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A Charged Analogue of the Vaidya–Tikekar Solution

L. K. Patel and Sharda S. Koppar

Department of Mathematics, Gujarat University,
Ahmedabad 380 009, India.

Abstract

We present here an interior solution of the Einstein–Maxwell equations for a charged static fluid sphere. The physical 3-space t = constant of the solution is spheroidal. The solution is interpreted as an exact relativistic model for a charged superdense star.

1. Introduction

Many exact interior solutions of the Einstein–Maxwell equations corresponding to a static charged sphere are available in the literature. The sphere of uniform density has been discussed by Kyle and Martin (1967) and by Mehra and Bohra (1979). The sphere of charged dust has been investigated by Bonnor and Wickramasuriya (1975), Raychaudhuri (1975) and Tikekar (1984). In all these cases, except Tikekar (1984), the physical 3-space t = constant is spherical. Patel and Pandya (1986) have obtained a Reissner–Nordstrom interior solution in which the physical 3-space t = constant is spheroidal. In all the above-mentioned solutions, the space–times are not conformally flat. Chang (1983) has obtained some conformally flat interior solutions of the Einstein–Maxwell equations for a charged static sphere, while Hajj-Boutros and Sfeila (1986) have presented a generation technique to derive new exact solutions for a charged fluid sphere. They have also listed those papers which deal with different aspects of interior Reissner–Nordstrom solutions; for the sake of brevity, we do not repeat these references here.

Furthermore, it has been generally suggested that the collapse of a spherically symmetric distribution of matter to a point singularity can be avoided if the matter is accompanied by charge. The gravitational attraction is then balanced by the electrostatic repulsion and by the pressure gradient. This shows the importance of the interior Reissner–Nordstrom solutions.

It is the purpose of the present paper to obtain an interior solution for a charged perfect fluid sphere with the following properties:

(i) the space–time describing the geometry of the solution is not conformally flat;

(ii) the associated physical 3-space t = constant is spheroidal.

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2. Metric Form and the Field Equations

Vaidya and Tikekar (1982) have discussed the space–times, with physical 3-space spheroidal, in some detail. They expressed the line element of such space–times in the form

$$ds^2 = \exp(\gamma)\,dt^2 - \left(1 - \frac{Kr^2}{R^2}\right)\left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2$$
$$- r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1}$$

where γ is a function of r only. Here R and K are constants and $K < 1$. The line element (1) is regular at all points where $r^2 < R^2$. We denote the coordinates by $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$ and $x^4 = t$. When $K = 0$, the physical 3-space $t =$ constant becomes spherical.

The combined Einstein–Maxwell field equations are (using the geometric units $c = G = 1$)

$$R^i_k - \tfrac{1}{2}R\delta^i_k = -8\pi T^i_k, \tag{2}$$

where the energy–momentum tensor splits into two parts:

$$T^i_k = M^i_k + E^i_k. \tag{3}$$

For a charged perfect fluid we have

$$M^i_k = (p+\rho)v^i v_k - p\delta^i_k, \tag{4}$$

$$E^i_k = \frac{1}{4\pi}\left(-F^{in}F_{kn} + \tfrac{1}{4}\delta^i_k F_{mn}F^{mn}\right), \tag{5}$$

where E^i_k is the electromagnetic energy tensor and v^i is the 4-vector velocity. The electromagnetic field tensor F_{ik} satisfies Maxwell's equations

$$F_{ik,j} + F_{kj,i} + F_{ji,k} = 0, \tag{6a}$$

$$\frac{\partial}{\partial x^k}\{(-g)^{\frac{1}{2}}F^{ik}\} = 4\pi(-g)^{\frac{1}{2}}J^i, \tag{6b}$$

J^i being the 4-current vector. The fluid has been assumed to have null conductivity, so that

$$J^i = \sigma v^i, \tag{7}$$

where σ denotes the charge density and, since the field is static, we have

$$v^i = (0, 0, 0, \exp(-\tfrac{1}{2}\gamma)). \tag{8}$$

Since there is only a radial electric field, the only surviving component of F_{ik} is F_{14}.

