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Four-wave Processes Involving Two Low Frequency Waves

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Abstract

The probability is calculated for a four-wave process in which a Langmuir wave combines with two low frequency (ion sound) waves to produce another Langmuir wave or a transverse wave. The major part of the calculation involves relevant approximations to the quadratic and cubic nonlinear response tensors. An estimate based on observational data from the interplanetary medium suggests that the four-wave process may be significant in solar radiophysics.

1. Introduction

Four-wave processes involving two low frequency waves are of interest when analogous three-wave processes involving one low frequency wave are not allowed due to kinematic restrictions. A three-wave process must satisfy relations $\omega = \omega_1+\omega_2$, $\boldsymbol{k} = \boldsymbol{k}_1+\boldsymbol{k}_2$ and a four-wave process must satisfy $\omega = \omega_1+\omega_2+\omega_3$, $\boldsymbol{k} = \boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3$, where ω, $\boldsymbol{k}$ etc. denote the frequency and wavevector of a wave. Consider the three-wave process in which a Langmuir wave $(\omega_1, \boldsymbol{k}_1)$ is converted into a transverse wave $(\omega, \boldsymbol{k})$ by scattering off an ion sound wave $(\pm\omega_2, \pm\boldsymbol{k}_2)$. This is a favourable process for fundamental plasma emission in solar radio bursts (see e.g. Melrose 1980, 1985). The transverse wave has a wavenumber much less than that of the Langmuir wave, i.e. $k \ll k_1$, and hence the ion sound wave must have $\boldsymbol{k}_2 \approx \mp\boldsymbol{k}_1$. This is a severe kinematic restriction and implies that only a very specific subclass of ion sound can be involved in the three-wave process. The kinematic restrictions on the three-wave process in which the final wave $(\omega, \boldsymbol{k})$ is another Langmuir wave are not as severe, but still select a specific subclass of the ion sound waves (e.g. Melrose 1982*a*). In the corresponding four-wave process the sum $(\omega_2+\omega_3, \boldsymbol{k}_2+\boldsymbol{k}_3)$ of the two low frequency waves plays the same role as the single low frequency wave in the three-wave process. Clearly a much larger subclass of the ion sound waves can satisfy the relevant condition $(\boldsymbol{k}_2+\boldsymbol{k}_3 \approx -\boldsymbol{k}_1)$ to allow a Langmuir wave to be scattered into a transverse wave through the four-wave process than through the three-wave process.

There are two reasons for interest in four-wave processes here. One is, as the foregoing argument implies, that they become candidates for driving the evolution of Langmuir turbulence when the kinematic restrictions on the three-wave process are not satisfied. The second reason concerns a specific theory for fine structures in type

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IIIb solar radio bursts (Melrose 1983*a*). This theory involves a four-wave process, but the estimates for the efficacy of this process were at best semiquantitative due to the lack of a detailed form for the probability of the process. An appropriate approximate form for the probability is derived here.

Inspection of the scattering amplitude for the four-wave process (cf. equation 18 below) shows that it is composed of four terms which may be interpreted as follows. One term involves the direct four-wave interaction due to the cubic nonlinear response of the plasma. This involves the cubic response tensor, here denoted by $\alpha^{\mu\nu\delta\sigma}(k, k_1, k_2, k_3)$ in a 4-tensor notation, in which two of the arguments $[k_2 = (\omega_2, \boldsymbol{k}_2)$ and $k_3 = (\omega_3, \boldsymbol{k}_3)]$ describe low frequency waves and the other two (k and k_1) describe high frequency waves. A second term involves a beat between the two low frequency waves to form a low frequency compound disturbance at k_2+k_3 through the quadratic nonlinear response of the plasma, with this beat then coupling to the two high frequency waves through another quadratic nonlinear response. The remaining two terms involve a beat between the initial Langmuir wave and one or other of the low frequency waves to form an intermediate Langmuir-wave-like disturbance, which then beats with the other low frequency wave. These last two terms involve the quadratic response tensor twice, with two high frequency and one low frequency disturbances being involved in both cases. An approximate form for the quadratic response tensor in this case is well known, being required to treat the three-wave process.

The first two of the aforementioned terms require approximate forms for the cubic response tensor when two of the waves are of low frequency, and for the quadratic response tensor when all three waves are of low frequency, respectively. The appropriate approximations do not appear to be known. The major part of this paper is devoted to the calculation of such approximate forms.

In Section 2 the formal theory of the relevant plasma processes is outlined in terms of a 4-tensor theory developed in earlier papers (Melrose 1982*b*, 1983*b*; Melrose and Kuijpers 1984). The relevant approximations to the nonlinear response tensors are derived in Section 3 and in some Appendixes, and are used in Section 4 to derive an approximate form for the probability for the four-wave process. A possible application of the four-wave process to processes in the interplanetary medium is discussed in Section 5 and the conclusions are given in Section 6.

2. Formal Theory

The weak turbulence expansion of the 4-current $J^\mu(k)$ in powers of the 4-potential $A^\mu(k)$ defines a hierarchy of response tensors:

$$J^\mu(k) = \alpha^{\mu\nu}(k)\, A_\nu(k) + \sum_{n=2}^{\infty} \int \mathrm{d}\lambda^{(n)}\, \alpha^{(n)\mu\nu_1\ldots\nu_n}(k, k_1, \ldots, k_n) \times A_{\nu_1}(k_1) \ldots A_{\nu_n}(k_n), \tag{1}$$

$$\mathrm{d}\lambda^{(n)} = \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \ldots \frac{\mathrm{d}^4 k_n}{(2\pi)^4}\,(2\pi)^4 \delta^4(k-k_1-\ldots-k_n). \tag{2}$$

A covariant version of kinetic theory (e.g. Melrose 1982*b*; Melrose and Kuijpers 1984) enables one to calculate the response tensors in terms of the eight-dimensional distribution function $F(u)$, where u is the 4-velocity:

$$\alpha^{\mu\nu}(k) = -\frac{q^2}{m}\int \mathrm{d}^4 u\, F(u)\, a^{\mu\nu}(k, k, u), \tag{3}$$

$$\begin{aligned}\alpha^{\mu\nu\rho}(k, k_1, k_2) = -\frac{q^3}{2m^2}\int \mathrm{d}^4 u\, F(u)\bigg(a^{\mu\nu}(k, k_1, u)\frac{k_{2\alpha}}{k_2 u}\, G^{\alpha\rho}(k_2, u)\\ + a^{\mu\rho}(k, k_2, u)\frac{k_{1\alpha}}{k_1 u}\, G^{\alpha\nu}(k_1, u) + a^{\nu\rho}(k_1, k_2, u)\frac{k_\alpha}{k u}\, G^{\alpha\mu}(k, u)\bigg),\end{aligned} \tag{4}$$

$$\begin{aligned}\alpha^{\mu\nu\rho\sigma}(k, k_1, k_2, k_3) = -\frac{q^4}{6m^3}\int \mathrm{d}^4 u\, F(u)\bigg\{\frac{(k_2+k_3)^2}{\{(k_2+k_3)u\}^2}\, a^{\mu\nu}(k, k_1, u)\, a^{\rho\sigma}(k_2, k_3, u)\\ + \frac{a^{\mu\nu}(k, k_1, u)}{(k-k_1)u}\left(\frac{k_{2\alpha}(k_{2\beta}+k_{3\beta})}{k_2 u} + \frac{(k_{2\alpha}+k_{3\alpha})k_{3\beta}}{k_3 u}\right) G^{\alpha\rho}(k_2, u)\, G^{\beta\sigma}(k_3, u)\\ + \frac{a^{\rho\sigma}(k_2, k_3, u)}{(k_2+k_3)u}\left(\frac{k_\alpha(k_\beta-k_{1\beta})}{k u} + \frac{(k_\alpha-k_{1\alpha})k_{1\beta}}{k_1 u}\right) G^{\alpha\mu}(k, u)\, G^{\beta\nu}(k_1, u)\\ + (\nu, k_1) \leftrightarrow (\rho, k_2) + (\nu, k_1) \leftrightarrow (\sigma, k_3)\bigg\}.\end{aligned} \tag{5}$$

