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Polariton Modes in Superlattices of Uniaxial Crystals

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Abstract

We present results of a study of the polariton modes, retarded or non-retarded, in superlattices consisting of alternating layers of uniaxial crystalline and isotropic materials. Novel features not existing in superlattices consisting solely of isotropic materials are predicted.

1. Introduction

The advent of artificial superlattices and their promising future as new device materials has stimulated interest in the study of polariton modes in layered media (Camley *et al.* 1983; Grunberg and Mika 1983; Camley and Mills 1984; Shi and Tsai 1984, 1985; Wen *et al.* 1985). Until now, only superlattice media consisting of isotropic materials have been treated theoretically, while experimentally, more often studied are superlattices consisting of anisotropic materials. We give, in the present article, calculations on polariton modes in a structure of alternating layers of uniaxial crystalline and isotropic materials. We assume the modulation wavelength to be much larger than the lattice parameters, so that the continuum approximation applies. In the following, we give firstly a general discussion (Section 2), then results in cases of special crystal axis orientations (Section 3), and thirdly non-retarded solutions (Section 4). We also predict possibilities for new experimental observations.

2. General Discussion

We consider a superlattice consisting of alternating layers of uniaxial and isotropic media of thicknesses d_a and d_b respectively. Without loss of generality, the crystal axis c of the uniaxial layers may be assumed to lie in the xz plane at an angle θ to the z-axis which is in the direction of superlattice modulation (see Fig. 1). The dielectric tensor of the b-layers (i.e. the uniaxial) can then be written as

$$\epsilon_b = \begin{bmatrix} \epsilon_{xx} & 0 & \epsilon_{xz} \\ 0 & \epsilon_{yy} & 0 \\ \epsilon_{zx} & 0 & \epsilon_{zz} \end{bmatrix}, \tag{1}$$

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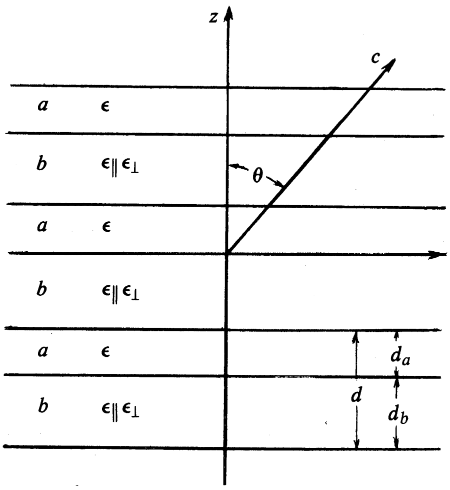

Fig. 1. Superlattice composed of isotropic layers a and uniaxial crystalline layers b of thicknesses d_a and d_b respectively. In the general case, the crystal axis c of the b-layers is at an angle θ with respect to the direction of superlattice modulation. The x-axis is chosen to lie in the plane fixed by z and c.

where

$$\epsilon_{xx} = \epsilon_\perp \cos^2\theta + \epsilon_\parallel \sin^2\theta, \quad \epsilon_{yy} = \epsilon_\perp, \quad \epsilon_{zz} = \epsilon_\perp \sin^2\theta + \epsilon_\parallel \cos^2\theta,$$

$$\epsilon_{xz} = \epsilon_{zx} = (\epsilon_\parallel - \epsilon_\perp)\cos\theta \sin\theta, \tag{2}$$

with $\epsilon_\parallel$ and $\epsilon_\perp$ being the dielectric constants in the directions parallel and perpendicular respectively to the crystal axis c. We consider both kinds of layers to be nonmagnetic with permeabilities $\mu_a = \mu_b = 1$.

Two types of electromagnetic wave are allowed to propagate in an uniaxial solid, namely ordinary waves (O-waves) with the electric field perpendicular to the plane containing the crystal axis c and the wavevector $\boldsymbol{k}$, and extraordinary waves (E-waves) with the electric field lying in that plane. Their wavevectors satisfy, respectively, the equations

$$k_x^2 + k_y^2 + k_z^2 = c^{-2}\omega^2\epsilon_\perp \quad (\mathrm{O}), \tag{3a}$$

$$\epsilon_\perp^{-1}(k_x \sin\theta + k_z \cos\theta)^2 + \epsilon_\parallel^{-1}(k_x\cos\theta - k_z \sin\theta)^2 + \epsilon_\parallel^{-1} k_y^2 = c^{-2}\omega^2 \quad (\mathrm{E}). \tag{3b}$$

Writing $\mathrm{i}\, k_z = \pm\alpha_{\mathrm{O,E}}$, equations (3) imply that

$$\alpha_{\mathrm{O}} = (k^2 - \epsilon_\perp c^{-2}\omega^2)^{\frac{1}{2}}, \tag{4a}$$

$$\alpha_{\mathrm{E}} = \pm\mathrm{i}(-\epsilon_{zz}^{-1}\epsilon_{xz} k_x) + |\epsilon_{zz}|^{-1}\epsilon_\perp^{\frac{1}{2}}\{\epsilon_\parallel k_x^2 + \epsilon_{zz}(k_y^2 - \epsilon_\parallel c^{-2}\omega^2)\}^{\frac{1}{2}}, \tag{4b}$$

with $k^2 = k_x^2 + k_y^2$. Writing the interface modes with fields confined in the immediate neighbourhoods of the interfaces in the form

$$E(x, y, z, t) = E(z)\exp\{\mathrm{i}(k_x x + k_y y - \omega t)\}, \tag{5}$$

then in the b-layers

$$E_b(z) = E^{O}_{b+} \exp(\alpha_O z) + E^{O}_{b-} \exp(-\alpha_O z) + \{E^{E}_{b+} \exp(\alpha'_E z) + E^{E}_{b-} \exp(-\alpha'_E z)\} \exp(i\alpha''_E). \quad (6)$$

The term $\alpha_E = \alpha'_E \pm i\alpha''_E$ for the electric field in the b-layers is, in general, a superposition of both O and E fields. Since $E^{O}_{b\pm}$ is parallel to $k \times c$, one can write

$$E^{O}_{b\pm} = \xi^{O}_{\pm}(k_y \cos\theta,\ -k_x \cos\theta \mp i\alpha_O \sin\theta,\ -k_y \sin\theta). \quad (7)$$

