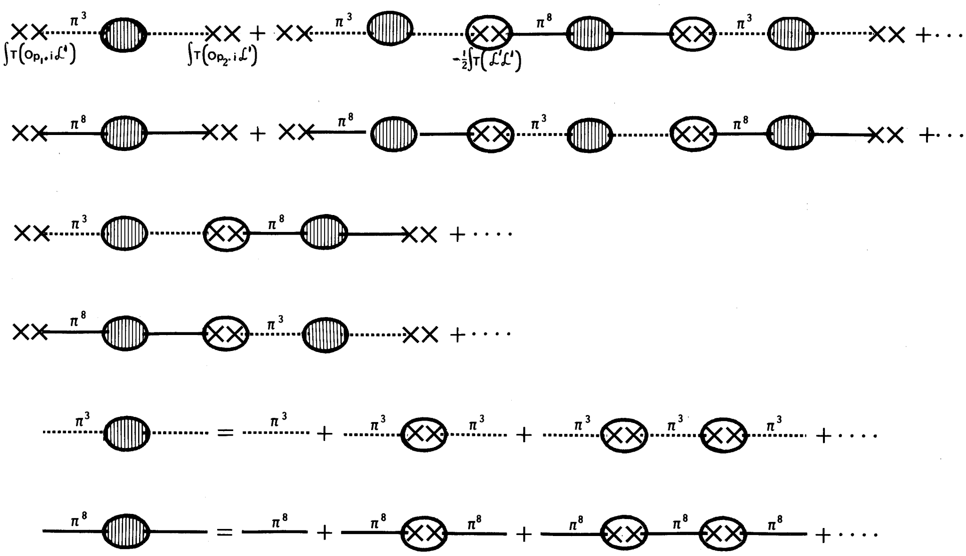

Fig. 5. Set of diagrams which give rise to the O(1) 'corrections' to $\int T\langle Op_1 \,.\, Op_2\rangle$ in the U(3)×U(3) case.

The interesting feature of equations (37) and (38) is that they satisfy

$$\langle \ldots\rangle_{\mathscr{L}}\,|_{m_q\to 0} \neq \langle \ldots\rangle_{\mathscr{L}_0}\,, \quad (39)$$

i.e., the result of calculating an amplitude in the non-chiral limit and then taking $m_q \to 0$ is not necessarily equivalent to what one would calculate in the *exact* chiral limit.

In hindsight this result is quite obvious. The pion pole can make a contribution to $\int T\langle \bar{u}\gamma_5\, u\,.\,\bar{u}\gamma_5\, u\rangle_{\mathscr{L}}$ which

$$\sim \langle 0|\bar{u}\gamma_5\, u|\pi\rangle \frac{(-\mathrm{i})}{m_\pi^2}\langle \pi|\bar{u}\gamma_5\, u|0\rangle$$

$$= \mathrm{O}(m_q)\,\mathrm{O}(1/m_q^2)\,\mathrm{O}(m_q) = \mathrm{O}(1),$$

but this is absent in the corresponding $\mathscr{L}_0$ term because $\langle 0|\bar{u}\gamma_5\, u|\pi\rangle_{\mathscr{L}_0} = 0$. These additional O(1) 'corrections' must be included whenever one uses the chiral Ward identities in the non-chiral limit.

In the chiral U(3)×U(3) situation the pseudoscalar–pseudoscalar amplitudes also receive a contribution from the η meson, plus possible π^0–η mixing contributions. The complete set of (singular) diagrams which contribute to say $\int_x T\langle Op_1(x)\,.\,Op_2(0)\rangle$, where Op_1 and Op_2 are some two-quark pseudoscalar operators (e.g. $\bar{u}\gamma_5\, u$) are given in Fig. 5. One finds that ($m_{\pi^3}^2 \equiv m_{33}^2$ and $m_{\pi^8}^2 \equiv m_{88}^2$)

$$\begin{aligned}
\int_x T\langle Op_1(x)\,.\,Op_2(0)\rangle_{\mathscr{L}} &= \int_x T\langle Op_1(x)\,.\,Op_2(0)\rangle_{\mathscr{L}_0} \\
&+\left(\int T\langle Op_1\,.\,\mathrm{i}\mathscr{L}'|\pi^3\rangle_{\mathscr{L}_0} \int T\langle Op_1\,.\,\mathrm{i}\mathscr{L}'|\pi^8\rangle_{\mathscr{L}_0}\right) \\
&\times \begin{bmatrix} \dfrac{(-\mathrm{i})}{m_{\pi^3}^2} & \dfrac{(-\mathrm{i})}{m_{\pi^3}^2}\langle \pi^3|-\tfrac{1}{2}\displaystyle\int T(\mathscr{L}'\mathscr{L}')|\pi^8\rangle\dfrac{(-\mathrm{i})}{m_{\pi^8}^2} \\ \dfrac{(-\mathrm{i})}{m_{\pi^8}^2}\langle \pi^8|-\tfrac{1}{2}\displaystyle\int T(\mathscr{L}'\mathscr{L}')|\pi^3\rangle\dfrac{(-\mathrm{i})}{m_{\pi^3}^2} & \dfrac{(-\mathrm{i})}{m_{\pi^8}^2} \end{bmatrix} \begin{bmatrix} \displaystyle\int T\langle \pi^3|Op_2\,.\,\mathrm{i}\mathscr{L}'\rangle_{\mathscr{L}_0} \\ \displaystyle\int T\langle \pi^8|Op_2\,.\,\mathrm{i}\mathscr{L}'\rangle_{\mathscr{L}_0} \end{bmatrix} \\
&\times \left(\frac{1}{1-|\langle \pi^3|-\frac{1}{2}\int T(\mathscr{L}'\mathscr{L}')|\pi^8\rangle|^2/m_{\pi^3}^2\, m_{\pi^8}^2}\right) + \mathrm{O}(m_q)
\end{aligned} \tag{40}$$