In equations (3)–(5), q and m are the charge and mass of a particular species of particle, and only the contribution of one species is retained. The quadratic response tensor (4) is symmetrised over (ν, k_1) and (ρ, k_2), and in the cubic response tensor the symmetrisation is indicated and is to be performed separately over each of the three terms written, giving nine terms in total. The other notation introduced is

$$G^{\alpha\mu}(k, u) = g^{\alpha\mu} - \frac{k^\alpha u^\mu}{k u}, \tag{6}$$

$$\begin{aligned}a^{\mu\nu}(k, k_1, u) &= G^{\alpha\mu}(k, u)\, G_\alpha{}^{\nu}(k_1, u)\\ &= g^{\mu\nu} - \frac{k_1^\mu u^\nu}{k_1 u} - \frac{k^\nu u^\mu}{k u} + \frac{k k_1\, u^\mu u^\nu}{(k u)(k_1 u)}.\end{aligned} \tag{7}$$

The linear response is included in the left-hand side of the wave equation

$$\Lambda^{\mu\nu}(k)\, A_\nu(k) = -\mu_0\, J^\mu_{\mathrm{ext}}(k), \tag{8}$$

$$\Lambda^{\mu\nu}(k) = k^2 g^{\mu\nu} - k^\mu k^\nu + \mu_0\, \alpha^{\mu\nu}(k), \tag{9}$$

and the extraneous (ext) current arises from the nonlinear responses. The photon propagator $D^{\mu\nu}(k)$ is such that the solution of (8) is

$$A^\mu(k) = D^\mu{}_\nu(k)\, J^\nu_{\mathrm{ext}}(k). \tag{10}$$

The plasma is assumed isotropic in an inertial frame with 4-velocity $\bar{u}$ (relative to the arbitrarily chosen reference frame). A separation into longitudinal (L) and transverse (T) parts may be achieved by introducing (Melrose 1982*b*)

$$L^{\mu\nu}(k, u) = \frac{k^2}{k^2-(ku)^2}\left\{a^{\mu\nu}(k, k, u) - \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\right\}$$

$$= \frac{1}{k^2-(ku)^2}\, k_\alpha\, G^{\alpha\mu}(k, u)\, k_\beta\, G^{\beta\nu}(k, u), \tag{11a}$$

$$T^{\mu\nu}(k, u) = \frac{1}{k^2-(ku)^2}\left\{-(ku)^2 a^{\mu\nu}(k, k, u) + k^2\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\right\}. \tag{11b}$$

Then we have

$$\alpha^{\mu\nu}(k) = L^{\mu\nu}(k, \bar{u})\,\alpha^{\mathrm{L}}(k) + T^{\mu\nu}(k, \bar{u})\alpha^{\mathrm{T}}(k), \tag{12}$$

and similarly for $\Lambda^{\mu\nu}(k)$ and $D^{\mu\nu}(k)$. We also have

$$\alpha^{\mathrm{L}}(k) = \frac{(k\bar{u})^4}{k^4}\, L_{\mu\nu}(k, \bar{u})\,\alpha^{\mu\nu}(k), \tag{13a}$$

$$\alpha^{\mathrm{T}}(k) = \tfrac{1}{2}\, T_{\mu\nu}(k, \bar{u})\,\alpha^{\mu\nu}(k). \tag{13b}$$

In terms of this theory we have the following expressions for the probabilities of three- and four-wave processes (e.g. Melrose 1986, p. 98). Let M, P, Q, R label wave modes and with a dispersion relation (for the mode M) $\omega = \omega_{\mathrm{M}}(\boldsymbol{k})$, a polarisation 4-vector $e^{\mu}_{\mathrm{M}}(\boldsymbol{k})$, normalised by $|\boldsymbol{e}_{\mathrm{M}}(\boldsymbol{k})|^2 = 1$ in the temporal gauge $e^{0}_{\mathrm{M}}(\boldsymbol{k}) = 0$, and a ratio $R_{\mathrm{M}}(\boldsymbol{k})$ of electric to total energy. Also let $k_{\mathrm{M}} = (\omega_{\mathrm{M}}(\boldsymbol{k}), \boldsymbol{k})$ denote the wave 4-vector satisfying the dispersion relation. The probability for the three-wave process $\mathrm{P+Q} \rightarrow \mathrm{M}$ is

$$w_{\mathrm{MPQ}}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{\hbar}{\epsilon_0^3}\, \frac{R_{\mathrm{M}}(\boldsymbol{k})\, R_{\mathrm{P}}(\boldsymbol{k}_1)\, R_{\mathrm{Q}}(\boldsymbol{k}_2)}{\omega_{\mathrm{M}}(\boldsymbol{k})\,\omega_{\mathrm{P}}(\boldsymbol{k}_1)\,\omega_{\mathrm{Q}}(\boldsymbol{k}_2)}\,|\alpha_{\mathrm{MPQ}}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)|^2$$

$$\times (2\pi)^4 \delta^4(k_{\mathrm{M}} - k_{\mathrm{P}} - k_{\mathrm{Q}}), \tag{14}$$

with

$$\alpha_{\mathrm{MPQ}}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) = e^{*}_{\mathrm{M}\mu}(\boldsymbol{k})\, e_{\mathrm{P}\nu}(\boldsymbol{k}_1)\, e_{\mathrm{Q}\rho}(\boldsymbol{k}_2)\, 2\alpha^{\mu\nu\rho}(k_{\mathrm{M}}, k_{1\mathrm{P}}, k_{2\mathrm{Q}}). \tag{15}$$

The analogous result for the four-wave process $\mathrm{P+Q+R} \rightarrow \mathrm{M}$ is

$$w_{\mathrm{MPQR}}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = \frac{\hbar^2}{\epsilon_0^4}\, \frac{R_{\mathrm{M}}(\boldsymbol{k})\, R_{\mathrm{P}}(\boldsymbol{k}_1)\, R_{\mathrm{Q}}(\boldsymbol{k}_2)\, R_{\mathrm{R}}(\boldsymbol{k}_3)}{\omega_{\mathrm{M}}(\boldsymbol{k})\,\omega_{\mathrm{P}}(\boldsymbol{k}_1)\,\omega_{\mathrm{Q}}(\boldsymbol{k}_2)\,\omega_{\mathrm{R}}(\boldsymbol{k}_3)} \tag{16}$$

$$\times |\alpha_{\mathrm{MPQR}}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)|^2 (2\pi)^4 \delta^4(k_{\mathrm{M}} - k_{\mathrm{P}} - k_{\mathrm{Q}} - k_{\mathrm{R}}),$$

with

$$\alpha_{\mathrm{MPQR}}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = e^{*}_{\mathrm{M}\mu}(\boldsymbol{k})\, e_{\mathrm{P}\nu}(\boldsymbol{k}_1)\, e_{\mathrm{Q}\rho}(\boldsymbol{k}_2)\, e_{\mathrm{R}\sigma}(\boldsymbol{k}_3)\, \alpha^{\mu\nu\rho\sigma}_{\mathrm{eff}}(k_{\mathrm{M}}, k_{1\mathrm{P}}, k_{2\mathrm{Q}}, k_{3\mathrm{R}}), \tag{17}$$

and with

$$
\begin{aligned}
\alpha_{\text{eff}}^{\mu\nu\rho\sigma}(k, k_1, k_2, k_3) = {} & 6\alpha^{\mu\nu\rho\sigma}(k, k_1, k_2, k_3) \\
& + 4\alpha^{\mu\nu\theta}(k, k_1, k-k_1)\, D_{\theta\eta}(k-k_1)\, \alpha^{\eta\rho\sigma}(k-k_1, k_2, k_3) \\
& + 4\alpha^{\mu\rho\theta}(k, k_2, k-k_2)\, D_{\theta\eta}(k-k_2)\, \alpha^{\eta\nu\sigma}(k-k_2, k_1, k_3) \\
& + 4\alpha^{\mu\sigma\theta}(k, k_3, k-k_3)\, D_{\theta\eta}(k-k_3)\, \alpha^{\eta\nu\rho}(k-k_3, k_1, k_2)\,. \qquad (18)
\end{aligned}
$$

Crossing symmetries allow one to write down the probabilities for crossed processes simply by reversing the sign of the wave 4-vector; for example, the probability for $\mathrm{Q}+\mathrm{R} \rightarrow \mathrm{M}+\mathrm{P}$ follows from (16) by replacing k_{P} by $-k_{\mathrm{P}}$.