On the other hand, since $E^{E}_{b\pm}$ lie in the plane containing k and c, the condition for a plane wave $D.k = 0$ together with $D = \epsilon . E$ implies that

$$E^{E}_{b\pm} = \xi^{E}_{\pm}(k_x\beta_\pm - c^{-1}\omega \sin\theta,\ k_y\beta_\pm,\ -c^{-1}\omega \sin\theta + (\alpha''_E \mp i\alpha'_E)\beta_\pm), \quad (8a)$$

$$\beta_\pm = (c^{-1}\omega\epsilon_\perp)^{-1}\{k_x \sin\theta + (\alpha''_E \mp i\alpha'_E)\cos\theta\}. \quad (8b)$$

In the isotropic layers, E_a may lie in any direction. The condition $\nabla . E_a = 0$ gives the relation among its components

$$E_{a\pm z} = \mp i\alpha_a^{-1}(k_x E_{a\pm x} + k_y E_{a\pm y}), \quad (9)$$

where

$$\alpha_a = (k^2 - \epsilon_a c^{-2}\omega^2)^{\frac{1}{2}}. \quad (10)$$

Superlattice periodicity in the z direction demands

$$E(z+nd) = E(z) \exp(i\,qnd), \quad (11)$$

where

$$\begin{aligned} E(z) &= E_{a-} \exp(-\alpha_a z) + E_{a+} \exp(\alpha_a z), && 0 < z < d_a \\ &= E^{O}_{b+} \exp(\alpha_O z) + E^{O}_{b-} \exp(-\alpha_O z) \\ &\quad + \{E^{E}_{b+} \exp(\alpha'_E z) + E^{E}_{b-} \exp(-\alpha'_E z)\} \exp(i\alpha''_E z), && -d_b < z < 0 \quad (12) \\ &= [E_{a-} \exp\{-\alpha_a(z+d)\} + E_{a+} \exp\{\alpha_a(z+d)\}] \\ &\quad \times \exp(-i\,qd), && -d < z < -d_b, \end{aligned}$$

with the corresponding magnetic field given by

$$\begin{aligned} H(z) &= H_{a-} \exp(-\alpha_a z) + H_{a+} \exp(\alpha_a z), && 0 < z < d_a \\ &= H^{O}_{b+} \exp(\alpha_O z) + H^{O}_{b-} \exp(-\alpha_O z) \\ &\quad + \{H^{E}_{b+} \exp(\alpha'_E z) + H^{E}_{b-} \exp(-\alpha'_E z)\} \exp(i\alpha''_E z), && -d_b < z < 0 \quad (13) \\ &= [H_{a-} \exp\{-\alpha_a(z+d)\} + H_{a+} \exp\{\alpha_a(z+d)\}] \\ &\quad \times \exp(-i\,qd), && -d < z < -d_b. \end{aligned}$$

One has with aid of Maxwell's equations

$$\boldsymbol{H}^{\mathrm{O}}_{a\pm} = \mp(\omega\alpha_a)^{-1}\mathrm{i}\,c\{k_x k_y E_{a\pm x} + (k_y^2-\alpha_a^2)E_{a\pm y},\ (\alpha_a^2-k_x^2)E_{a\pm x} - k_x k_y E_{a\pm y},$$
$$\mp\mathrm{i}\,\alpha_a(k_y E_{a\pm x} - k_x E_{a\pm y})\}\,, \tag{14}$$

$$\boldsymbol{H}^{\mathrm{O}}_{b\pm} = \omega^{-1} c\xi^{\mathrm{O}}_{\pm}\{(\alpha_{\mathrm{O}}^2 - k_y^2)\sin\theta \mp \mathrm{i}\,k_x\alpha_{\mathrm{O}}\cos\theta,\ k_y(k_x\sin\theta \mp \mathrm{i}\,\alpha_{\mathrm{O}}\cos\theta),$$
$$-k^2\cos\theta \mp \mathrm{i}\,\alpha_{\mathrm{O}}k_x\sin\theta\}\,, \tag{15}$$

$$\boldsymbol{H}^{\mathrm{E}}_{b\pm} = \xi^{\mathrm{E}}_{\pm}\{-k_y\cos\theta,\ k_x\cos\theta - (\alpha''_{\mathrm{E}} \mp \mathrm{i}\,\alpha'_{\mathrm{E}})\sin\theta,\ k_y\sin\theta\}\,. \tag{16}$$

It is easy to see that $\boldsymbol{H}^{\mathrm{E}}$ and $\boldsymbol{H}^{\mathrm{O}}$ are perpendicular and parallel to the plane containing $\boldsymbol{k}$ and $\boldsymbol{c}$ respectively.

Continuity of the tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$ at $z = 0$ and $-d_b$ gives eight equations for the eight unknowns: $\xi^{\mathrm{O}}_{b\pm}$, $\xi^{\mathrm{E}}_{b\pm}$, $E_{a\pm x}$ and $E_{a\pm y}$. After eliminating $E_{a\pm x}$ and $E_{a\pm y}$, we are left with the following four equations:

$$\xi^{\mathrm{O}}_{+}\{-k_y\cos\theta\exp(-\alpha_{\mathrm{O}}d_b + \mathrm{i}\,qd) + A^{-}_{+}\} + \xi^{\mathrm{O}}_{-}\{-k_y\cos\theta\exp(\alpha_{\mathrm{O}}d_b + \mathrm{i}\,qd) + A^{+}_{+}\}$$
$$+\xi^{\mathrm{E}}_{+}\{-(k_x\beta_{+} - c^{-1}\omega\sin\theta)\exp(-\mathrm{i}\,\alpha''_{\mathrm{E}}d_b - \alpha'_{\mathrm{E}}d_b + \mathrm{i}\,qd) + B^{+}_{+}\}$$
$$+\xi^{\mathrm{E}}_{-}\{-(k_x\beta_{-} - c^{-1}\omega\sin\theta)\exp(-\mathrm{i}\,\alpha''_{\mathrm{E}}d_b + \alpha'_{\mathrm{E}}d_b + \mathrm{i}\,qd) + B^{-}_{+}\} = 0,$$