$$\begin{aligned}
&= \int T\langle Op_1\,.\,Op_2\rangle_{\mathscr{L}_0} + \left(\int T\langle Op_1\,.\,\mathrm{i}\mathscr{L}'|\pi^0\rangle_{\mathscr{L}_0} \int T\langle Op_1\,.\,\mathrm{i}\mathscr{L}'|\eta\rangle_{\mathscr{L}_0}\right) \\
&\times \begin{bmatrix} \dfrac{(-\mathrm{i})}{m_{\pi^0}^2} & 0 \\ 0 & \dfrac{(-\mathrm{i})}{m_\eta^2} \end{bmatrix} \begin{bmatrix} \displaystyle\int T\langle \pi^0|Op_2\,.\,\mathrm{i}\mathscr{L}'\rangle_{\mathscr{L}_0} \\ \displaystyle\int T\langle \eta|Op_2\,.\,\mathrm{i}\mathscr{L}'\rangle_{\mathscr{L}_0} \end{bmatrix} + \mathrm{O}(m_q).
\end{aligned} \tag{41}$$

Using equation (40) it is straightforward (although tedious) to calculate the O(1) 'corrections' to the pseudoscalar–pseudoscalar amplitudes. Since these results are not particularly enlightening we do not include them here. We leave it as an exercise to the interested reader to check that, with the O(1) 'corrections', equation (26) reduces to (13) and the left-hand side of equation (30) vanishes.

5. Quark Mass Ratios

Dashen's (1969) electromagnetic theorem and the result of Das *et al.* (1967) are unaffected by the change in the scheme of SCSB. As a result we take

$$(m^2_{\pi^\pm})_{\rm em} \approx (m^2_{\rm K^\pm})_{\rm em} \approx \frac{6e^2}{(4\pi)^2}\, m^2_\rho \ln 2 \approx 0\cdot 00143\ {\rm GeV}^2\,. \tag{42}$$

In the usual scheme of SCSB the quark mass ratios can be determined from equations (Ala), (Alb), (A1c) and (42) (see equations A2a and A2b). In the AS, however, the masses $m^2_{\pi^\pm}$, $m^2_{\rm K^\pm}$ and $m^2_{\rm K^0,\bar K^0}$ involve two symmetry breaking parameters (A and $Z_{\rm s}$) and this is no longer possible. Instead, what can be done is to determine the quark mass ratios for specific values of $\xi_{\rm s} \equiv Z_{\rm s}/A$.

Table 1. Values of r, R, $\hat m^2 A$, m_1, m_2, A and $Z_{\rm s}$ for specific values of $\xi_{\rm s}$

$\xi_{\rm s} = \dfrac{Z_{\rm s}}{A}$	$r = \dfrac{2m_3}{m_1+m_2}$	$R = \dfrac{m_2-m_1}{m_1+m_2}$	$\hat m^2 A$ (MeV^2)	Taking $m_3 \approx 150$ MeV m_1 (MeV)	m_2 (MeV)	A	$Z_{\rm s}$
0	6·368	0·04059	$1\cdot 805\times 10^4$	22·6	24·5	32·5	0
1/12	8·209	0·05502	$9\cdot 752\times 10^3$	17·3	19·3	29·2	2·43
1/6	9·896	0·06927	$6\cdot 051\times 10^3$	14·1	16·2	25·0	4·17
1/3	12·64	0·09496	$3\cdot 069\times 10^3$	10·7	13·0	21·8	7·27
1/2	14·67	0·1163	$1\cdot 933\times 10^3$	9·04	11·4	18·5	9·25
1	18·31	0·1609	$8\cdot 472\times 10^3$	6·88	9·52	12·6	12·6
2	21·37	0·2066	$3\cdot 780\times 10^2$	5·57	8·47	7·67	15·3
5	23·96	0·2528	$1\cdot 380\times 10^2$	4·68	7·84	3·52	17·6
∞	26·15	0·2991	0[A]	4·02	7·45	0	19·5

[A] With $\hat m^2 Z_{\rm s} \approx 6\cdot 413\times 10^2$.

From equations (21a), (21b) and (21c) it follows that

$$\frac{m^2_{\rm K^0}+(m^2_{\rm K^+}-{\rm em})-(m^2_{\pi^+}-{\rm em})}{m^2_{\pi^+}-{\rm em}} = \frac{(r^2+2r-1+R^2)+2r(2+r)\xi_{\rm s}}{2+2(2+r)\xi_{\rm s}}$$

$$\approx 26\cdot 146\ ({\rm exp})\,, \tag{43}$$

where we have defined $r \equiv 2m_3/(m_1+m_2)$ and $R \equiv (m_2-m_1)/(m_1+m_2)$. The numerical value in (43) is obtained on inserting the experimental values of $m^2_{\rm K^0}$, $m^2_{\rm K^+}$ and $m^2_{\pi^+}$ and the estimate (42). If we neglect the R^2 term in (43) we can use it to solve for r in terms of $\xi_{\rm s}$. The values of r for some specific values of $\xi_{\rm s}$ are given in Table 1.