3. Approximations

Two types of physical approximation are made here in simplifying the expressions (3)–(5) for the response tensors. One type of approximation is to the distribution function. We have already assumed that the particles are isotropic [in the rest frame $\bar{u} = (1, \mathbf{0})$], for example, in writing down the form (12). Here we further assume that the particles are nonrelativistic. The implications of this assumption are summarised in Appendix 1. In calculating explicit forms for the response tensors we use the rest frame and assume a Maxwellian distribution of velocities proportional to $\exp(-v^2/2\,V^2)$, where V is the 'thermal speed'.

(a) Cold Plasma Limit

The other type of approximation involves the ratio of the phase speed $\omega/|\boldsymbol{k}|$ of the waves to the thermal speed V in the rest frame. A disturbance is said to be fast for $\omega/|\boldsymbol{k}|\,V \gg 1$. When all the disturbances are fast, including the beats $k-k_1$ etc. which appear explicitly in (5), we may assume $ku \approx k\bar{u}$, $k_1\,u \approx k_1\,\bar{u}$ etc. in (3)–(5). We also make the nonrelativistic approximation. The resulting approximations to (3)–(5) are obtained from them by replacing $\int \mathrm{d}^4 u\, F(u)$ by n (cf. Appendix 1), and setting $u = \bar{u}$ elsewhere.

These approximation forms correspond to those obtained using cold plasma theory, i.e. using a theory in which the particles are described as a fluid with 4-velocity $\bar{u}$.

(b) One Slow Disturbance

Another class of known approximations may be obtained by assuming that one disturbance is slow and the other two are fast. For the linear response (3), where only one wave 4-vector k is involved, this corresponds to assuming $|\omega| \ll |\boldsymbol{k}|\,V$ in the rest frame. For the quadratic response we are free to identify k_2 as the slow disturbance, with k and k_1 corresponding to fast disturbances. The relevant approximation for the cubic response corresponds to all of k, k_1, k_2, k_3 describing fast disturbances, but with one of the beats, $k-k_1 = k_2+k_3$ say, corresponding to a slow disturbance.

The approximations made in treating these cases involve (i) assuming the relevant ku [i.e. ku in (3), $k_2\,u$ in (4) and $(k-k_1)u$ in (5)] to be small, and retaining only the term with the highest power of this factor in the denominator, and (ii) assuming the relevant disturbance to be longitudinal. The details of these approximations are described in Appendix 2. The results involve

$$\alpha^{\mathrm{L}}(k) \approx -\frac{q^2}{m}\frac{k^2(k\bar{u})^2}{k^2-(k\bar{u})^2}\int \frac{\mathrm{d}^4 u\, F(u)}{(ku)^2}. \tag{19}$$

For a Maxwellian distribution in the rest frame one finds

$$\alpha^{\mathrm{L}}(k) \approx \frac{q^2 n}{m}\frac{\omega^2}{|\boldsymbol{k}|^2 V^2}\left\{1-\phi\left(\frac{\omega}{\sqrt{2}|\boldsymbol{k}|V}\right)\right\}, \tag{20}$$

where

$$\phi(z) = -\frac{z}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{d}t\,\frac{\mathrm{e}^{-t^2}}{t-z} = 2z\mathrm{e}^{-z^2}\int_0^z \mathrm{d}t\,\mathrm{e}^{t^2} \tag{21}$$

is a form of the plasma dispersion function. It is relevant to note that for $\omega = 0$ in the rest frame the integral in (19) appears to be positive definite, and yet its value implied by the well-known result (20) is negative. This point is discussed in Appendix 3.

The explicit approximations obtained for this case are

$$\alpha^{\mu\nu}(k) \approx \frac{k_\alpha\, G^{\alpha\mu}(k,\bar{u})\, k_\beta\, G^{\beta\nu}(k,\bar{u})}{k^2-(k\bar{u})^2}\,\alpha^{\mathrm{L}}(k), \tag{22}$$

$$\alpha^{\mu\nu\rho}(k,k_1,k_2) \approx \frac{q}{2m}\,a^{\mu\nu}(k,k_1,\bar{u})\frac{k_{2\alpha}}{k_2\,\bar{u}}\,G^{\alpha\rho}(k_2,\bar{u})\,\alpha^{\mathrm{L}}(k_2), \tag{23}$$

$$\alpha^{\mu\nu\rho\sigma}(k,k_1,k_2,k_3) \approx \frac{q^2}{6m^2}\,a^{\mu\nu}(k,k_1,\bar{u})\,a^{\rho\sigma}(k_2,k_3,\bar{u})\frac{(k-k_1)^2}{\{(k-k_1)\bar{u}\}^2}\,\alpha^{\mathrm{L}}(k-k_1). \tag{24}$$

(c) Three Slow Disturbances

Suppose both k_1 and k_2 describe slow disturbances; then under most circumstances k_1+k_2 also corresponds to a slow disturbance. When this is the case we require an approximation to $\alpha^{\mu\nu\rho}(k,k_1,k_2)$ in which all disturbances are slow. Similarly if k_1, k_3 and k_2+k_3 describe slow disturbances with k and k_1 describing fast disturbances (but not $k-k_1 = k_2+k_3$) then we require an analogous approximation to $\alpha^{\mu\nu\rho\sigma}(k,k_1,k_2,k_3)$. The method of deriving these approximations is the same as in the case of one slow disturbance. However, now the highest powers of ku etc. in the denominator correspond to the fourth power of v. The specific combination which appears is of the form

$$\phi(k,k_1,k_2) = \int \mathrm{d}^4 u\, F(u)\left(\frac{(k_1\,k_2)k^2}{k_1 u\, k_2 u(ku)^2}+\frac{(kk_1)k_2^2}{ku\, k_1 u(k_2\,u)^2}+\frac{(kk_2)k_1^2}{ku\, k_2 u(k_1\,u)^2}\right), \tag{25}$$

with $k = k_1+k_2$.

The integrals in (25) may be evaluated explicitly in the limit of zero phase speed (i.e. for $\omega = \omega_1 = \omega_2 = 0$). The details are outlined in Appendix 4. The result is

$$\phi(k,k_1,k_2) = n/V^4. \tag{26}$$

The analysis in Appendix 4 leading to (26) is surprisingly cumbersome in view of the simplicity of the result. A simple physical model for the response leads to the form (26). This model involves assuming that the nonlinear response to a static electric potential ϕ involves a variation in the number density proportional to $\exp(-q\phi/mV^2)$. The details are outlined in Appendix 5.