$$\xi^{\mathrm{O}}_{+}\{-k_y\cos\theta\exp(-\mathrm{i}\,qd) + A^{-}_{-}\exp(-\alpha_{\mathrm{O}}d_b)\}$$
$$+\xi^{\mathrm{O}}_{-}\{-k_y\cos\theta\exp(-\mathrm{i}\,qd) - A^{+}_{-}\exp(\alpha_{\mathrm{O}}d_b)\}$$
$$+\xi^{\mathrm{E}}_{+}\{-(k_x\beta_{+} - c^{-1}\omega\sin\theta)\exp(-\mathrm{i}\,qd) + B^{+}_{-}\exp(-\mathrm{i}\,\alpha''_{\mathrm{E}}d_b - \alpha'_{\mathrm{E}}d_b)\}$$
$$+\xi^{\mathrm{E}}_{-}\{-(k_x\beta_{-} - c^{-1}\omega\sin\theta)\exp(-\mathrm{i}\,qd) + B^{-}_{-}\exp(-\mathrm{i}\,\alpha''_{\mathrm{E}}d_b + \alpha'_{\mathrm{E}}d_b)\} = 0,$$

$$\xi^{\mathrm{O}}_{+}\{(k_x\cos\theta + \mathrm{i}\,\alpha_{\mathrm{O}}\sin\theta)\exp(-\alpha_{\mathrm{O}}d_b + \mathrm{i}\,qd) + C^{+}_{+}\}$$
$$+\xi^{\mathrm{O}}_{-}\{(k_x\cos\theta - \mathrm{i}\,\alpha_{\mathrm{O}}\sin\theta)\exp(\alpha_{\mathrm{O}}d_b + \mathrm{i}\,qd) + C^{-}_{+}\}$$
$$+\xi^{\mathrm{E}}_{+}\{-k_y\beta_{+}\exp(-\mathrm{i}\,\alpha''_{\mathrm{E}}d_b - \alpha'_{\mathrm{E}}d_b + \mathrm{i}\,qd) + D^{+}_{+}\}$$
$$+\xi^{\mathrm{E}}_{-}\{-k_y\beta_{-}\exp(-\mathrm{i}\,\alpha''_{\mathrm{E}}d_b - \alpha'_{\mathrm{E}}d_b + \mathrm{i}\,qd) + D^{-}_{+}\} = 0,$$

$$\xi^{\mathrm{O}}_{+}\{(k_x\cos\theta + \mathrm{i}\,\alpha_{\mathrm{O}}\sin\theta)\exp(-\mathrm{i}\,qd) + C^{+}_{-}\exp(-\alpha_{\mathrm{O}}d_b)\}$$
$$+\xi^{\mathrm{O}}_{-}\{(k_x\cos\theta - \mathrm{i}\,\alpha_{\mathrm{O}}\sin\theta)\exp(-\mathrm{i}\,qd) + C^{-}_{-}\exp(\alpha_{\mathrm{O}}d_b)\}$$
$$+\xi^{\mathrm{E}}_{+}\{-k_y\beta_{+}\exp(-\mathrm{i}\,qd) + D^{+}_{-}\exp(-\mathrm{i}\,\alpha''_{\mathrm{E}}d_b - \alpha'_{\mathrm{E}}d_b)\}$$
$$+\xi^{\mathrm{E}}_{-}\{-k_y\beta_{-}\exp(-\mathrm{i}\,qd) + D^{-}_{-}\exp(-\mathrm{i}\,\alpha''_{\mathrm{E}}d_b + \alpha'_{\mathrm{E}}d_b)\} = 0, \tag{17}$$

where

$$A_{\pm}^{+} = k_y \cos\theta \cosh(\alpha_a d_a) \pm [\mathrm{i}\, c^2\omega^{-2}(\epsilon_a\alpha_a)^{-1}\sinh(\alpha_a d_a) \times \{k_x k_y(\alpha_{\mathrm{O}}^2-\alpha_a^2)\sin\theta + \mathrm{i}\, k_y \alpha_{\mathrm{O}}(k^2-\alpha_a^2)\cos\theta\}],$$

$$A_{\pm}^{-} = k_y \cos\theta \cosh(\alpha_a d_a) \pm [\mathrm{i}\, c^2\omega^{-2}(\epsilon_a\alpha_a)^{-1}\sinh(\alpha_a d_a) \times \{k_x k_y(\alpha_{\mathrm{O}}^2-\alpha_a^2)\sin\theta - \mathrm{i}\, k_y \alpha_{\mathrm{O}}(k^2-\alpha_a^2)\cos\theta\}],$$

$$B_{\pm}^{+} = (k_x\beta_+ - c^{-1}\omega\sin\theta)\cosh(\alpha_a d_a) \pm [\mathrm{i}\, c(\omega\epsilon_a\alpha_a)^{-1}\sinh(\alpha_a d_a) \times \{(\alpha_a^2 - k_y^2)(\alpha_{\mathrm{E}}'' - \mathrm{i}\,\alpha_{\mathrm{E}}')\sin\theta - k_y\alpha_a^2\cos\theta\}],$$

$$B_{\pm}^{-} = (k_x\beta_- - c^{-1}\omega\sin\theta)\cosh(\alpha_a d_a) \pm [\mathrm{i}\, c(\omega\epsilon_a\alpha_a)^{-1}\sinh(\alpha_a d_a) \times \{(\alpha_a^2 - k_y^2)(\alpha_{\mathrm{E}}'' + \mathrm{i}\,\alpha_{\mathrm{E}}')\sin\theta - k_y\alpha_a^2\cos\theta\}],$$

$$C_{\pm}^{+} = -(k_x\cos\theta + \mathrm{i}\,\alpha_{\mathrm{O}}\sin\theta)\cosh(\alpha_a d_a) \pm [\mathrm{i}(\epsilon_a\alpha_a)^{-1}\sinh(\alpha_a d_a) \times \{(-\epsilon_a\alpha_{\mathrm{O}}^2 + \epsilon_a k_y^2 - \epsilon_\perp k_y^2)\sin\theta + \mathrm{i}\, k_x\epsilon_a\alpha_{\mathrm{O}}\cos\theta\}],$$

$$C_{\pm}^{-} = -(k_x\cos\theta - \mathrm{i}\,\alpha_{\mathrm{O}}\sin\theta)\cosh(\alpha_a d_a) \pm [\mathrm{i}(\epsilon_a\alpha_a)^{-1}\sinh(\alpha_a d_a) \times \{(-\epsilon_a\alpha_{\mathrm{O}}^2 + \epsilon_a k_y^2 - \epsilon_\perp k_y^2)\sin\theta - \mathrm{i}\, k_x\epsilon_a\alpha_{\mathrm{O}}\cos\theta\}],$$