The Maxwell equations (6) lead us to write

$$F_{14} = \frac{\exp(\frac{1}{2}\gamma)}{r^2}\left(\frac{1 - Kr^2/R^2}{1 - r^2/R^2}\right)^{\frac{1}{2}} \int_0^{\gamma} 4\pi\sigma\gamma^2\left(\frac{1 - Kr^2/R^2}{1 - r^2/R^2}\right)^{\frac{1}{2}} \mathrm{d}r, \tag{9}$$

and also

$$-F_{14}\,F^{14} = E^2(r), \tag{10}$$

where $E(r)$ can be interpreted as the field intensity. The results (9) and (10) imply that

$$4\pi\sigma = \frac{1}{r^2}\left(\frac{\mathrm{d}}{\mathrm{d}r}(r^2 E)\right)\left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}}\left(1 - \frac{Kr^2}{R^2}\right)^{-\frac{1}{2}}, \tag{11}$$

and, subsequently,

$$Q(r) = 4\pi \int_0^{\gamma}\left(1 - \frac{Kr^2}{R^2}\right)^{\frac{1}{2}}\left(1 - \frac{r^2}{R^2}\right)^{-\frac{1}{2}} \sigma r^2\, \mathrm{d}r, \tag{12}$$

which represents the total charge contained within the sphere of radius r.

The Einstein–Maxwell equations (2), along with (3), (4) and (5), give

$$-8\pi p + E^2 = \left\{\frac{1-K}{R^2} - \frac{\gamma'}{r}\left(1 - \frac{r^2}{R^2}\right)\right\}\left(1 - \frac{Kr^2}{R^2}\right)^{-1}, \tag{13}$$

$$8\pi\rho + E^2 = \frac{3(1-K)}{R^2}\left(1 - \frac{Kr^2}{3R^2}\right)\left(1 - \frac{Kr^2}{R^2}\right)^{-2}, \tag{14}$$

$$\begin{aligned} -8\pi p - E^2 = {} & -\left(\frac{\gamma''}{2} + \frac{\gamma'^2}{4} + \frac{\gamma'}{2r}\right)\left(1 - \frac{r^2}{R^2}\right)\left(1 - \frac{Kr^2}{R^2}\right)^{-1} \\ & + \frac{(1-K)r}{R^2}\left(\frac{\gamma'}{2} + \frac{1}{r}\right)\left(1 - \frac{Kr^2}{R^2}\right)^{-2}. \end{aligned} \tag{15}$$

Here a prime indicates differentiation with respect to r.

3. Solution of the Field Equations

We have the three equations (13), (14) and (15) and the four variables p, ρ, γ and E^2; hence, we have only one free variable. Patel and Pandya (1986) have used this freedom to obtain the solution by taking a specific form of the function E^2. In this section we take another and new specific form of E^2 and solve the three equations for the remaining three functions. Let us assume that

$$E^2(r) = \frac{\beta^2 r^2 \exp(-\frac{1}{2}\gamma)}{R^4(1 - Kr^2/R^2)^2}, \tag{16}$$

where β is a constant. Clearly E^2 is positive. Equating the expressions for the pressure p from (13) and (15), we get the differential equation

$$(1-K+Kz^2)\frac{\mathrm{d}^2F}{\mathrm{d}z^2}-Kz\frac{\mathrm{d}F}{\mathrm{d}z}+K(K-1)F=0, \tag{17}$$

where $z^2 = 1-r^2/R^2$ and $F = \exp(\frac{1}{2}\gamma)-2\beta^2/K(K-1)$. Equation (17) has been solved by Vaidya and Tikekar (1982) in the uncharged case (i.e. $\beta = 0$) for $K = -2$. Following the same method, the closed-form solution of (17) for $K = -2$ in the charged case can be expressed as

$$\exp(\tfrac{1}{2}\gamma) = \tfrac{1}{3}\beta^2+Az(1-\tfrac{4}{9}z^2)+B(1-\tfrac{2}{3}z^2)^{\frac{3}{2}}, \tag{18}$$

where A and B are integration constants. The closed-form solutions of (17) have also been obtained for $K = -7$ and -14 etc., but are not reported here.

The matter density and the fluid pressure are found to be

$$8\pi\rho = \frac{3}{R^2(3-2z^2)^2}\left(5-2z^2-\frac{\beta^2(1-z^2)}{\beta^2+3Az(1-\frac{4}{9}z^2)+3B(1-\frac{2}{3}z^2)^{\frac{3}{2}}}\right), \tag{19}$$

$$8\pi pR^2(3-2z^2)\{\tfrac{1}{3}\beta^2+Az(1-\tfrac{4}{9}z^2)+B(1-\tfrac{2}{3}z^2)^{\frac{3}{2}}\}$$
$$=\frac{\beta^2(1-z^2)}{3-2z^2}-\beta^2+B(6z^2-3)(1-\tfrac{2}{3}z^2)^{\frac{1}{2}}+Az(4z^2-5). \tag{20}$$