The resulting approximations to the quadratic and cubic response tensors in this limit of essentially zero frequencies for three disturbances are, respectively,

$$\alpha^{\mu\nu\rho}(k, k_1, k_2) \approx -\frac{q^3 n}{2m^2 V^4} \frac{k\bar{u}\, k_1\bar{u}\, k_2\bar{u}}{k^2 k_1^2 k_2^2} k_\alpha\, G^{\alpha\mu}(k, \bar{u})\, k_{1\beta}\, G^{\beta\nu}(k_1, \bar{u})\, k_{2\gamma}\, G^{\beta\delta}(k_2, \bar{u}), \tag{27}$$

$$\alpha^{\mu\nu\rho\sigma}(k, k_1, k_2, k_3) \approx -\frac{q^4 n}{6m^3 V^4} a^{\mu\nu}(k, k_1, \bar{u}) \frac{k_2\bar{u}\, k_3\bar{u}}{k_2^2 k_3^2} k_{2\alpha}\, G^{\alpha\rho}(k_2, \bar{u})\, k_{3\beta}\, G^{\beta\rho}(k_3, \bar{u}). \tag{28}$$

(d) Photon Propagator

We retain only the longitudinal part of the photon propagator, so that (12) with (11a) implies

$$D^{\mu\nu}(k) \approx \frac{1}{k^2-(k\bar{u})^2} k_\alpha\, G^{\alpha\mu}(k, \bar{u})\, k_\beta\, G^{\beta\nu}(k, \bar{u})\, D^{\mathrm{L}}(k). \tag{29}$$

The relation between $D^{\mathrm{L}}(k)$ and the longitudinal (in the rest frame) part of the dielectric tensor $K^{\mathrm{L}}(k)$ is

$$D^{\mathrm{L}}(k) = -\mu_0 \frac{(k\bar{u})^2}{k^4} \frac{1}{K^{\mathrm{L}}(k)}. \tag{30}$$

In the rest frame we assume

$$K^{\mathrm{L}}(k) \approx 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}\left(1 + \frac{3|\boldsymbol{k}|^2 V_{\mathrm{e}}^2}{\omega^2}\right), \quad \omega \approx \omega_{\mathrm{p}}, \quad \omega/|\boldsymbol{k}| \gg V_{\mathrm{e}} \tag{31a}$$

$$\approx 1 + \frac{1}{|\boldsymbol{k}|^2 \lambda_{\mathrm{De}}^2} \frac{\omega_{\mathrm{pi}}^2}{\omega^2}, \qquad \omega \lesssim \omega_{\mathrm{pi}}, \quad V_{\mathrm{i}} \ll \frac{\omega}{|\boldsymbol{k}|} \ll V_{\mathrm{e}}. \tag{31b}$$

4. Approximate Probability

The four-wave probability (16) involves a scattering amplitude (17) with four different terms (18). For the case of two low frequency waves causing the scattering of a Langmuir wave into another Langmuir wave or a transverse wave, these four terms may be interpreted as explained in the Introduction. We need to find approximate forms for each of these four terms to determine which is the dominant one.

The first two terms in (18) may be combined as follows. We use the approximations (28) for the cubic response (27) (with $k \to k_2 + k_3$, $k_1 \to k_2$, $k_2 \to k_3$) for the beat between the two low frequency waves, (30) with (31b) for the photon propagator (with $k \to k - k_1$) and (23) (with $k_2 \to k - k_1$) for the remaining beat in the second term in (18). We further assume

$$\alpha^{\mathrm{L}}(k-k_1) \approx \frac{q^2 n}{m V^2} \frac{\{(k-k_1)\bar{u}\}^2}{(k-k_1)^2} \tag{32}$$

for the low frequency disturbance at $k-k_1 = k_2+k_3$, and use the identity

$$k_\alpha\, G^{\alpha\mu}(k,\bar{u})\, k^\beta\, G_{\beta\mu}(k,\bar{u}) = \frac{k^2\{k^2-(k\bar{u})^2\}}{(k\bar{u})^2}. \tag{33}$$

Finally, we retain only the contribution of the electrons ($q = -e$, $m = m_{\mathrm{e}}$) to $\alpha^{\mu\nu\rho}$ and $\alpha^{\mu\nu\rho\sigma}$. Then in (18) we have

$$\begin{aligned}6\alpha^{\mu\nu\rho\sigma}(k,k_1,k_2,k_3) &+ 4\alpha^{\mu\nu\theta}(k,k_1,k-k_1)\,D_{\theta\eta}(k-k_1)\,\alpha^{\eta\rho\sigma}(k-k_1,k_2,k_3)\\ &\approx \frac{e^4 n_{\mathrm{e}}}{m_{\mathrm{e}}^3 V_{\mathrm{e}}^4}\left(1+\frac{\omega_{\mathrm{p}}^2}{(k-k_1)^2 V_{\mathrm{e}}^2 K^{\mathrm{L}}(k-k_1)}\right)\alpha^{\mu\nu}(k,k_1,\bar{u})\,\frac{k_2\bar{u}\,k_3\bar{u}}{k_2^2\,k_3^2}\\ &\quad\times k_{2\alpha}\,G^{\alpha\rho}(k_2,\bar{u})\,k_{3\beta}\,G^{\beta\sigma}(k_3,\bar{u}).\end{aligned} \tag{34}$$

The quantity in large parentheses may be rewritten in the following form in the rest frame:

$$\left(1+\frac{\omega_{\mathrm{p}}^2}{(k-k_1)^2 V_{\mathrm{e}}^2 K^{\mathrm{L}}(k-k_1)}\right) \approx \left(\frac{1+\chi_{\mathrm{i}}(k-k_1)}{1+\chi_{\mathrm{e}}(k-k_1)+\chi_{\mathrm{i}}(k-k_1)}\right), \tag{35}$$

with the electron and ion susceptibilities approximated by

$$\chi_{\mathrm{e}}(k-k_1) \approx 1/|\boldsymbol{k}-\boldsymbol{k}_1|^2\lambda_{\mathrm{De}}^2, \qquad \chi_{\mathrm{i}}(k-k_1) \approx -\omega_{\mathrm{pi}}^2/(\omega-\omega_1)^2,$$

respectively (cf. equation 31b).

The assumption that the electronic contribution to the quadratic nonlinear response tensor (27) dominates over the ionic contribution is not necessarily well satisfied. Inspection of the approximations derived in Section 3 shows that, in passing from the fast to the slow limits for any particular disturbance, the magnitude of the quadratic response tensor is reduced by a factor of the order of the fourth power of the ratio of the (slow) phase speed to the thermal speed. On comparing the electronic contribution from (27) for $\omega_2/|\boldsymbol{k}_2|\,V_{\mathrm{e}} \ll 1$, $\omega_3/|\boldsymbol{k}_3|\,V_{\mathrm{e}} \ll 1$ with the ionic contribution for $\omega_2/|\boldsymbol{k}_2|\,V_{\mathrm{i}} \gg 1$, $\omega_3/|\boldsymbol{k}_3|\,V_{\mathrm{i}} \gg 1$, i.e. for cold ions, one finds them roughly in the ratio $(m_{\mathrm{i}}^2/m_{\mathrm{e}}^2)(\omega_2/|\boldsymbol{k}_2|\,V_{\mathrm{e}})^2(\omega_3/|\boldsymbol{k}_3|\,V_{\mathrm{e}})^2$, which is of order unity for ion sound waves with $\omega_2 = |\boldsymbol{k}_2|\,v_{\mathrm{s}}$, $v_{\mathrm{s}}^2/V_{\mathrm{e}}^2 = m_{\mathrm{e}}/m_{\mathrm{i}}$. Thus the electronic and ionic contributions to $\alpha^{\eta\rho\sigma}(k-k_1,k_2,k_3)$ should be comparable. However, we now argue that the final two terms in (18) are larger in magnitude than the first two terms, now approximated by (34), and for present purposes it suffices to note that this ionic contribution, which should be included in (34), is no larger in magnitude than the electronic contribution to $\alpha^{\eta\rho\sigma}(k-k_1,k_2,k_3)$ which has been retained in (34).