$$D_{\pm}^{+} = k_y\beta_+\cosh(\alpha_a d_a) \pm [\mathrm{i}\, c^{-1}\omega(\epsilon_a\alpha_a)^{-1}\sinh(\alpha_a d_a) \times \{k_x k_y(\alpha_{\mathrm{E}}'' - \mathrm{i}\,\alpha_{\mathrm{E}}')\sin\theta - k_y\alpha_a^2\cos\theta\}],$$

$$D_{\pm}^{-} = k_y\beta_-\cosh(\alpha_a d_a) \pm [\mathrm{i}\, c^{-1}\omega(\epsilon_a\alpha_a)^{-1}\sinh(\alpha_a d_a) \times \{k_x k_y(\alpha_{\mathrm{E}}'' + \mathrm{i}\,\alpha_{\mathrm{E}}')\sin\theta - k_y\alpha_a^2\cos\theta\}]. \quad (18)$$

The condition for the existence of a nontrivial solution of equations (17) yields an implicit dispersion relation of polaritons with k parallel to the interfaces. It is, in general, very complicated and at an arbitrary orientation of the crystal axis, so that there no longer exists pure transverse magnetic (TM) and/or transverse electric (TE) modes.

3. Certain Special Cases

Let us consider now three specieal cases:

(1) $\theta = 0$ (c perpendicular to the interfaces). Without loss of generality one may set $k_y = 0$, then

$$\alpha_{\mathrm{E}}'' = 0, \qquad \alpha_{\mathrm{E}}' = (\epsilon_\perp/\epsilon_\parallel)^{\frac{1}{2}}(k^2 - \epsilon_\parallel c^{-2}\omega^2)^{\frac{1}{2}}, \quad (19a)$$

$$\beta_\pm = \mp\mathrm{i}\, c\omega^{-1}(\epsilon_\parallel\epsilon_\perp)^{\frac{1}{2}}(k^2 - \epsilon_\parallel c^{-2}\omega^2)^{\frac{1}{2}}. \quad (19b)$$

TM modes: For such modes, $\boldsymbol{E}^{\mathrm{O}}_{\pm} = \xi^{\mathrm{O}}_{\pm}(0, -k_x, 0)$ and $\xi^{\mathrm{O}}_{\pm} = 0$. The first two equations of (17) yield with the aid of (19)

$$\{-k_x\beta_+ \exp(-\alpha'_{\mathrm{E}}\, d_b + \mathrm{i}\, qd) + B^+_+\}\{-k_x\beta_- \exp(\mathrm{i}\, qd) + B^-_- \exp(\alpha'_{\mathrm{E}}\, d_b)\}$$
$$-\{-k_x\beta_- \exp(\alpha'_{\mathrm{E}}\, d_b + \mathrm{i}\, qd) + B^-_+\}\{-k_x\beta_+ \exp(-\mathrm{i}\, qd)$$
$$+ B^+_- \exp(-\alpha'_{\mathrm{E}}\, d_b)\} = 0, \qquad (20)$$

where

$$B^+_{\pm} = k_x\beta_+ \cosh(\alpha_a\, d_a) \pm \mathrm{i}\, c(\omega\epsilon_a)^{-1}\alpha_a\, k_x \sinh(\alpha_a\, d_a), \qquad (21a)$$

$$B^-_{\pm} = k_x\beta_- \cosh(\alpha_a\, d_a) \pm \mathrm{i}\, c(\omega\epsilon_a)^{-1}\alpha_a\, k_x \sinh(\alpha_a\, d_a). \qquad (21b)$$

The last two equations of (17) become identities in this case. Substitution of equations (21) into (20) leads to the dispersion relation

$$\cos(qd) = \cosh(\alpha_a\, d_a)\cosh(\alpha'_{\mathrm{E}}\, d_b)$$
$$+ (2\alpha_a \alpha'_{\mathrm{E}}\, \epsilon_a \epsilon_{\perp})^{-1}(\epsilon^2_{\parallel}\, \alpha^2_a + \epsilon^2_a \alpha'^2_{\mathrm{E}}) \sinh(\alpha_a\, d_a)\sinh(\alpha'_{\mathrm{E}}\, d_b). \qquad (22)$$

Equation (22) is similar to the dispersion relation of TM waves in a superlattice with isotropic b-layers of dielectric constant $\epsilon_{\perp}$ (Shi and Tsai 1984, 1985).

TE modes: In contrast to the TM case, $\xi^{\mathrm{E}}_{\pm} = 0$ and the last two equations of (17) lead to

$$\cos(qd) = \cosh(\alpha_a\, d_a)\cosh(\alpha_{\mathrm{O}}\, d_b)$$
$$+ \tfrac{1}{2}(\alpha^{-1}_{\mathrm{O}}\, \alpha_a + \alpha^{-1}_a\, \alpha_{\mathrm{O}})\sinh(\alpha_a\, d_a)\sinh(\alpha_{\mathrm{O}}\, d_b). \qquad (23)$$

This is identical to the dispersion relation of TE modes if the b-layers were isotropic with dielectric constant $\epsilon_{\perp}$.