The density ρ and the pressure p at the centre ($r = 0$) attain the values

$$8\pi\rho_0 = \frac{9}{R^2}, \tag{21}$$

$$8\pi p_0 = \frac{9}{R^2}\left(\frac{B\sqrt{3}-A-\beta^2}{B\sqrt{3}+5A+3\beta^2}\right). \tag{22}$$

The charge density σ can be determined from (11) in the form

$$8\pi\sigma R^2\left(1+\frac{2r^2}{R^2}\right)^{\frac{5}{2}}\exp(\tfrac{3}{4}\gamma) = \beta\left\{\frac{A}{9}\left(30+\frac{11r^2}{R^2}-\frac{22r^4}{R^4}+\frac{8r^6}{R^6}\right)\right.$$
$$\left.+\frac{2B}{\sqrt{3}}\left(1-\frac{r^2}{R^2}\right)^{\frac{1}{2}}\left(1+\frac{2r^2}{R^2}\right)^{\frac{3}{2}}\left(1-\frac{r^2}{3R^2}\right)+\tfrac{2}{3}\beta^2\left(3+\frac{2r^2}{R^2}\right)\left(1-\frac{r^2}{R^2}\right)^{\frac{1}{2}}\right\}, \tag{23}$$

where $\exp(\gamma)$ is given by (18). It is easy to see that $E^2 = 0$ at the centre $r = 0$.

We consider a situation where the spherical charged perfect fluid distribution extends to a finite radius $a < R$. The interior metric (1) with $\exp(\gamma)$ given by (18) should then match with the external Reissner–Nordstrom metric

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 - \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2$$
$$- r^2(d\theta^2 + \sin^2\theta \, d\phi^2), \tag{24}$$

across the boundary $r = a$. Here m and q, respectively, denote the total mass and the total charge of the sphere. The relations (12) and $Q(a) = q$ give us

$$\beta^2 a^6 \Big/ R^4 \left(1 + \frac{2a^2}{R^2}\right)^2 \left\{\frac{\beta^2}{3} + \frac{A}{9}\left(1 - \frac{a^2}{R^2}\right)^{\frac{1}{2}}\left(5 + \frac{4a^2}{R^2}\right) + \frac{B}{3\sqrt{3}}\left(1 + \frac{2a^2}{R^2}\right)^{\frac{3}{2}}\right\} = q^2. \tag{25}$$

The appropriate boundary conditions are

$$\exp\{\gamma(a)\} = \left(1 - \frac{a^2}{R^2}\right)\left(1 + \frac{2a^2}{R^2}\right)^{-1} = 1 - \frac{2m}{a} + \frac{q^2}{a^2},$$

and the pressure at $r = a$ vanishes. These conditions determine the constants A, B and m to be

$$A = \tfrac{3}{2}\left(1 - \frac{2a^2}{R^2}\right)\left(1 + \frac{2a^2}{R^2}\right)^{-\frac{1}{2}} - \frac{\beta^2}{6}\left(4 - \frac{5a^2}{R^2}\right)\left(1 - \frac{a^2}{R^2}\right)^{-\frac{1}{2}}, \tag{26}$$

$$B = \frac{\sqrt{3}}{2}\left(1 + \frac{4a^2}{R^2}\right)\left(1 - \frac{a^2}{R^2}\right)^{\frac{1}{2}}\left(1 + \frac{2a^2}{R^2}\right)^{-1}$$
$$+ \frac{\beta^2}{6\sqrt{3}}\left(2 - \frac{9a^2}{R^2} - \frac{20a^4}{R^4}\right)\left(1 + \frac{2a^2}{R^2}\right)^{-\frac{3}{2}}, \tag{27}$$

$$\frac{m}{a} = \frac{3a^2}{2R^2}\left(1 + \frac{2a^2}{R^2}\right)^{-1} + \frac{\beta^2 a^4}{2R^4}\left(1 - \frac{a^2}{R^2}\right)^{-\frac{1}{2}}\left(1 + \frac{2a^2}{R^2}\right)^{-\frac{3}{2}}. \tag{28}$$

From (28) it is evident that the mass parameter m is positive.