Similarly, using the relation

$$\frac{m^2_{\rm K^0}-(m^2_{\rm K^+}-{\rm em})}{m^2_{\pi^+}-{\rm em}} = \frac{R\{1+r+(2+r)\xi_{\rm s}\}}{1+(2+r)\xi_{\rm s}} \approx 0\cdot 29909\ ({\rm exp})\,, \tag{44}$$

and the values of r calculated above we can also determine R as a function of $\xi_{\rm s}$ (see Table 1).

The values of R calculated in this way can then be used to check the consistency of the initial approximation of neglecting the R^2 term in (43). Since

$$R^2/r^2 \lesssim 10^{-4}, \tag{45}$$

the results of Table 1 remain practically unchanged.

The cases $\xi_s = 0$ (i.e. $Z_s = 0$) and $\xi_s = \infty$ (i.e. $A = 0$) can be solved analytically, with

$$r = \frac{2m_3}{m_1+m_2} = \frac{m_{K^0}+(m_{K^+}^2-\mathrm{em})^{\frac{1}{2}}-(m_{\pi^+}^2-\mathrm{em})^{\frac{1}{2}}}{(m_{\pi^+}^2-\mathrm{em})^{\frac{1}{2}}} \quad (\xi_s = 0) \tag{46a}$$

$$= \frac{m_{K^0}^2+(m_{K^+}^2-\mathrm{em})-(m_{\pi^+}^2-\mathrm{em})}{m_{\pi^+}^2-\mathrm{em}} \quad (\xi_s = \infty), \tag{46b}$$

$$R = \frac{m_2-m_1}{m_1+m_2} = \frac{m_{K^0}-(m_{K^+}^2-\mathrm{em})^{\frac{1}{2}}}{(m_{\pi^+}^2-\mathrm{em})^{\frac{1}{2}}} \quad (\xi_s = 0) \tag{47a}$$

$$= \frac{m_{K^0}^2-(m_{K^+}^2-\mathrm{em})}{m_{\pi^+}^2-\mathrm{em}} \quad (\xi_s = \infty). \tag{47b}$$

It is interesting to note that the expressions for $\xi_s = \infty$ coincide with those obtained in the usual scheme of SCSB (see equations A2a and A2b).

An estimate of the condensate parameters can be obtained from equation (21a):

$$\hat{m}^2 A = \frac{m_{\pi^+}^2-\mathrm{em}}{1+(2+r)\xi_s}, \tag{48}$$

where $\hat{m} = \frac{1}{2}(m_1+m_2)$ (see Table 1).

As mentioned earlier $Z_s \neq 0$ entails a violation of Zweig's rule in the 0^+ channel, and therefore, Z_s is expected to be quite small. Consequently (see Table 1) the values of r and R are smaller than the values in the usual scheme. It is interesting to note that the likely value of r calculated from the pseudoscalar sector in the AS ($r \lesssim 10$) is in fairly good agreement with the value one obtains from a consideration of baryon masses (J. Stern 1982, unpublished).

6. π^0–η–η' Mixing and the Decays $\psi\to\pi^0\gamma, \eta\gamma, \eta'\gamma$ and $\psi'\to\psi\pi^0, \psi\eta$

The π^0–η–η' mixing matrix elements (21d)–(21i) can be expressed in terms of $r = m_3/\hat{m}$, $R = (m_2-m_1)/(m_1+m_2)$, $\xi_s = Z_s/A$, $\xi_p = Z_p/A$ and $\hat{m}^2 A$:

$$\begin{aligned}
m_{33}^2 &= (\hat{m}^2 A)\{1+R^2+(2+r)\xi_s-2R^2\xi_p\},\\
m_{38}^2 &= -2\sqrt{\tfrac{1}{3}}\,(\hat{m}^2 A)R\{1+(1+\tfrac{1}{2}r)\xi_s-(1-r)\xi_p\},\\
m_{39}^2 &= -2\sqrt{\tfrac{2}{3}}\,\alpha(\hat{m}^2 A)R\{1+(1+\tfrac{1}{2}r)(\xi_s-\xi_p)\},\\
m_{88}^2 &= \tfrac{1}{3}(\hat{m}^2 A)\{1+R^2+2r^2+(2+r)(1+2r)\xi_s-2(1-r)^2\xi_p\},\\
m_{99}^2 &= \tfrac{2}{3}\alpha^2(\hat{m}^2 A)\{1+R^2+\tfrac{1}{2}r^2+2(1+\tfrac{1}{2}r)^2(\xi_s-\xi_p)\}+\chi^2/N_c,\\
m_{89}^2 &= \tfrac{1}{3}\sqrt{2}\,\alpha(\hat{m}^2 A)\{1+R^2-r^2+(1-r)(2+r)(\xi_s-\xi_p)\}.
\end{aligned} \tag{49}$$