(2) $\theta = \frac{1}{2}\pi$ and $\boldsymbol{k}$ parallel to the crystal axis (the x-axis). We have, in this case,

$$\alpha''_{\mathrm{E}} = 0, \quad \alpha'_{\mathrm{E}} = (\epsilon_{\parallel}/\epsilon_{\perp})^{\frac{1}{2}}(k^2 - \epsilon_{\perp}\, c^{-2}\omega^2)^{\frac{1}{2}}, \quad \beta_{\pm} = \beta = (\omega\epsilon_{\perp})^{-1} c k_x. \qquad (24)$$

TM modes: $\xi^{\mathrm{O}}_{\pm} = 0$ and from the last two equations of (17) and equations (24) we have

$$\cos(qd) = \cosh(\alpha_a\, d_a)\cosh(\alpha'_{\mathrm{E}}\, d_b)$$
$$+ (2\epsilon_a \epsilon_{\parallel}\, \alpha_a \alpha'_{\mathrm{E}})^{-1}(\epsilon^2_a \alpha'^2_{\mathrm{E}} + \epsilon^2_{\parallel}\, \alpha^2_a) \sinh(\alpha_a\, d_a) \sinh(\alpha'_{\mathrm{E}}\, d_b). \qquad (25)$$

TE modes: $\xi^{\mathrm{E}}_{\pm} = 0$ and the resulting dispersion relation is

$$\cos(qd) = \cosh(\alpha_a\, d_a)\cosh(\alpha_{\mathrm{O}}\, d_b)$$
$$+ \tfrac{1}{2}(\alpha^{-1}_{\mathrm{O}}\, \alpha_a + \alpha^{-1}_a\, \alpha_{\mathrm{O}}) \sinh(\alpha_a\, d_a) \sinh(\alpha_{\mathrm{O}}\, d_b). \qquad (26)$$

The same comment concerning equation (23) applies here.

(3) $\theta = \frac{1}{2}\pi$ and $\boldsymbol{k}$ perpendicular to the crystal axis ($k_x = 0$). We now have

$$\alpha''_{\mathrm{E}} = 0, \quad \alpha'_{\mathrm{E}} = (k^2 - \epsilon_{\parallel}\, c^{-2}\omega^2)^{\frac{1}{2}}, \quad \beta_{\pm} = 0. \qquad (27)$$

TM modes: $\xi_{\pm}^{\mathrm{E}} = 0$ in the present configuration, and the dispersion relation has the form

$$\cos(qd) = \cosh(\alpha_a d_a)\cosh(\alpha_{\mathrm{O}} d_b) + (2\alpha_{\mathrm{O}}\alpha_a \epsilon_\perp \epsilon_a)^{-1}(\epsilon_a^2 \alpha_{\mathrm{O}}^2 + \epsilon_\perp^2 \alpha_a^2)\sinh(\alpha_a d_a)\sinh(\alpha_{\mathrm{O}} d_b). \quad (28)$$

TE modes: $\xi_{\pm}^{\mathrm{O}} = 0$ and the dispersion relation is

$$\cos(qd) = \cosh(\alpha_a d_a)\cosh(\alpha'_{\mathrm{E}} d_b) + \tfrac{1}{2}(\alpha_{\mathrm{E}}'^{-1}\alpha_a + \alpha_a^{-1}\alpha'_{\mathrm{E}})\sinh(\alpha_a d_a)\sinh(\alpha'_{\mathrm{E}} d_b). \quad (29)$$

The above results can be incorporated into the following unified equations:
TM modes:

$$\cos(qd) = \cosh(\alpha_a d_a)\cosh(\alpha d_b) + (2\epsilon_a \alpha_a \epsilon\alpha)^{-1}(\epsilon_a^2\alpha^2 + \epsilon^2\alpha_a^2)\sinh(\alpha_a d_a)\sinh(\alpha d_b), \quad (30)$$

where $\alpha = \alpha'_{\mathrm{E}}$ in cases (1) and (2), and $\alpha = \alpha_{\mathrm{O}}$ in case (3), while $\epsilon = \epsilon_\perp$ in cases (1) and (3), and $\epsilon = \epsilon_\parallel$ in case (2).

TE modes:

$$\cos(qd) = \cosh(\alpha_a d_a)\cosh(\alpha d_b) + \tfrac{1}{2}(\alpha^{-1}\alpha_a + \alpha_a^{-1}\alpha)\sinh(\alpha_a d_a)\sinh(\alpha d_b), \quad (31)$$

where $\alpha = \alpha_{\mathrm{O}}$ in cases (1) and (2), and $\alpha = \alpha'_{\mathrm{E}}$ in case (3).

It can be seen from equation (30), in the case of TM modes, that for α_a and α to be both positive, ϵ_a and ϵ must differ in sign. We assume $\epsilon_a > 0$ in the following discussion, so that ϵ has to be negative. In the case of TE modes, no solutions exist with both α_a and α positive. Only layer localised waves, i.e. TE waves localised in one sort of layer with either α_a or α imaginary are allowed (Shi and Tsai 1984, 1985). Needless to say, the TM waves may also be localised in one kind of layer.

Let us assume that the b-layers are made of polar semiconductors, so that

$$\epsilon_{\perp,\parallel}(\omega) = \epsilon_{\perp,\parallel}(\infty)(\omega_{\mathrm{T}\perp,\parallel}^2 - \omega^2)^{-1}(\omega_{\mathrm{L}\perp,\parallel}^2 - \omega^2), \quad (32)$$

where ω_{T} and ω_{L} are the TO and LO phonon frequencies respectively, with the subscripts $\parallel$ and $\perp$ being used to distinguish frequencies of phonons propagating along and transverse to the crystal axis, and with $\epsilon_{\perp,\parallel}(\infty)$ the high frequency dielectric constants. We assume that $\omega_{\mathrm{T}\parallel} < \omega_{\mathrm{T}\perp} < \omega_{\mathrm{L}\parallel} < \omega_{\mathrm{L}\perp}$ which is, for example, the case for CdS. A numerical evaluation of the dispersion curves has been performed. From equation (30), we can see that in order to derive a solution $\epsilon_\perp$ and $\epsilon_\parallel$ must both be negative in cases (1) and (2), whereas only $\epsilon_\perp < 0$ is required in case (3). In the former case, ω must be limited within the range $\omega_{\mathrm{T}\perp} < \omega < \omega_{\mathrm{L}\parallel}$, while in the latter case the allowed range of ω is less restrictive, $\omega < \omega_{\mathrm{L}\perp}$.