The final two terms in (18) are of the same form with k_2 and k_3 interchanged. Using the symmetry property

$$\alpha^{\mu\nu\rho}(k,k_1,k_2) = \alpha^{\mu\rho\nu}(k,k_2,k_1), \tag{36}$$

we have the following approximations from (23) with (32) to the tensors in the third term in (18):

$$\alpha^{\mu\rho\theta}(k, k_2, k-k_2) \approx -\frac{q^3}{2m^2 V^2} a^{\mu\theta}(k, k-k_2, \bar{u}) \frac{k_2 \bar{u}}{k_2^2} k_{2\alpha} G^{\alpha\rho}(k_2, \bar{u}), \quad (37a)$$

$$\alpha^{\eta\nu\sigma}(k-k_2, k_1, k_3) \approx -\frac{q^3}{2m^2 V^2} a^{\eta\nu}(k-k_2, k_1, \bar{u}) \frac{k_3 \bar{u}}{k_3^2} k_{3\beta} G^{\beta\sigma}(k_3, \bar{u}). \quad (37b)$$

Assuming that only the longitudinal part of the photon propagator (cf. equation 30) is retained, we then find

$$a^{\mu\theta}(k, k-k_2, \bar{u}) D_{\theta\eta}(k-k_2) a^{\eta\nu}(k-k_2, k_1, \bar{u})$$

$$\approx -\frac{\mu_0}{K^{\mathrm{L}}(k-k_2)} \frac{(k-k_2)_\alpha G^{\alpha\mu}(k, \bar{u})(k-k_2)_\beta G^{\beta\nu}(k_1, \bar{u})}{[(k-k_2)^2-\{(k-k_2)\bar{u}\}^2]\{(k-k_2)\bar{u}\}^2}. \quad (38)$$

Then this term in (18) reduces to, for electrons,

$$4\alpha^{\mu\rho\theta}(k, k_2, k-k_2) D_{\theta\eta}(k-k_2) \alpha^{\eta\nu\sigma}(k-k_2, k_1, k_3)$$

$$\approx -\frac{e^4 n_e}{m_e^3 V_e^4} \frac{\omega_p^2}{K^{\mathrm{L}}(k-k_2)} \frac{(k-k_2)_\alpha G^{\alpha\mu}(k, \bar{u})}{[(k-k_2)^2-\{(k-k_2)\bar{u}\}^2]} \frac{(k-k_2)_\beta G^{\beta\nu}(k_1, \bar{u})}{\{(k-k_2)\bar{u}\}^2}$$

$$\times \frac{k_2 \bar{u}\, k_3 \bar{u}}{k_2^2\, k_3^2} k_{2\gamma} G^{\gamma\rho}(k_2, \bar{u})\, k_{3\delta} G^{\delta\sigma}(k_3, \bar{u}). \quad (39)$$

Comparison of (34) and (39) for $\omega \approx \omega_1 \approx \omega_p$ shows that they are roughly in the ratio

$$\frac{1+\chi_i(k-k_1)}{1+\chi_e(k-k_1)+\chi_i(k-k_1)} : -\frac{1}{K^{\mathrm{L}}(k-k_2)}$$

$$\approx \frac{|k-k_1|^2 v_s^2}{(\omega-\omega_1)^2-|\boldsymbol{k}-\boldsymbol{k}_1|^2 v_s^2} : \frac{\omega_p^2}{(\omega-\omega_2)^2-\omega_p^2-3|\boldsymbol{k}-\boldsymbol{k}_2|^2 V_e^2}, \quad (40)$$

where we use the approximations (31). The case of interest here is when $|\boldsymbol{k}_2| \approx |\boldsymbol{k}_3|$ is much greater than $|\boldsymbol{k}_1|$ or $|\boldsymbol{k}|$. For ion sound waves we then have $\omega-\omega_1 = \omega_2+\omega_3 \approx 2|\boldsymbol{k}_2| v_s$, and the ratio (40) is much less than unity. Hence in the following we retain only the approximation (39) to (18).

The two low frequency waves are assumed longitudinal, and the two high frequency waves may be described by their polarisation 3-vectors $\boldsymbol{e}_M$ and $\boldsymbol{e}_P$ in the temporal gauge. Then (17) with (39) simplifies in the rest frame to

$$\alpha_{\mathrm{MPss'}}(k, k_1, k_2, k_3) \approx -\frac{2e^4 n_e}{3m_e^3 V_e^4} \frac{\omega_{pi}^3}{|\boldsymbol{k}_2|^2} \boldsymbol{e}_M^* \cdot \boldsymbol{\kappa}_2\, \boldsymbol{e}_P \cdot \boldsymbol{\kappa}_2, \quad (41)$$

where $\boldsymbol{\kappa}_2$ is a unit vector along $\boldsymbol{k}_2$, and where we assume $|\boldsymbol{k}_2| \approx |\boldsymbol{k}_3| \gg |\boldsymbol{k}|, |\boldsymbol{k}_1|$ and that the low frequency waves are ion sound waves with $\omega_2 = |\boldsymbol{k}_2| v_s$ and $v_s^2 = V_e^2 \omega_{pi}^2/\omega_p^2$.

Making the same approximations for the low frequency waves in the probability (16), with $R_s(k_2) \approx \omega_2^2/2\omega_{pi}^2$ and with $R_M \approx R_P \approx \frac{1}{2}$ for the high frequency waves, one finds

$$w_{MPss'}(k, k_1, k_2, k_3) \approx \frac{1}{36}\left(\frac{\hbar\omega_p\,\omega_{pi}}{n_e\,m_e\,V_e^2|k_2|\lambda_{De}}\right)^2$$

$$\times |e_M^* . \kappa_2|^2 |e_P . \kappa_2|^2 (2\pi)^4 \delta^4(k_M - k_P - k_s - k_{s'}), \tag{42}$$

with $k_s = (\omega_s(k_2), k_2)$ and $k_{s'} = (\omega_s(k_3), k_3)$. For comparison the probability (14) when only one ion sound wave is involved is

$$w_{MPs}(k, k_1, k_2) = \frac{\hbar\omega_p^2}{8\,n_e\,m_e\,V_e^2}|k_2|\,v_s|e_M^* . e_P|^2(2\pi)^4\delta^4(k_M - k_P - k_s), \tag{43}$$

with $k_s = (\omega_s(k_2), k_2)$.

5. Application to the Interplanetary Plasma

Suppose that Langmuir waves and short-wavelength ion sound waves are present together in a region of linear dimension L. Let the Langmuir waves be described by a characteristic wavenumber k_L and a range Δk_L, a range $\Delta\Omega_L$ of solid angles to which the directions of k_L are effectively confined, and an effective temperature T_L. Let the ion sound waves be described by a similar set of parameters. We may determine conditions under which the four-wave process is effective as follows. The formal theory is used to estimate the time required for the four-wave process to increase the effective temperature T_T of the resulting transverse waves to the value T_L at which it saturates. Let this time be t_s. The effective optical depth τ_4 for the four-wave process may then be identified as

$$\tau_4 = L/v_g\,t_s, \tag{44}$$

where $v_g \approx \sqrt{3}\,c k_L\,V_e/\omega_p$ is the group speed for the transverse waves (whose frequency is close to $\omega_L = \omega_p + 3k_L^2\,V_e^2/2\omega_p$).

The kinetic equation for the four-wave process P+Q+R $\rightarrow$ M is, in semiclassical form,

$$\frac{dN_M(k)}{dt} = \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{d^3k_3}{(2\pi)^3}\,w_{MPQR}(k, k_1, k_2, k_3)$$

$$\times [N_P(k_1)\,N_Q(k_2)\,N_R(k_3) - N_M(k)\{N_P(k_1)\,N_Q(k_2)$$

$$+ N_P(k_1)\,N_R(k_3) + N_Q(k_2)\,N_R(k_3)\}], \tag{45}$$

with the effective temperature $T_M(k)$ related to the occupation number $N_M(k)$ by

$$T_M(k) = \hbar\omega_M(k)\,N_M(k) \tag{46}$$

for each mode. In estimating t_s only the first term in the braces in (45) is retained, and then with the approximations made above (and M $\to$ T and P $\to$ L) the identification

$$t_s^{-1} = T_L^{-1}\langle \mathrm{d}T_T(k)/\mathrm{d}t\rangle \tag{47}$$

is made, where the average (denoted by angle brackets) is over all angles of emission.