Recall that $\alpha = F_\pi / F_s$ is the ratio of the pion to the singlet decay constant. Although $\alpha = 1$ in the combined SU(3) and large N_c limits (Witten 1979; Di Vecchia *et al.* 1981; Christos 1984*b*), it turns out that large N_c corrections must be quite large. Consequently, for a much better fit it proves more appropriate to simply allow α to differ from one. The χ^2/N_c term in (49) arises in the same way as in the usual scheme of SCSB. One assumes that the 'topological susceptibility'

$$\langle\langle \nu^2 \rangle\rangle \equiv -\mathrm{i} \int \mathrm{d}^4 x\, \partial_x^\mu\, \partial^\nu\, T\langle K_\mu(x)\, K_\nu(0)\rangle \tag{50}$$

is nonzero and positive (Witten 1979; Veneziano 1979) in pure Yang–Mills theory (no quarks). The requirement that $\langle\langle \nu^2 \rangle\rangle_{\mathrm{QCD}}$ should vanish (Crewther 1979) leads to the result that the flavour singlet pseudoscalar acquires a contribution to its mass squared, which does not vanish in the chiral limit, given by the Witten formula

$$m_{99}^2 \approx 6\langle\langle \nu^2 \rangle\rangle_{\mathrm{YM}}/F_\pi^2 \equiv \chi^2/N_c \qquad (m_q \to 0)\,. \tag{51}$$

For a thorough explanation of this mechanism and the underlying assumptions see Christos (1984*a*).

Instead of dealing with the π^0–η–η' system as a whole, it proves convenient to consider initially η–η' mixing alone, in isolation of the π^0, i.e. in the limit $m_1 = m_2 = \hat{m}$: $R = 0$. This is reasonable because isospin breaking is so much smaller than SU(3) breaking (see equation 45).

In this case (with the inclusion of $\alpha \neq 1$ corrections) the AS description of η–η' mixing can be fitted by the three parameters (for each value of ξ_s) χ^2/N_c, α and ξ_p. (This should be compared with the usual scheme where there are only two parameters, χ^2/N_c and α.) Consequently there are no predictions for η–η' mixing in the AS of SCSB. The analysis is however not without interest because it gives an estimate of important quantities like the topological susceptibility, the ratio of singlet to pion decay constants and the magnitude of Zweig violation in the 0^- channel.

The relevant equations are

$$m_{88}^2 + m_{99}^2 = m_\eta^2 + m_{\eta'}^2 \approx 1{\cdot}2182\ \mathrm{GeV}^2\ (\mathrm{exp})\,, \tag{52a}$$

$$m_{88}^2\, m_{99}^2 - (m_{89}^2)^2 = m_\eta^2\, m_{\eta'}^2 \approx 0{\cdot}2762\ \mathrm{GeV}^2\ (\mathrm{exp})\,, \tag{52b}$$

$$\tan\theta = (m_{88}^2 - m_\eta^2)/(-m_{89}^2)\,, \tag{52c}$$

where the mixing angle θ is defined by

$$|\eta\rangle = \cos\theta\,|8\rangle + \sin\theta\,|s\rangle\,, \qquad |\eta'\rangle = -\sin\theta\,|8\rangle + \cos\theta\,|s\rangle\,. \tag{53}$$

Inserting (52a) and (52c) into (52b) to eliminate m_{99}^2 and m_{89}^2 respectively gives

$$m_\eta^2\, m_{\eta'}^2 = m_{88}^2(m_\eta^2 + m_{\eta'}^2 - m_{88}^2) - (m_{88}^2 - m_\eta^2)^2/\tan\theta\,. \tag{54}$$

Since m_{88}^2 is a (linear) function of ξ_p only, equation (54) is quadratic in ξ_p. The interesting solution is the smaller of the two; the other gives $\alpha = 0$. The value of α can then be determined from equation (52c), followed by the value of χ^2/N_c from (49). In performing these calculations we have used a (conservative) value of $\theta \approx 10°\pm4°$. The results are given in Table 2. These should be compared with the values obtained in the usual scheme of SCSB given by equations (A5).

Table 2. Values of ξ_p, F_s/F_π and χ^2/N_c for specific values of ξ_s as determined from fitting the η–η' spectrum, with $\theta = 10°\pm4°$