Thus, the final form of the metric of our solution is

$$ds^2 = \tfrac{1}{9}\left\{\beta^2 + \frac{A}{3}\left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}}\left(5 + \frac{4r^2}{R^2}\right) + \frac{B}{\sqrt{3}}\left(1 + \frac{2r^2}{R^2}\right)^{\frac{3}{2}}\right\}^2 dt^2$$
$$- \frac{1 + 2r^2/R^2}{1 - r^2/R^2} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\phi^2), \tag{29}$$

where A and B are given by (26) and (27). When $\beta = 0$, the electromagnetic field disappears and we get the solution discussed by Vaidya and Tikekar (1982) in connection with the exact relativistic model for a superdense star. Thus, our solution is a charged analogue of the Vaidya–Tikekar solution, and describes the interior field of a charged superdense star.

Table 1. Variation of parameters with a/b for (*a*) $\beta^2 = 0{\cdot}1$ and (*b*) $\beta^2 = 2{\cdot}0$

No.	a/R	λ	A	B	$f(A,B)$	$1-3f(A,B)$	$1-f(A,B)$
				(*a*) $\beta^2 = 0{\cdot}1$			
1	0.025000	0.997913	1.430550	0.924300	0.007774	0.976679	0.992226
2	0.050000	0.991697	1.422240	0.925902	0.009037	0.972889	0.990963
3	0.075000	0.981479	1.408470	0.928513	0.011146	0.966562	0.988854
4	0.100000	0.967468	1.389350	0.932047	0.014107	0.957680	0.985893
5	0.125000	0.949940	1.365060	0.936390	0.017928	0.946216	0.982072
6	0.150000	0.929229	1.335780	0.941399	0.022622	0.932133	0.977378
7	0.175000	0.905713	1.301770	0.946909	0.028208	0.915376	0.971792
8	0.200000	0.879797	1.263270	0.952740	0.034710	0.895871	0.965290
9	0.225000	0.851901	1.220570	0.958698	0.042160	0.873521	0.957840
10	0.250000	0.822444	1.173960	0.964582	0.050601	0.848196	0.949399
11	0.275000	0.791835	1.123770	0.970186	0.060091	0.819727	0.939909
12	0.300000	0.760458	1.070280	0.975308	0.070702	0.787895	0.929298
13	0.325000	0.728670	1.013830	0.979746	0.082527	0.752420	0.917473
14	0.350000	0.696792	0.954700	0.983306	0.095687	0.712939	0.904313
15	0.375000	0.665108	0.893198	0.985803	0.110337	0.668989	0.889663
16	0.400000	0.633860	0.829605	0.987057	0.126676	0.619971	0.873324
17	0.425000	0.603253	0.764187	0.986903	0.144963	0.565111	0.855037
18	0.450000	0.573454	0.697201	0.985179	0.165535	0.503396	0.834465
19	0.475000	0.544594	0.628881	0.981736	0.188838	0.433487	0.811162
20	0.500000	0.516773	0.559449	0.976430	0.215471	0.353586	0.784529
21	0.525000	0.490060	0.489106	0.969120	0.246256	0.261233	0.753744
22	0.550000	0.464502	0.418041	0.959670	0.282340	0.152980	0.717660
23	0.575000	0.440123	0.346425	0.947943	0.325383	0.023850	0.674617
24	0.600000	0.416929	0.274414	0.933799	0.377866	-0.133598	0.622134
				(*b*) $\beta^2 = 2{\cdot}0$			
1	0.025000	0.997782	0.164481	2.016130	0.128708	0.613875	0.871292
2	0.050000	0.991172	0.157952	2.002370	0.127731	0.616807	0.872269
3	0.075000	0.980308	0.147154	1.979580	0.126083	0.621750	0.873917
4	0.100000	0.965409	0.132217	1.947920	0.123736	0.628793	0.876264
5	0.125000	0.946765	0.113312	1.907660	0.120646	0.638063	0.879354
6	0.150000	0.924732	0.090655	1.859110	0.116755	0.649735	0.883245
7	0.175000	0.899707	0.064497	1.802600	0.111989	0.664033	0.888011
8	0.200000	0.872118	0.035119	1.738540	0.106252	0.681243	0.893748
9	0.225000	0.842409	0.002831	1.667340	0.099425	0.701725	0.900575
10	0.250000	0.811024	-0.032041	1.589430	0.091357	0.725928	0.908643
11	0.275000	0.778392	-0.069156	1.505230	0.081863	0.754412	0.918137
12	0.300000	0.744919	-0.108163	1.415190	0.070710	0.787871	0.929290
13	0.325000	0.710982	-0.148710	1.319700	0.057608	0.827175	0.942392
14	0.350000	0.676919	-0.190439	1.219170	0.042196	0.873413	0.957805
15	0.375000	0.643026	-0.232996	1.113960	0.024012	0.927963	0.975988
16	0.400000	0.609559	-0.276033	1.004420	0.002474	0.992578	0.997526
17	0.425000	0.576729	-0.319202	0.890850	-0.023171	1.069510	1.023170
18	0.450000	0.544710	-0.362163	0.773535	-0.053904	1.161710	1.053900
19	0.475000	0.513634	-0.404575	0.652714	-0.091018	1.273050	1.091020
20	0.500000	0.483600	-0.446103	0.528595	-0.136251	1.408750	1.136250
21	0.525000	0.454674	-0.486406	0.401350	0.191980	1.575940	1.191980
22	0.550000	0.426896	-0.525136	0.271117	-0.261524	1.784570	1.261520
23	0.575000	0.400279	-0.561934	0.137997	-0.349643	2.048930	1.349640
24	0.600000	0.374815	-0.596420	0.002057	-0.463357	2.390070	1.463360