On inserting the probability (42) in (45), the average over $|e_T^* . \kappa_2|^2$ is performed giving a factor $\frac{2}{3}$. Suppose the k_3 integral is performed over $\delta^3(k - k_1 - k_2 - k_3)$. Then we have $k_3 = -k_2 - (k_1 - k)$. In the limit $|k_2| \approx |k_3| \gg |k|, |k_1|$ we may then set $k_3 \approx -k_2$ in the integrand. The integral over k_1 is separated into one over solid angle (giving $\Delta\Omega_2$) and one over $k_1 = |k_1|$. This latter integral is performed over the δ-function for frequency. This gives

$$\int \mathrm{d}k_1\, k_1^2\, \delta(\omega_T - \omega_L - \omega_s \mp \omega_{s'}) \approx \omega_p\, k_L/3\, V_e^2, \tag{48}$$

where the group speed $3V_e^2 k_L/\omega_p$ of the Langmuir waves is assumed much greater than that ($\approx v_s$) of the ion sound waves. Then one finds

$$\frac{1}{t_s} \approx \frac{\Delta\Omega_L\, \Delta\Omega_s}{162(2\pi)^5}\, \frac{\omega_p^6\, k_L}{n_e^2\, V_e^5}\, \frac{|k_1 . \kappa_2|^2}{k_s \lambda_{De}}\, \frac{\Delta k_s}{k_s}\left(\frac{T_s}{T_e}\right)^2, \tag{49}$$

and (44) gives

$$\tau_4 \approx \frac{\Delta\Omega_L\, \Delta\Omega_s}{162(2\pi)^5}\, \frac{\omega_p^7\, L}{n_e^2\, V_e^6\, c}\, \frac{|\kappa_1 . \kappa_2|^2}{k_s \lambda_{De}}\, \frac{\Delta k_s}{k_s}\left(\frac{T_s}{T_e}\right)^2. \tag{50}$$

Note that the factor $|\kappa_1 . \kappa_2|^2$ is the square of the cosine of the angle between the Langmuir wavevector k_L and that of either of the two short wavelength ion sound waves, i.e. k_2 or k_3. Except when these two directions are close to orthogonal, this factor may be replaced by unity for semiquantitative purposes.

It is convenient for practical purposes to express the ratio T_s/T_e in terms of the ratio of the r.m.s. electric energy density $\epsilon_0|E_s|^2$ to the thermal energy density in the electrons $n_e T_e$, or in terms of the r.m.s. fractional density fluctuations $\delta n_e/n_e$. The ratio of electric to total energy in ion sound waves is $\frac{1}{2}k_s^2\lambda_{De}^2$, and hence T_s and $|E_s|^2$ are related by

$$\frac{\Delta\Omega_s\, k_s^3}{(2\pi)^3}\, \frac{\Delta k_s}{k_s}\, T_s = \frac{\epsilon_0|E_s|^2}{k_s^2\lambda_{De}^2}. \tag{51}$$

Also we have

$$\left(\frac{\delta n_e}{n_e}\right)^2 = \frac{\epsilon_0|E_s|^2}{n_e\, T_e}\, \frac{1}{(k_s\lambda_{De})^2}. \tag{52}$$

Eliminating T_s between (50) and (51) and using (52) gives

$$\tau_4 \approx \frac{\Delta\Omega_L}{\Delta\Omega_s}\, \frac{\pi}{81\sqrt{3}}\, \frac{\omega_p L}{c}\, \frac{k_s}{\Delta k_s}\, |\kappa_1 . \kappa_2|^2 \left\{ \begin{array}{l} \dfrac{1}{(k_s\lambda_{De})^{11}}\left(\dfrac{\epsilon_0|E_s|^2}{n_e\, T_e}\right)^2 \\[2ex] \dfrac{1}{(k_s\lambda_{De})^7}\left(\dfrac{\delta n_e}{n_e}\right)^4. \end{array} \right. \tag{53}$$

It is clear that τ_4 is largest for the smallest allowed values of k_s. We have assumed that $k_s \gg k_2$, and hence the most favourable limit of (53) is for $k_s \approx k_L$.

For comparison, an analogous calculation for the three-wave L $\pm$ s $\rightarrow$ T gives

$$\tau_3 \approx \frac{\Delta\Omega_L}{\Delta\Omega_s} \frac{\pi}{18\sqrt{3}} \frac{\omega_p L}{c} \frac{k_s}{\Delta k_s} \frac{1}{(k_L \lambda_{De})^5} \frac{\epsilon_0 |E_s|^2}{n_e T_e}, \tag{54}$$

where now we require $k_s \approx k_L$ from the three-wave condition $\boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2$ with $|\boldsymbol{k}| \ll |\boldsymbol{k}_1|, |\boldsymbol{k}_2|$.

It is apparent that the very strong inverse dependence of (53) on $k_s \lambda_{De}$ favours the smallest value of k_s consistent with our assumptions; this value is $k_s \approx k_L$. Therefore let us compare (53) and (54) for $k_s \approx k_L$. (The four-wave process then involves two ion sound waves each at about 60° to $\boldsymbol{k}_L$, and the three-wave process requires an ion sound wave along $\boldsymbol{k}_L$.) One finds that

$$\frac{\tau_4}{\tau_3} \approx \frac{1}{4} \frac{1}{(k_L \lambda_{De})^6} \frac{\epsilon_0 |E_s|^2}{n_e T_e}. \tag{55}$$

Lin *et al.* (1986) have estimated the parameters of the Langmuir waves and the ion sound waves, as well as of the background plasma and the fast electrons, in two events in the interplanetary plasma. For an event on 11 March 1979, their parameters give $f_p = 13$ kHz, $k_L \lambda_{De} \approx 5{\cdot}1 \times 10^{-2}$, $k_s \lambda_{De} \approx 4{\cdot}0 \times 10^{-2}$, $\epsilon_0 E_s^2/n_e T_e \approx 2{\cdot}5 \times 10^{-10}$, and for an event on 8 February 1979, their parameters give $f_p = 24$ kHz, $k_L \lambda_{De} \approx 4{\cdot}7 \times 10^{-2}$, $k_s \lambda_{De} \approx 4{\cdot}4 \times 10^{-2}$, $\epsilon_0 E_s^2/n_e T_e \approx 7 \times 10^{-13}$. On approximating the factor $(\Delta\Omega_L/\Delta\Omega_s)(k_s/\Delta k_s)|\boldsymbol{\kappa}_1 . \boldsymbol{\kappa}_2|^2$ in (53) by unity and assuming $L \approx 10^6$ m, corresponding to a burst of waves of ≈ 2 s duration being convected past the spacecraft, one finds that $\tau_4 \approx 10^{-3}$ for the first of these events, and a much smaller value for the second of the events. These parameters in (55) give $\tau_4/\tau_3 \approx 3 \times 10^{-3}$ for the more favourable case.

It is interesting that the optical depth (τ_4) for the four-wave process can be nearly comparable with that (τ_3) for the three-wave process for observed values of E_s. It is assumed that the three-wave process is strongly saturated in most theories for fundamental plasma emission in which it is invoked. The foregoing estimate of τ_4/τ_3 then suggests that the four-wave process may also saturate, and may lead to bright emission under favourable conditions. In particular, in the theory for stria bursts (Melrose 1983*a*) based on the four-wave process, it seems plausible that one has $\tau_4 \gtrsim 1$ in localised regions to produce the localised emission in each stria. However, the requirements that the low frequency waves have frequencies near the lower hybrid frequency (Melrose 1983*a*) and that $k_s \lambda_{De}$ be relatively small (for τ_4 to be large) are difficult to reconcile.

It is reasonable to conclude from this discussion that the four-wave process could lead to detectable emission in solar radio bursts, but that the argument for this is not a compelling one. The estimates are subject to considerable uncertainty, and the most favourable case ($k_s \lesssim k_L$) is at the limit of validity of the approximation $k_s \gg k_L$ made in deriving the approximate formulas used.

6. Conclusions

The main results of the present paper can be summarised as follows.

(1) The approximate expression for the quadratic nonlinear response tensor for three slow disturbances is that given by a simple model based on Debye shielding (Appendix 5). An analogous approximation applies for the cubic response tensor when two of the disturbances are slow.

(2) For short wavelength ion sound waves the four-wave process is dominated by the matrix elements which correspond to two consecutive three-wave processes with the intermediate state being a virtual Langmuir wave.

(3) In the short wavelength limit the optical depth for the four-wave process is a strongly increasing function of decreasing wavenumber k_s of the ion sound waves.

(4) At the limit of validity of the approximations made here, specifically for $k_s \approx k_L$, where k_L is the wavenumber of the Langmuir waves, the ratio of the optical depth for the four-wave process to that for the three-wave process is given by (55). For parameters based on observational data from the interplanetary plasma, this ratio is not particularly small, suggesting that the four-wave process may be important in practice.