$\xi_s = Z_s/A$	$\xi_p = Z_p/A$	$\alpha^{-1} = F_s/F_\pi$	χ^2/N_c (GeV^2)
0	$0{\cdot}5025^{-0{\cdot}0504}_{+0{\cdot}0342}$	$1{\cdot}37^{-0{\cdot}24}_{+0{\cdot}68}$	$0{\cdot}8748^{-0{\cdot}0450}_{+0{\cdot}0283}$
1/12	$0{\cdot}5026^{-0{\cdot}0535}_{+0{\cdot}0350}$	$1{\cdot}55^{-0{\cdot}30}_{+0{\cdot}82}$	$0{\cdot}8637^{-0{\cdot}0474}_{+0{\cdot}0337}$
1/6	$0{\cdot}5027^{-0{\cdot}0547}_{+0{\cdot}0371}$	$1{\cdot}66^{-0{\cdot}33}_{+0{\cdot}90}$	$0{\cdot}8601^{-0{\cdot}0482}_{+0{\cdot}0356}$
1/3	$0{\cdot}5031^{-0{\cdot}0630}_{+0{\cdot}0427}$	$1{\cdot}78^{-0{\cdot}37}_{+0{\cdot}99}$	$0{\cdot}8581^{-0{\cdot}0490}_{+0{\cdot}0366}$
1/2	$0{\cdot}5036^{-0{\cdot}0725}_{+0{\cdot}0492}$	$1{\cdot}85^{-0{\cdot}40}_{+1{\cdot}03}$	$0{\cdot}8575^{-0{\cdot}0491}_{+0{\cdot}0369}$
1	$0{\cdot}5051^{-0{\cdot}1035}_{+0{\cdot}0700}$	$1{\cdot}92^{-0{\cdot}42}_{+1{\cdot}10}$	$0{\cdot}8572^{-0{\cdot}0492}_{+0{\cdot}0371}$
2	$0{\cdot}5083^{-0{\cdot}1670}_{+0{\cdot}1132}$	$1{\cdot}97^{-0{\cdot}43}_{+1{\cdot}13}$	$0{\cdot}8572^{-0{\cdot}0492}_{+0{\cdot}0371}$
5	$0{\cdot}5179^{-0{\cdot}3603}_{+0{\cdot}2442}$	$2{\cdot}00^{-0{\cdot}44}_{+1{\cdot}15}$	$0{\cdot}8572^{-0{\cdot}0492}_{+0{\cdot}0371}$

It is interesting to briefly consider the hypothetical case where $\theta \approx 0$ and ask: when does the Gell-Mann–Okubo (GMO) mass formula hold? It is easy to see from equations (21) or (49) (here $m_\eta^2 = m_{88}^2$) that (for $m_1 \approx m_2 \approx \hat{m}$) the quadratic GMO formula $m_\eta^2 = \frac{4}{3}m_K^2 - \frac{1}{3}m_\pi^2$ holds if $\xi_p = \frac{1}{2}$, while the linear GMO formula $m_\eta = \frac{4}{3}m_K - \frac{1}{3}m_\pi$ holds if $\xi_p = \frac{1}{3}$ and $\xi_s = 0$ (J. Stern 1982, unpublished). In the AS there is no natural explanation of the quadratic GMO formula, which works quite well.

Having determined these parameters we can now return to the π^0–η–η' analysis and predict the values of $m_{\pi^0}^2$ and the other two mixing angles for each value of ξ_s. The value of $m_{\pi^0}^2$ is determined by the secular equation

$$\det\begin{bmatrix} m_{33}^2 - m_{\pi^0}^2 & m_{38}^2 & m_{39}^2 \\ & m_{88}^2 - m_{\pi^0}^2 & m_{89}^2 \\ \text{symm.} & & m_{99}^2 - m_{\pi^0}^2 \end{bmatrix} = 0. \tag{55}$$

Assuming that

$$m_{\pi^0}^2 \ll m_{88}^2,\ m_{99}^2, \tag{56}$$

it follows that

$$\Delta m_{\pi^0}^2 \equiv m_{\pi^0}^2 - m_{33}^2 \approx \frac{2m_{38}^2\, m_{39}^2\, m_{89}^2 - (m_{39}^2)^2 m_{88}^2 - (m_{38}^2)^2 m_{99}^2}{m_{88}^2\, m_{99}^2 - (m_{89}^2)^2}, \tag{57}$$

from which one can determine a value of $m_{33}^2 - m_{\pi^0}^2$ and $m_{\pi^0}^2$. An improvement to this value can be obtained by resubstituting the last value of $m_{\pi^0}^2$ into the parts of (55) where (56) was assumed. The result converges to five significant figures after only three iterations. It is then a simple matter to determine the corresponding eigenvector

$$|\pi^0\rangle = a_{\pi^0 3}|\pi^3\rangle + a_{\pi^0 8}|\pi^8\rangle + a_{\pi^0 s}|s\rangle. \tag{58}$$

The results are given in Table 3.

Table 3. Values of $\Delta m^2_{\pi^0} = m^2_{\pi^0} - m^2_{33}$, $m^2_{\pi^0}$, $1 - a_{\pi^0 3}$, $a_{\pi^0 8}$ and $a_{\pi^0 s}$ for specific values of ξ_s