Fig. 2a illustrates the TM dispersion curves of case (1) ($\theta = 0$). Fig. 2b is for case (2) ($\theta = \frac{1}{2}\pi$, $k_y = 0$); there is only one allowed band in this case. Fig. 2c

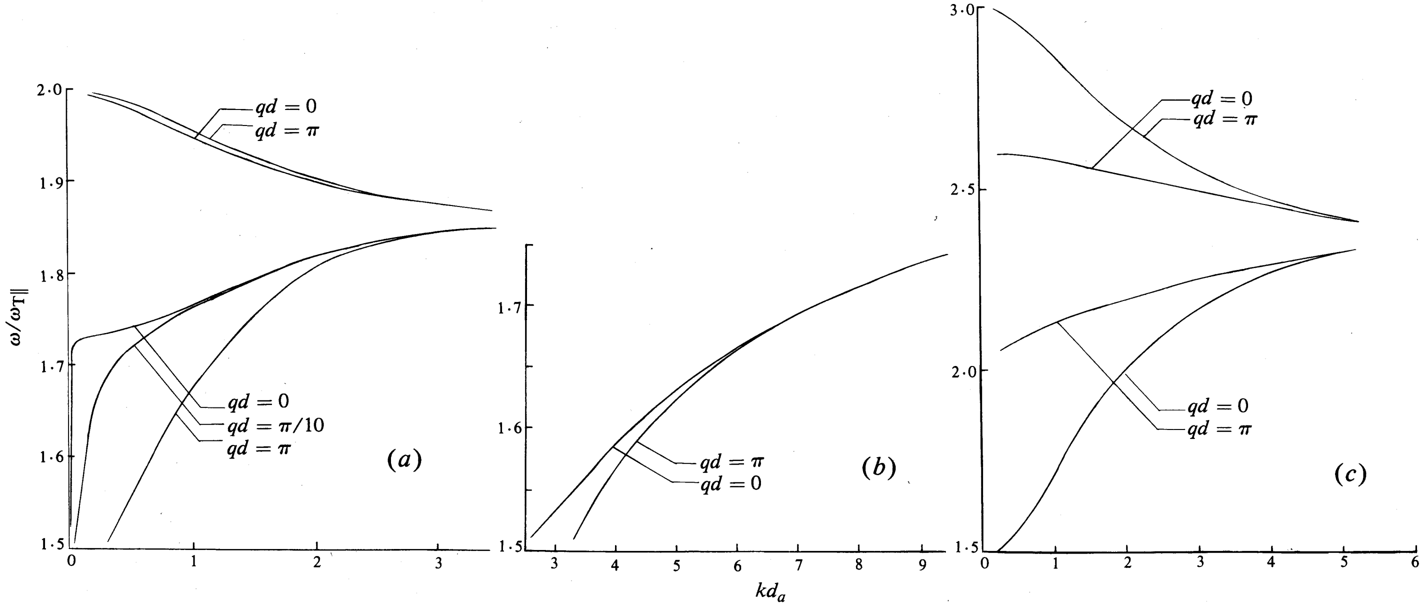

Fig. 2. TM mode dispersion curves in (*a*) case (1) $\theta = 0$ (equation 22), (*b*) case (2) $\theta = \frac{1}{2}\pi$ and $\boldsymbol{k}$ parallel to $\boldsymbol{c}$ (equation 25) and (*c*) case (3) $\theta = \frac{1}{2}\pi$ and $\boldsymbol{k}$ perpendicular to $\boldsymbol{c}$ (equation 28). The parameters adopted are $\omega_{T\parallel} = 10\times10^{13}\ \mathrm{cm}^{-1}$, $\omega_{T\perp} = 1\cdot5\omega_{T\parallel}$, $\omega_{L\parallel} = 2\cdot0\omega_{T\parallel}$ and $\omega_{L\perp} = 3\cdot0\omega_{T\parallel}$; $d_a = 2d_b = 1000$ Å; and $\epsilon_a = \epsilon_\perp(\infty) = \epsilon_\parallel(\infty) = 1$.

corresponds to case (3) ($\theta = \frac{1}{2}\pi$, $k_x = 0$). To give an impression of the relative positions of polariton bands in all three cases, we reproduce them together in Fig. 3, where the limiting frequencies at large k of the various branches are also indicated.

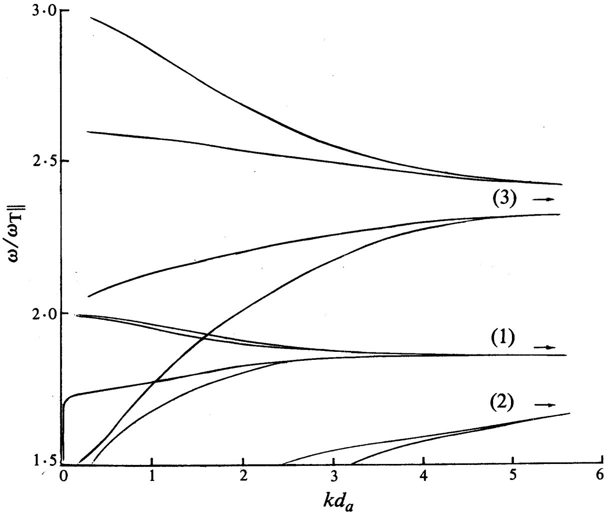

Fig. 3. Reproduction of the dispersion curves of Fig. 2 to illustrate the relative positions. The limiting frequencies at large k (arrows) of the various branches are fixed by (1) $\epsilon_\perp + (\epsilon_\perp/\epsilon_\parallel)^{1/2} = 0$; (2) $\epsilon_\perp + 1 = 0$ or $\omega = \{\frac{1}{2}(\omega_{L\perp}^2 + \omega_{T\perp}^2)\}^{1/2}$; and (3) $\epsilon_\parallel + (\epsilon_\perp/\epsilon_\parallel)^{1/2} = 0$.

4. Non-retarded Solutions

We now turn to non-retarded solutions. In the electrostatic limit, one has

$$E = -\nabla\psi, \qquad \psi(x, y, z, t) = \phi(z)\exp\{i(k_x x + k_y y - \omega t)\}.$$

Within a b-layer, $\nabla.(\epsilon\nabla\psi) = 0$ leads to

$$\epsilon_{zz}\frac{d^2\phi_b(z)}{dz^2} + \frac{d\phi_b(z)}{dz}(2i\,k_x\,\epsilon_{xz}) - (k_x^2\,\epsilon_{xx} + k_y^2\,\epsilon_{yy})\phi_b(z) = 0.$$

Taking $\phi_b(z)$ as the superposition of $\phi_b^+\exp(\alpha z)$ and $\phi_b^-\exp(-\alpha z)$, we get

$$\alpha = \epsilon_\perp^{\frac{1}{2}}(\epsilon_\parallel\,k_x^2 + \epsilon_{zz}\,k_y^2)^{\frac{1}{2}}|\epsilon_{zz}|^{-1} \mp i\,k_x\,\epsilon_{zz}^{-1}\,\epsilon_{xz} = \alpha_E' \pm i\alpha_E''. \tag{33}$$