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Appendix 1. Nonrelativistic Approximation

The integrals

$$\int \mathrm{d}^4 u\, F(u)(u\bar{u})^N = n^{(N)} \tag{A1}$$

correspond to the proper number density for $N = 0$ and the actual number density in the rest frame for $N = 1$. For a distribution which is isotropic in the rest frame one can evaluate integrals of the form

$$N^{\nu_1 \dots \nu_n} = \int \mathrm{d}^4 u\, F(u) u^{\nu_1} \dots u^{\nu_n} \tag{A2}$$

explicitly, e.g.

$$N^{\nu} = n^{(1)} \bar{u}^{\mu}, \tag{A3a}$$

$$N^{\mu\nu} = (\tfrac{4}{3} n^{(2)} - \tfrac{1}{3} n^{(0)}) \bar{u}^{\mu} \bar{u}^{\nu} - \tfrac{1}{3}(n^{(2)} - n^{(0)}) g^{\mu\nu}, \tag{A3b}$$

and so on.

In the nonrelativistic limit one sets the Lorentz factor of the particles in their rest frame equal to unity. This corresponds to $u\bar{u} = 1$. The the $n^{(N)}$ in (A1) are all to be approximated by the same value n. The explicit evaluations, e.g. as in (A3), then lead to

$$N^{\nu_1 \dots \nu_n} = n \bar{u}^{\nu_1} \dots \bar{u}^{\nu_n}. \tag{A4}$$

Appendix 2. One Slow Disturbance

For k_2 a slow disturbance and both k and k_1 fast disturbances, we set $ku = k\bar{u}$ and $k_1 u = k_1 \bar{u}$ in (4) and retain only the term with $(k_2 u)^2$ in the denominator. However, this procedure ignores the charge continuity and gauge invariance conditions:

$$k_\mu\, \alpha^{\mu\nu\rho}(k, k_1, k_2) = k_{1\nu}\, \alpha^{\mu\nu\rho}(k, k_1, k_2) = k_{2\rho}\, \alpha^{\mu\nu\rho}(k, k_1, k_2) = 0\,. \tag{A5}$$

To overcome this we assume that the disturbance k_2 is longitudinal in the rest frame $\bar{u} = (1, \mathbf{0})$ and use the Coulomb gauge $A = (\phi, \mathbf{0})$ to describe it. Then all relevant information is contained in the component

$$\bar{u}_\rho\, \alpha^{\mu\nu\rho}(k, k_1, k_2) \approx \frac{q^3}{2m^2} \int \mathrm{d}^4 u\, F(u) \left(g^{\mu\nu} - \frac{k_1^\mu\, u^\nu}{k_1\, \bar{u}} - \frac{k^\nu\, u^\mu}{k \bar{u}} + \frac{k\, k_1\, u^\mu\, u^\nu}{k \bar{u}\, k_1\, \bar{u}} \right) k_2^2\, \frac{u \bar{u}}{(k_2\, u)^2}\,. \tag{A6}$$

The nonrelativistic approximation involves setting $u\bar{u} = 1$. The approximation

$$\int \mathrm{d}^4 u\, \frac{F(u)}{(k_2\, u)^2}\, u^{\nu_1} \ldots u^{\nu_n} \approx \bar{u}^{\nu_1} \ldots \bar{u}^{\nu_n} \int \mathrm{d}^4 u\, \frac{F(u)}{(k_2\, u)^2} \tag{A7}$$

is required to be consistent with the nonrelativistic approximation and with the neglect of terms with lower powers of $k_2\, u$ in the denominator. Thus we find

$$\begin{aligned} \bar{u}_\rho\, \alpha^{\mu\nu\rho}(k, k_1, k_2) &\approx \frac{q^3}{2m^2}\, k_2^2\, a^{\mu\nu}(k, k_1, \bar{u}) \int \mathrm{d}^4 u\, \frac{F(u)}{(k_2\, u)^2} \\ &\approx \frac{q}{2m}\, k_2^2\, \frac{a^{\mu\nu}(k, k_1, \bar{u})}{(k_2\, \bar{u})^2}\, \alpha^{\mathrm{L}}(k_2)\,, \end{aligned} \tag{A8}$$

where we use (19) and note that the approximation $k_2^2 - (k_2\, \bar{u})^2 \approx k_2^2$ is consistent with our other approximations.

We need to impose the gauge invariance condition, which requires that the index ρ appears in the form $k_{2\alpha}\, G^{\alpha\rho}(k_2, \bar{u})$. In the rest frame the left-hand side of (A8) corresponds to $\alpha^{\mu\nu 0}$, and we are to multiply $k_{2\alpha}\, G^{\alpha\rho}(k_2, \bar{u})$ by the appropriate factor $[k_2\, \bar{u}/\{(k_2\, \bar{u})^2 - k_2^2\} \approx k_2\, \bar{u}/k_2^2]$ so that its $\rho = 0$ component is equal to unity. The result (23) follows.

In the case of the cubic response with $k - k_1 = k_2 + k_3$ a slow disturbance, the first term with $\{(k_2 + k_3)u\}^2$ in the denominator is the only one which contributes. Then using (A7), the integral again reduces to the form (19), and the result (24) follows on making the approximation $(k - k_1)^2 - \{(k - k_1)\bar{u}\}^2 \approx (k - k_1)^2$.

Appendix 3. Singular Integrals

Consider the evaluation of the mean value $\langle 1/v_z^2 \rangle$ of $1/v_z^2$ for a nonrelativistic Maxwellian distribution by the following partial integration

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \frac{\mathrm{d}v_z}{\sqrt{2\pi}\, V}\, \exp(-v_z^2/2V^2) \\ &= -\frac{V^2}{\sqrt{2\pi}\, V}\, \frac{\exp(-v_z^2/2V^2)}{v_z} \Bigg|_{-\infty}^{\infty} - V^2 \int_{-\infty}^{\infty} \frac{\mathrm{d}v_z}{\sqrt{2\pi}\, V}\, \frac{\exp(-v_z^2/2V^2)}{v_z^2}\,. \end{aligned} \tag{A9}$$

The integrated term appears to vanish and the implied result $\langle 1/v_z^2\rangle = -1/V^2$ is what is required in the evaluation of the limit $\omega \to 0$ of (20). In this sense the result $\langle 1/v_z^2\rangle = -1/V^2$ is 'correct'. However, the result is nonsensical because it requires a positive-definite integral to have a negative-definite value.

The resolution of this paradox is evidently associated with the interpretation of the integral as a principal value. Then the integrated term in (A9) diverges due to the contributions from $v_z = +\eta$ and $v_z = -\eta$ in the limit $\eta \to 0$. The existence of this singular contribution removes the inconsistency with the sign of the finite part of the integral. However, we require a further singular contribution to cancel the one implied in (A10) in order to obtain a meaningful result. Evidently this is obtained by requiring that the principal value be imposed on the left-hand member of

$$\int \mathrm{d}^4 u\, \frac{1}{ku}\, k\, \frac{\partial F(u)}{\partial u} = [\text{integrated term}] + k^2 \int \mathrm{d}^4 u\, \frac{F(u)}{(ku)^2}\,. \tag{A10}$$

The principal value then makes this integrated term singular and such that it cancels that implied in (A9).