ξ_s	$\Delta m^2_{\pi^0}$ (MeV^2)	$m^2_{\pi^0}$ (MeV^2)	$1 - a_{\pi^0 3}$ ($\times 10^{-5}$)	$a_{\pi^0 8}$	$a_{\pi^0 s}$ $\theta = 6°$	$\theta = 10°$	$\theta = 14°$
0	$-32{\cdot}4^{+5\cdot9}_{-4\cdot8}$	18016^{+9}_{-7}	$5{\cdot}4^{-1\cdot1}_{+1\cdot0}$	$1{\cdot}04^{-0\cdot11}_{+0\cdot09}$	$8{\cdot}35\times10^{-7}$	$1{\cdot}54\times10^{-4}$	$4{\cdot}65\times10^{-4}$
1/12	$-32{\cdot}7^{+5\cdot8}_{-4\cdot6}$	18016^{+9}_{-7}	$5{\cdot}6^{-1\cdot1}_{+0\cdot9}$	$1{\cdot}06^{-0\cdot11}_{+0\cdot08}$	$2{\cdot}70\times10^{-7}$	$5{\cdot}31\times10^{-4}$	$8{\cdot}76\times10^{-4}$
1/6	$-33{\cdot}0^{+5\cdot7}_{-4\cdot5}$	18016^{+9}_{-7}	$5{\cdot}7^{-1\cdot0}_{+0\cdot9}$	$1{\cdot}07^{-0\cdot10}_{+0\cdot08}$	$4{\cdot}55\times10^{-4}$	$8{\cdot}07\times10^{-4}$	$1{\cdot}21\times10^{-3}$
1/3	$-33{\cdot}7^{+5\cdot3}_{-4\cdot2}$	18015^{+9}_{-6}	$5{\cdot}9^{-0\cdot9}_{+0\cdot8}$	$1{\cdot}08^{-0\cdot10}_{+0\cdot07}$	$7{\cdot}20\times10^{-4}$	$1{\cdot}22\times10^{-3}$	$1{\cdot}73\times10^{-3}$
1/2	$-34{\cdot}4^{+4\cdot9}_{-4\cdot0}$	18014^{+9}_{-6}	$6{\cdot}1^{-0\cdot9}_{+0\cdot7}$	$1{\cdot}09^{-0\cdot09}_{+0\cdot07}$	$9{\cdot}12\times10^{-4}$	$1{\cdot}52\times10^{-3}$	$2{\cdot}13\times10^{-3}$
1	$-36{\cdot}3^{+4\cdot0}_{-3\cdot2}$	18012^{+9}_{-6}	$6{\cdot}4^{-0\cdot7}_{+0\cdot6}$	$1{\cdot}11^{-0\cdot09}_{+0\cdot06}$	$1{\cdot}28\times10^{-3}$	$2{\cdot}12\times10^{-3}$	$2{\cdot}93\times10^{-3}$
2	$-38{\cdot}7^{+2\cdot6}_{-2\cdot1}$	18010^{+8}_{-6}	$6{\cdot}8^{-0\cdot6}_{+0\cdot4}$	$1{\cdot}13^{-0\cdot08}_{+0\cdot06}$	$1{\cdot}64\times10^{-3}$	$2{\cdot}71\times10^{-3}$	$3{\cdot}73\times10^{-3}$
5	$-41{\cdot}7^{+0\cdot8}_{-0\cdot7}$	18007^{+7}_{-5}	$7{\cdot}2^{-0\cdot3}_{+0\cdot2}$	$1{\cdot}15^{-0\cdot07}_{+0\cdot05}$	$2{\cdot}00\times10^{-3}$	$3{\cdot}29\times10^{-3}$	$4{\cdot}52\times10^{-3}$

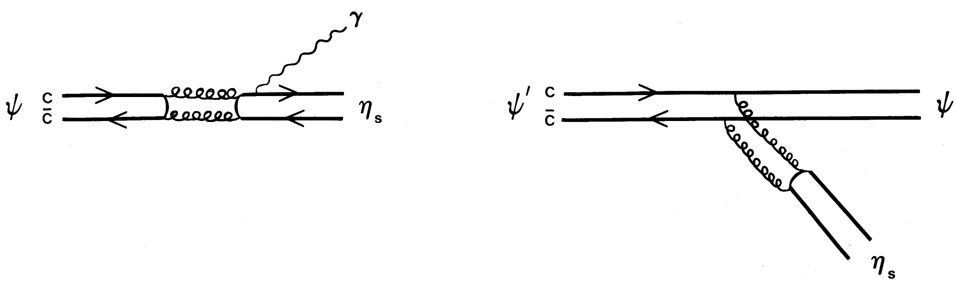

Fig. 6. Gluon annihilation diagrams by which the decays $\psi \to \pi^0\gamma, \eta\gamma, \eta'\gamma$ and $\psi' \to \psi\pi^0, \psi\eta$ can proceed. Here $\eta_s = \pi^9$ is the flavour singlet pseudoscalar meson.

Assuming that the decays $\psi \to \pi^0\gamma, \eta\gamma, \eta'\gamma$ and $\psi' \to \psi\pi^0, \psi\eta$ proceed predominantly through the gluon annihilation diagrams of Fig. 6, we find that

$$\begin{aligned} &\Gamma(\psi\to\pi^0\gamma) : \Gamma(\psi\to\eta\gamma) : \Gamma(\psi\to\eta'\gamma) \\ &\qquad \approx (a^2_{\pi^0 s})^2(m^2_\psi - m^2_{\pi^0})^3 : \sin^2\theta(m^2_\psi - m^2_\eta)^3 : \cos^2\theta(m^2_\psi - m^2_{\eta'})^3, \end{aligned} \tag{59}$$

$$\frac{\Gamma(\psi'\to\psi\pi^0)}{\Gamma(\psi'\to\psi\eta)} \approx \frac{(a_{\pi^0 s})^2}{\sin^2\theta}\,\frac{\{m^2_{\psi'} - (m_\psi + m_{\pi^0})^2\}^{\frac{3}{2}}\{m^2_{\psi'} - (m_\psi - m_{\pi^0})^2\}^{\frac{3}{2}}}{\{m^2_{\psi'} - (m_\psi + m_\eta)^2\}^{\frac{3}{2}}\{m^2_{\psi'} - (m_\psi - m_\eta)^2\}^{\frac{3}{2}}}. \tag{60}$$