If ρ_a is the value of ρ at the boundary $r = a$ and $\lambda = \rho_a/\rho_0$, then a straightforward calculation yields

$$\lambda = \tfrac{1}{3}\left(3 + \frac{2a^2}{R^2}\right)\left(1 + \frac{2a^2}{R^2}\right)^{-2} - \frac{\beta^2 a^2}{9R^2}\left(1 - \frac{a^2}{R^2}\right)^{-\frac{1}{2}}\left(1 + \frac{2a^2}{R^2}\right)^{-\frac{3}{2}}. \tag{30}$$

It is not hard to see that $\lambda < 1$. Thus, the central density ρ_0 is greater than the density ρ_a at the boundary. For a physically significant model we must have

$$\rho_0 > 0, \quad p_0 > 0, \quad \rho_0 - p_0 \geqslant 0, \quad \rho_0 - 3p_0 \geqslant 0. \tag{31}$$

If we set $f(A, B) = (B\sqrt{3} - A - \beta^2)/(B\sqrt{3} + 5A + 3\beta^2)$, we see that the conditions (31) are equivalent to

$$f(A, B) > 0, \qquad 1 - f(A, B) \geqslant 0, \qquad 1 - 3f(A, B) \geqslant 0. \tag{32}$$

If a/R is given, then A, B and λ can be determined from (26), (27) and (30) respectively for a fixed value of β. For the uncharged case, Vaidya and Tikekar (1982) have noted that a/R must be less than $0 \cdot 5567$ for a physically significant model. For $\beta^2 = 0 \cdot 1$ and $\beta^2 = 2 \cdot 0$, and for $0 \cdot 025 \leqslant a/R \leqslant 0 \cdot 60$, the values of λ, A, B, $f(A, B)$, $1 - 3f(A, B)$ and $1 - f(A, B)$ are given in Table 1. From this table it is clear that, when a/R increases, λ decreases. From Table 1*a* it is clear that for $\beta^2 = 0 \cdot 1$, $1 - 3f(A, B)$ becomes negative for $a/R = 0 \cdot 60$. Thus, for $\beta^2 = 0 \cdot 1$, the conditions (31) are satisfied for $0 \cdot 025 \leqslant a/R \leqslant 0 \cdot 575$. From Table 1*b* it is evident that $f(A, B)$ becomes negative for $a/R = 0 \cdot 425$. Thus, for $\beta^2 = 2 \cdot 0$, the conditions (31) are satisfied for $0 \cdot 025 \leqslant a/R \leqslant 0 \cdot 40$. Though the numerical calculations have been carried out for the exact solution corresponding $K = -2$, the method is quite general and can be used for a whole series of models with $K < 1$.

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References

Bonnor, W. B., and Wickramasuriya, S. B. P. (1975). *Mon. Not. R. Astron. Soc.* **170**, 643.
Chang, Z. S. (1983). *Gen. Relat. Gravit.* **15**, 293.
Hajj-Boutros, J., and Sfeila, J. (1986). *Gen. Relat. Gravit.* **18**, 395.
Kyle, C. F., and Martin, A. W. (1967). *Nuovo Cimento* **50**, 583.
Mehra, A. L., and Bohra, M. (1979). *Gen. Relat. Gravit.* **11**, 333.
Patel, L. K., and Pandya, B. M. (1986). *Acta Phys. Hung.* **60**, 57.
Raychaudhuri, A. K. (1975). *Ann. Inst. Henri Poincare* A **22**, 229.
Tikekar, R. (1984). *Gen. Relat. Gravit.* **16**, 445.
Vaidya, P. C., and Tikekar, R. (1982). *J. Astrophys. Astron.* **3**, 325.

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