Appendix 4. Evaluation of $\Phi(k_0, k_1, k_2)$

In the limit $\omega = \omega_1 = 0$, with $k = k_1 + k_2$, equation (25) for a nonrelativistic Maxwellian distribution in the rest frame gives

$$\Phi(k, k_1, k_2) = n \int \frac{\mathrm{d}^3 \boldsymbol{v}\, \exp(-v^2/V^2)}{(2\pi)^{3/2} V^3} \left(\frac{\boldsymbol{k}_1 . \boldsymbol{k}_2 |\boldsymbol{k}|^2}{\boldsymbol{k}_1 . \boldsymbol{v}\, \boldsymbol{k}_2 . \boldsymbol{v} |\boldsymbol{k} . \boldsymbol{v}|^2} + \frac{\boldsymbol{k} . \boldsymbol{k}_1 |\boldsymbol{k}_2|^2}{\boldsymbol{k} . \boldsymbol{v}\, \boldsymbol{k}_1 . \boldsymbol{v} |\boldsymbol{k}_2 . \boldsymbol{v}|^2} + \frac{\boldsymbol{k} . \boldsymbol{k}_2 |\boldsymbol{k}_1|^2}{\boldsymbol{k} . \boldsymbol{v}\, \boldsymbol{k}_2 . \boldsymbol{v} |\boldsymbol{k}_1 . \boldsymbol{v}|^2} \right). \tag{A11}$$

Consider the second term in the integrand. Let $\boldsymbol{k}_2 = (0, k_2, 0)$ be along the y-axis and $\boldsymbol{k}_1 = (k_1 \sin\psi, k_1 \cos\psi, 0)$ be in the xy plane, so that we have $\boldsymbol{k} = (k_1 \sin\psi, k_2 + k_1 \cos\psi, 0)$. The v_z integral is then trivial. After we write

$$\frac{1}{\boldsymbol{k}_1 . \boldsymbol{v}\, \boldsymbol{k}_2 . \boldsymbol{v}} = \frac{1}{k_2 v_z} \left(\frac{1}{k_1 \sin\psi\, v_x + k_1 \cos\psi\, v_y} - \frac{1}{k_1 \sin\psi\, v_x + (k_1 \cos\psi + k_2) v_y} \right), \tag{A12}$$

the v_x integral may be evaluated in terms of the function $\phi(z)$ defined by (21). The other two terms in (A11) lead to results of the same form. Combining the three terms and writing the result in a coordinate-independent form gives

$$\Phi(k, k_1, k_2) = n \int \frac{\mathrm{d}v}{(2\pi)^{\frac{1}{2}} V} \frac{\exp(-v^2/V^2)}{v^4} \left\{ \frac{|\boldsymbol{k}|^2}{\boldsymbol{k}_1 . \boldsymbol{k}_2} \phi\left(\frac{\boldsymbol{k}_1 . \boldsymbol{k}_2\, v}{\sqrt{2} |\boldsymbol{k}_1 \times \boldsymbol{k}_2| V} \right) - \frac{|\boldsymbol{k}_2|^2}{\boldsymbol{k} . \boldsymbol{k}_1} \phi\left(\frac{\boldsymbol{k} . \boldsymbol{k}_1\, v}{\sqrt{2} |\boldsymbol{k} \times \boldsymbol{k}_1| v} \right) - \frac{|\boldsymbol{k}_1|^2}{\boldsymbol{k} . \boldsymbol{k}_2} \phi\left(\frac{\boldsymbol{k} . \boldsymbol{k}_2\, v}{\sqrt{2} |\boldsymbol{k} \times \boldsymbol{k}_2| v} \right) \right\}, \tag{A13}$$

where we omit the subscript y on the remaining variable of integration v_y.

Consider the integrals

$$I_n(\alpha,\beta) = \alpha \int \frac{\mathrm{d}v\, v}{v^n}\, \mathrm{e}^{(\alpha^2+\beta)v^2} \int_0^{\alpha v} \mathrm{d}t\, \mathrm{e}^{t^2}. \tag{A14}$$

We require $I_4(\alpha,\beta)$ with $\beta = 1/2\,V^2$ in the evaluation of (A13). We have

$$\partial I_n(\alpha,\beta)/\partial\beta = -I_{n-2}(\alpha,\beta), \tag{A15}$$

so that I_4 may be obtained from I_0 by two integrations. We can evaluate I_0 as follows. Inserting the final form from (21) in (A14) we partially integrate to find

$$I_0(\alpha,\beta) = \alpha \int_{-\infty}^{\infty} \mathrm{d}v\, v\,\mathrm{e}^{-(\alpha^2+\beta)v^2} \int_0^{\alpha v} \mathrm{d}t\, \mathrm{e}^{t^2}$$

$$= \frac{\alpha}{2(\alpha^2+\beta)} \int_{-\infty}^{\infty} \mathrm{d}v\, \mathrm{e}^{-\beta v^2} = \frac{\sqrt{\pi}}{2}\, \frac{\alpha^2}{\beta^{\frac{1}{2}}(\alpha^2+\beta)}. \tag{A16}$$

Two integrations with respect to β then give

$$I_4(\alpha,\beta) = \sqrt{\pi}\,\alpha\{(\alpha^2+\beta)\arctan(\beta^{\frac{1}{2}}/\alpha) - \alpha\beta^{\frac{1}{2}}\}. \tag{A17}$$

In writing the final result it is convenient to introduce $\boldsymbol{k}_0 = -\boldsymbol{k}$ so that there is a symmetry involving the indices 0 to 2, with $\boldsymbol{k}_0 + \boldsymbol{k}_1 + \boldsymbol{k}_2 = 0$. Let ψ_{jl} be the angle between $\boldsymbol{k}_j$ and $\boldsymbol{k}_l$. Then the final result is

$$\Phi(-\boldsymbol{k}_0, \boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{n}{2V^4} \sum \frac{|\boldsymbol{k}_2|}{|\boldsymbol{k}_j \times \boldsymbol{k}_l|} \left(\frac{\psi_{jl}}{\sin^2\psi_{jl}} - \cos\psi_{jl} \right), \tag{A18}$$

where the sum is over the three permutations $ijl = 012, 120, 201$. The signs of the angles can be fixed by considering the limiting case when $\boldsymbol{k}_0$, $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are parallel. On taking the limit $\psi_{jl} \to 0$ one finds $\Phi = n/V^4$ from (A18), as required by (A11). Then as one allows ψ_{jl} to be nonzero, one requires the sum $\psi_{01}+\psi_{12}+\psi_{20}$, which is the sum of the angles of a triangle modulo π, to remain equal to zero. Thus one finds

$$\sum \frac{|\boldsymbol{k}_i|^2}{|\boldsymbol{k}_j \times \boldsymbol{k}_l|} \frac{\psi_{jl}}{\sin^2\psi_{jl}} = \frac{|\boldsymbol{k}_0|^2 |\boldsymbol{k}_1|^2 |\boldsymbol{k}_2|^2}{|\boldsymbol{k}_0 \times \boldsymbol{k}_1|^3} \sum \psi_{jl} = 0, \tag{A19}$$

where one uses $|\boldsymbol{k}_0 \times \boldsymbol{k}_1| = |\boldsymbol{k}_1 \times \boldsymbol{k}_2| = |\boldsymbol{k}_0 \times \boldsymbol{k}_2|$. Also we have

$$\sum \frac{|\boldsymbol{k}_i|^2}{|\boldsymbol{k}_j \times \boldsymbol{k}_l|} \cot\psi_{jl} = \frac{1}{|\boldsymbol{k}_0 \times \boldsymbol{k}_1|^2} \sum |\boldsymbol{k}_i|^2 \boldsymbol{k}_j . \boldsymbol{k}_l = -2. \tag{A20}$$

Thus (A18) reduces to

$$\Phi(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) = n/V^4. \tag{A21}$$

Appendix 5.

If we suppose the response of the plasma is electrostatic, the charge density is then

$$\rho(x) = q\bar{n}\exp\{-q\phi(x)/mV^2\}. \tag{A22}$$

The second order response is obtained by expanding the exponential to second order,

$$\rho^{(2)}(x) = -\frac{q^3\bar{n}}{2m^2V^4}|\phi(x)|^2, \tag{A23}$$

and Fourier transforming

$$\rho^{(2)}(k) = \frac{q^3\bar{n}}{2m^2V^4}\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}(2\pi)^3\delta^3(k-k_1-k_2)\phi(k_1)\phi(k_2). \tag{A24}$$

Now $\rho^{(2)}$ and ϕ can be regarded as time components of respective 4-vectors, and (A24) then implies a quadratic response tensor (in the limit $\omega = \omega_1 = \omega_2 = 0$ in the rest frame)

$$\alpha^{000}(k, k_1, k_2) = q^3\bar{n}/2m^2V^4. \tag{A25}$$

This result is reproduced by the $\mu = \nu = \rho = 0$ term of (27).

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