The predicted ratios of these decay rates for the AS are given in Table 4. These should be compared with the experimental values

$$\begin{aligned} &\Gamma(\psi\to\pi^0\gamma) : \Gamma(\psi\to\eta\gamma) : \Gamma(\psi\to\eta'\gamma) \\ &\qquad \approx 0{\cdot}08\pm0{\cdot}06 : 1{\cdot}0\pm0{\cdot}1 : 4{\cdot}2\pm0{\cdot}6, \end{aligned} \tag{61}$$

$$\frac{\Gamma(\psi'\to\psi\pi^0)}{\Gamma(\psi'\to\psi\eta)} \approx (3{\cdot}6\pm1{\cdot}8)\times10^{-2}, \tag{62}$$

and to the values one obtains in the usual scheme of SCSB (see equations A6). One notices that although the usual scheme does not give a very good account of these

Table 4. Ratios of decay rates for specific values of ξ_s, with $\theta = 6°$, $10°$ and $14°$

ξ_s	$\Gamma(\psi\to\pi^0\gamma)/\Gamma(\psi\to\eta\gamma)$ $\theta=6°$	$\theta=10°$	$\theta=14°$	$\Gamma(\psi'\to\psi\pi^0)/\Gamma(\psi'\to\psi\eta)$ $\theta=6°$	$\theta=10°$	$\theta=14°$
0	$6\cdot97\times10^{-11}$	$8\cdot65\times10^{-7}$	$4\cdot05\times10^{-6}$	$1\cdot23\times10^{-9}$	$1\cdot53\times10^{-5}$	$7\cdot17\times10^{-5}$
1/12	$7\cdot29\times10^{-6}$	$1\cdot02\times10^{-5}$	$1\cdot43\times10^{-5}$	$1\cdot29\times10^{-4}$	$1\cdot81\times10^{-4}$	$2\cdot54\times10^{-4}$
1/6	$2\cdot08\times10^{-5}$	$2\cdot36\times10^{-5}$	$2\cdot72\times10^{-5}$	$3\cdot68\times10^{-4}$	$4\cdot18\times10^{-4}$	$4\cdot82\times10^{-4}$
1/3	$5\cdot19\times10^{-5}$	$5\cdot38\times10^{-5}$	$5\cdot59\times10^{-5}$	$9\cdot20\times10^{-4}$	$9\cdot53\times10^{-4}$	$9\cdot90\times10^{-4}$
1/2	$8\cdot33\times10^{-5}$	$8\cdot42\times10^{-5}$	$8\cdot49\times10^{-5}$	$1\cdot48\times10^{-3}$	$1\cdot49\times10^{-3}$	$1\cdot50\times10^{-3}$
1	$1\cdot65\times10^{-4}$	$1\cdot63\times10^{-4}$	$1\cdot61\times10^{-4}$	$2\cdot91\times10^{-3}$	$2\cdot89\times10^{-3}$	$2\cdot84\times10^{-3}$
2	$2\cdot70\times10^{-4}$	$2\cdot66\times10^{-4}$	$2\cdot60\times10^{-4}$	$4\cdot78\times10^{-3}$	$4\cdot71\times10^{-3}$	$4\cdot60\times10^{-3}$
5	$4\cdot00\times10^{-4}$	$3\cdot93\times10^{-4}$	$3\cdot82\times10^{-4}$	$7\cdot08\times10^{-3}$	$6\cdot96\times10^{-3}$	$6\cdot77\times10^{-3}$

ratios, those obtained in the AS [for reasonable (i.e. small) values of ξ_s] are further removed by at least another order of magnitude.

Because of the comparatively large experimental errors this cannot be used to rule out the AS of SCSB. It is interesting to note however that models with $\xi_s = 0$ ($\gamma = 0$), which are extensively used in the literature, give results which differ from the experimental values by many orders of magnitude.

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Appendix: Quark Mass Ratios and π^0–η–η' Mixing in the Usual Scheme of SCSB

In the usual scheme of SCSB (where to leading order $F_\pi \approx F_K \approx F_8$ and $\langle \bar{u}u\rangle \approx \langle \bar{d}d\rangle \approx \langle \bar{s}s\rangle \approx \frac{1}{3}\langle \bar{q}q\rangle$), we have

$$m_{\pi^\pm}^2 - (m_{\pi^\pm}^2)_{\rm em} = -\frac{1}{F_\pi^2}(m_1+m_2)\langle \bar{u}u\rangle + O(m_q^2), \tag{A1a}$$

$$m_{K^\pm}^2 - (m_{K^\pm}^2)_{\rm em} = -\frac{1}{F_\pi^2}(m_1+m_3)\langle \bar{u}u\rangle + O(m_q^2), \tag{A1b}$$

$$m_{K^0,\bar{K}^0}^2 = -\frac{1}{F_\pi^2}(m_2+m_3)\langle \bar{u}u\rangle + O(m_q^2), \tag{A1c}$$

$$m_{88}^2 = -\frac{1}{3F_\pi^2}(m_1+m_2+4m_3)\langle \bar{u}u\rangle + O(m_q^2), \tag{A1d}$$