Equation (33) can also be obtained directly from (4) in the limit $c \to \infty$. In the a-layers, $\phi_a(z)$ satisfies the Laplace equation

$$\left(\frac{d^2}{dz^2} - k^2\right)\phi_a(z) = 0, \qquad k^2 = k_x^2 + k_y^2.$$

Periodicity in the z direction implies

$$\begin{aligned}\phi(z) &= \phi_a^+ \exp(kz) + \phi_a^- \exp(-kz), & 0 < z < d_a\\ &= \{\phi_b^+ \exp(\alpha_{\mathrm{E}}' z) + \phi_b^- \exp(-\alpha_{\mathrm{E}}' z)\} \exp(\mathrm{i}\,\alpha_{\mathrm{E}}''), & -d_b < z < 0\\ &= \{\phi_a^+ \exp(kz + kd) + \phi_a^- \exp(-kz - kd)\} \exp(-\mathrm{i}\,qd), & -d < z < -d_b,\end{aligned} \tag{34}$$

and continuity of ϕ and D_z at the interfaces

$$\phi_b(z)|_{z=0,-d_b} = \phi_a(z)|_{z=0,-d_b},$$

$$\{\mathrm{i}\, k_x \epsilon_{zx} \phi_b(z) + \phi_b'(z)\epsilon_{zz}\}_{z=0,-d_b} = \epsilon_a \phi_a'(z)|_{z=0,-d_b}$$

gives rise, for the coefficients $\phi_a^\pm$ and $\phi_b^\pm$, to a set of four algebraic relations, with the secular equation

$$\begin{aligned}\cos(qd - \alpha_{\mathrm{E}}'' d_b) &= \cosh(kd_a)\cosh(\alpha_{\mathrm{E}}' d_b)\\ &\quad + (2\epsilon_a \epsilon_{zz} k\alpha_{\mathrm{E}}')^{-1}(\epsilon_a^2 k^2 + \alpha_{\mathrm{E}}'^2 \epsilon_{zz})\sinh(kd_a)\sinh(\alpha_{\mathrm{E}}' d_b).\end{aligned} \tag{35}$$

For (35) to be valid, one must have $\epsilon_{zz} < 0$ (when $\alpha_{\mathrm{E}}' > 0$), or equivalently, in view of (33) and $\epsilon_{zz} = \epsilon_\perp \sin^2\theta + \epsilon_\| \cos^2\theta$,

$$\epsilon_\perp(\omega) < 0, \tag{36}$$

$$\epsilon_\|(\omega) < |\epsilon_\perp|(1 + \tan^2\theta + k_x^{-2} k_y^2)^{-1} k_x^{-2} k_y^2 \tan^2\theta. \tag{37}$$

In the cases of special orientations of the crystal axis considered in Section 3, equation (35) can be obtained directly from (30) in the limit $c \to \infty$. [No non-retarded solutions corresponding to equation (31) exist.] As a result of the anisotropy, equation (35) contains θ as well as the angle between $\boldsymbol{c}$ and $\boldsymbol{k}$. In the isotropic limit ($\epsilon_\perp = \epsilon_\|$), (35) reduces correctly to the dispersion relation obtained by Camley and Mills (1984). Setting $d_a = \infty$, equation (35) leads to the dispersion relations

$$\epsilon_{zz} = -\alpha_{\mathrm{E}}'^{-1} \epsilon_a k \tanh(\tfrac{1}{2}\alpha_{\mathrm{E}}' d_b), \tag{38}$$

$$\epsilon_{zz} = -\alpha_{\mathrm{E}}'^{-1} \epsilon_a k \operatorname{cotanh}(\tfrac{1}{2}\alpha_{\mathrm{E}}' d_b), \tag{39}$$

in accordance with earlier results by Tarkhanian (1975).

At a general orientation of the crystal axis, the anisotropy of the b-layers results in an asymmetry of the dispersion curves with respect to k_x, i.e. equation (35) is not invariant under a change of sign of k_x. A similar observation occurred in the case of a single interface (Wallis *et al.* 1974).

Another feature of interest is that, in contrast to the isotropic case, non-retarded layer localised modes with fields limited mainly in one sort of layer becomes possible in the anisotropic case. [Only retarded but not non-retarded layer localised modes are allowed in isotropic superlattices (Camley and Mills 1984; Shi and Tsai 1984, 1985).]