$$m_{99}^2 - \frac{\chi^2}{N_c} = -\frac{2\alpha^2}{3F_\pi^2}(m_1+m_2+m_3)\langle \bar{u}u\rangle + O(m_q^2), \tag{A1e}$$

$$m_{89}^2 = -\frac{\sqrt{2}\,\alpha}{3F_\pi^2}(m_1+m_2-2m_3)\langle \bar{u}u\rangle + O(m_q^2), \tag{A1f}$$

$$m_{33}^2 = -\frac{1}{F_\pi^2}(m_1+m_2)\langle \bar{u}u\rangle + O(m_q^2), \tag{A1g}$$

$$m_{38}^2 = -\frac{1}{\sqrt{3}F_\pi^2}(m_1-m_2)\langle \bar{u}u\rangle + O(m_q^2), \tag{A1h}$$

$$m_{39}^2 = -\frac{\sqrt{2}\,\alpha}{\sqrt{3}F_\pi^2}(m_1-m_2)\langle \bar{u}u\rangle + O(m_q^2), \tag{A1i}$$

where $(m_{\pi^\pm}^2)_{\rm em}$ and $(m_{K^\pm}^2)_{\rm em}$ are the electromagnetic contributions to m_π^2 and m_K^2 respectively, $\alpha \equiv F_\pi/F_s$, and χ^2/N_c is the 'anomaly' contribution to the singlet pseudoscalar mass squared.

The quark mass ratios can be determined from equations (A1a), (A1b), (A1c) and (42):

$$\frac{2m_3}{m_1+m_2} \approx \frac{m_{K^0}^2+(m_{K^+}^2-\mathrm{em})-(m_{\pi^+}^2-\mathrm{em})}{m_{\pi^+}^2-\mathrm{em}} \approx 26\cdot146, \tag{A2a}$$

$$\frac{m_2-m_1}{m_1+m_2} \approx \frac{m_{K^0}^2-(m_{K^+}^2-\mathrm{em})}{m_{\pi^+}^2-\mathrm{em}} \approx 0\cdot2991. \tag{A2b}$$

The π^0–η–η' matrix elements (A1d)–(A1i) are then a function of only two variables, $\alpha = F_\pi/F_s$ and χ^2/N_c. The equation (the trace of the mixing matrix)

$$m_{33}^2+m_{88}^2+m_{99}^2 = m_{\pi^0}^2+m_\eta^2+m_{\eta'}^2 \approx 1\cdot2363\ \mathrm{GeV}^2\ (\exp) \tag{A3}$$

can be used to eliminate χ^2/N_c. The equation (the determinant of the mixing matrix)

$$\begin{aligned} &m_{33}^2\, m_{88}^2\, m_{99}^2 + 2m_{38}^2\, m_{39}^2\, m_{89}^2 - (m_{39}^2)^2 m_{88}^2 \\ &-(m_{38}^2) m_{99}^2 - (m_{89}^2)^2 m_{33}^2 = m_{\pi^0}^2\, m_\eta^2\, m_{\eta'}^2 \\ &\approx 0\cdot5030\times10^{-2}\ \mathrm{GeV}^6\ (\exp) \end{aligned} \tag{A4}$$

can then be used to determine α. The mixing matrix can then be diagonalized. The results are

$$m_{\pi^0}^2 \approx 0\cdot0180074\ \mathrm{GeV}^2 \quad (\exp\ 0\cdot018215\ \mathrm{GeV}^2),$$

$$m_\eta^2 \approx 0\cdot3063\ \mathrm{GeV}^2 \quad (\exp\ 0\cdot3012\ \mathrm{GeV}^2),$$

$$m_{\eta'}^2 \approx 0\cdot9120\ \mathrm{GeV}^2 \quad (\exp\ 0\cdot9169\ \mathrm{GeV}^2);$$

$$|\pi^0\rangle \approx (1-7\cdot0\times10^{-5})|\pi^3\rangle+1\cdot132\times10^{-2}|\pi^8\rangle+3\cdot40\times10^{-3}|s\rangle,$$

$$|\eta\rangle \approx -1\cdot171\times10^{-2}|\pi^3\rangle+0\cdot9880|\pi^8\rangle+0\cdot1540|s\rangle,$$

$$|\eta'\rangle \approx -1\cdot613\times10^{-3}|\pi^3\rangle-0\cdot1540|\pi^8\rangle+0\cdot9881|s\rangle;$$

$$F_s/F_\pi = \alpha^{-1} \approx 2\cdot32, \quad \chi^2/N_c \approx 0\cdot866\ \mathrm{GeV}^2, \quad \theta \approx 8\cdot86°. \tag{A5}$$

Inserting the value of $a_{\pi^0 s} \approx 3\cdot40\times10^{-3}$ into equations (59) and (60) gives

$$\Gamma(\psi\to\pi^0\gamma):\Gamma(\psi\to\eta\gamma):\Gamma(\psi\to\eta'\gamma) \approx 5\times10^{-4}:1:33\cdot5, \tag{A6a}$$

$$\Gamma(\psi'\to\psi\pi^0)/\Gamma(\psi'\to\psi\eta) \approx 9\cdot4\times10^{-3}. \tag{A6b}$$

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