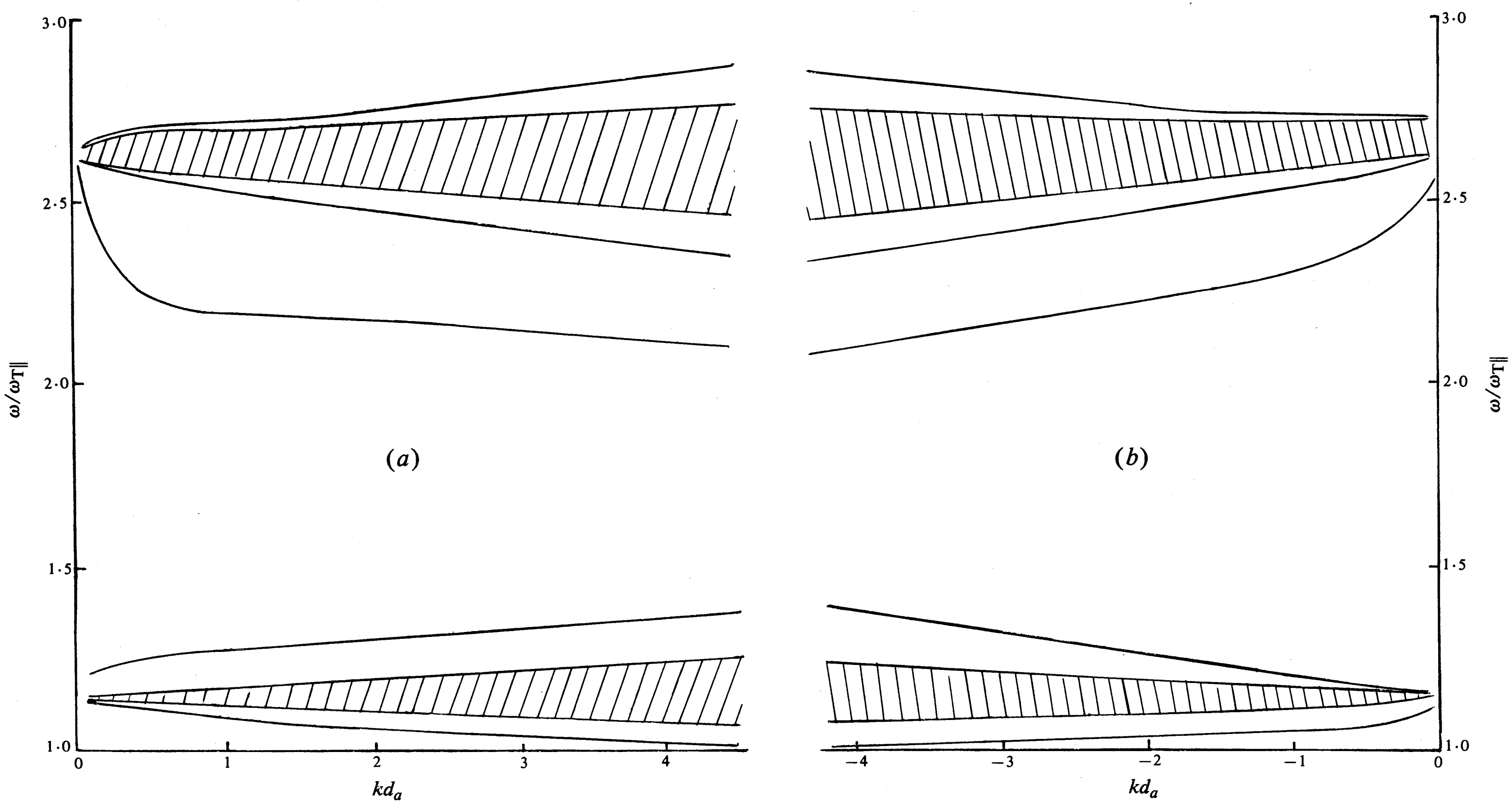

Fig. 4. Dispersion curves obtained by solving equation (40) for (*a*) positive $k = k_x$ and (*b*) negative $k = k_x$ at $\theta = \pi/4$ and $qd = \pi/5$. The same parameters as in Fig. 2 are adopted. The asymmetry with respect to positive and negative k_x is evident.

Setting $\alpha = -i\alpha'_E$ in (35), we have

$$\cos(qd - \alpha''_E d_b) = \cosh(kd_a)\cosh(\alpha d_b) + (2\epsilon_a \epsilon_{zz} \alpha k)^{-1}(\epsilon_a^2 k^2 - \epsilon_{zz}^2 \alpha'^2_E)\sinh(kd_a)\sinh(\alpha''_E d_b). \quad (40)$$

For α to be real, α'_E must be imaginary. At a given angle θ this is possible in suitable frequency ranges. In the cases (1) and (2) of Section 3, the frequency ranges are fixed by $\epsilon_\perp, \epsilon_\| < 0$, while in the case (3), since $\alpha'_E = k$, there can be no undulatory solutions, as in the isotropic case.

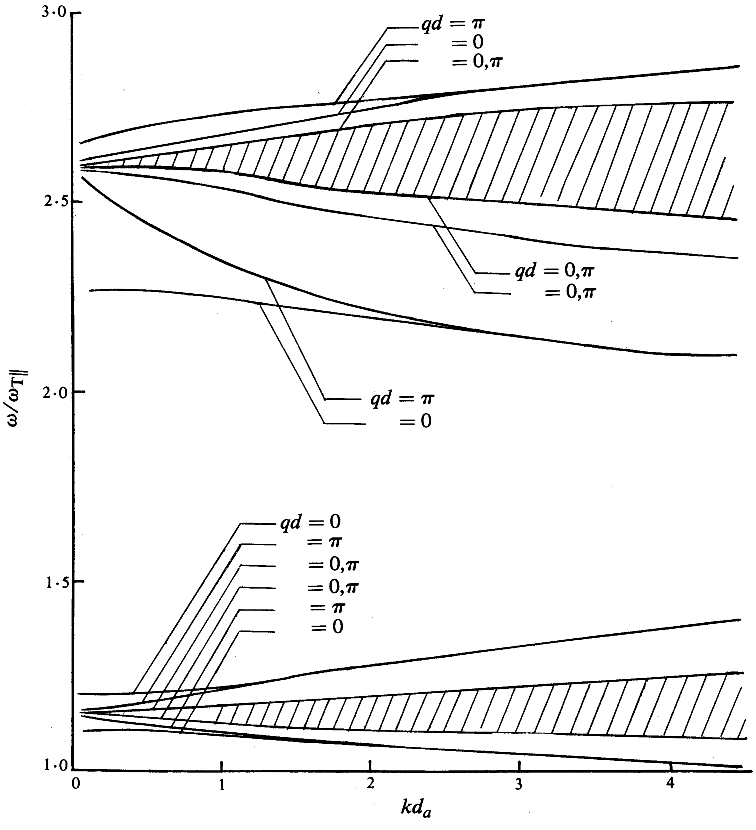

Fig. 5. Dispersion curves obtained by solving equation (40) with $\theta = \pi/4$ and $qd = 0, \pi$. The same parameters as in Fig. 2 are adopted.

At a general angle θ, equation (40) is also non-invariant under a change of sign of k_x. Figs 4*a* and 4*b* show the dispersion curves for $\pm k$, $k = k_x$, respectively, when $\theta = \pi/4$ and $qd = \pi/5$. The shaded areas are regions of crowded dispersion curves. A glance at these two figures shows clearly the asymmetry with respect to $\pm k = k_x$. Fig. 5 illustrates the dispersion curves according to equation (40) for $\theta = \pi/4$ and $qd = 0$ and π. The shaded areas again represent regions of crowded dispersion curves. The curves for $qd = 0$ and π almost coincide except at the boundaries of the two allowed regions.

The asymmetry with respect to $\pm k_x$ and the existence of layer localised modes are two new features unique to superlattices consisting of uniaxial crystalline and isotropic layers, and thus can be expected to be subjected to, for example, light scattering investigations.

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