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A Bosonisation of QCD and Realisations of Chiral Symmetry

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Abstract

We employ the functional integral formalism to study quantum chromodynamics (QCD) with N_f quarks of zero bare mass. In addition to local $SU(N_c)$ colour symmetry, this theory possesses exact global $G = U_L(N_f) \otimes U_R(N_f)$ chiral symmetry. We obtain an exact bilocal Bose field representation of the generating functional which, as we prove after establishing the manner in which the bilocal fields transform under G, preserves the global chiral symmetry. We demonstrate how a local Bose field representation of the generating functional may be obtained from the bilocal bosonisation. This provides a direct link between QCD and low energy meson phenomenological models. We utilise the bilocal bosonisation in the study of the dynamical breakdown of the global chiral symmetry group G. We derive the vacuum field equation from the exact bilocal Bose field effective action and discuss two alternative solutions: one corresponding to a Wigner–Weyl realisation of the global symmetry group G in which the vacuum configuration is invariant under G; the other to a mixed realisation in which the vacuum manifold is the coset space $G/H = U_A(N_f)$, where $H = U_V(N_f)$ is a subgroup of G.

1. Introduction

In a previous paper (Cahill and Roberts 1985), using the functional integral formalism, we developed an approximate bilocal field formulation of QCD (Marciano and Pagels 1978), which we used to establish a quantitative connection between QCD and a number of hadron phenomenologies. An important step in that work is a consideration of the vacuum structure in QCD. In the present paper we present a significant extension of the derivation presented in Cahill and Roberts (1985) which enables us to obtain an exact bilocal Bose field formulation of QCD in four dimensions. That is, we show that QCD may be exactly reformulated in terms of bilocal Bose fields and an effective action which describes their dynamics.

Expanding the bilocal fields in terms of a complete orthonormal set of local functions, where the expansion coefficients are also local functions, we derive an exact local field bosonisation of QCD. The expansion coefficients are interpreted as mesonic fluctuations about the vacuum configuration of the theory and we thus obtain a direct link between QCD and the phenomena of meson interactions. We have determined elsewhere (Cahill and Roberts 1985; Praschifka *et al.* 1986*a*, 1986*b*) the explicit form of the effective action for these local meson fields and established that it reproduces the standard meson phenomenology including the anomalous interactions and the results of current algebra.

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We also extend the discussion of the vacuum structure found in Cahill and Roberts (1985) to the case of general N_f flavour QCD with the bare mass of each quark set to zero; then, in addition to local $\mathrm{SU}(N_c)$ symmetry, the action of the theory is invariant under the transformations of the global chiral group $\mathrm{G} = \mathrm{U_L}(N_f)\otimes\mathrm{U_R}(N_f)$.

In the functional formulation the vacuum field configuration is defined to be the dynamically determined field configuration; that is, the field configuration of least action, about which any perturbative expansion of vacuum expectation values in the theory should be performed. The connection between this definition of the vacuum and that in the second quantised formulation of QCD is clear when it is realised that the usual perturbative expansion in the latter is recovered in the functional approach by expanding the vacuum expectation values in the theory about the trivial field configurations $\bar{q}(x) = 0$, $q(x) = 0$ and $A_\mu^a(x) = 0$.

We work in the Euclidean metric so that the functional integrals which we consider are well defined. Since the vacuum expectation value in the Minkowski metric of any quantity of physical interest may be obtained from its Euclidean counterpart through an analytic continuation in the Euclidean time variable (Wightman 1976), our approach is completely equivalent to the formulation of QCD in Minkowski metric.

In Cahill and Roberts (1985) we employed an approximation procedure in which the generating functional of connected gluon Green's functions was truncated at the two-point level. This enabled us to obtain an approximate representation of the generating functional of QCD as a functional integral over a finite set of hermitian bilocal fields. These bilocal fields carry the colour, flavour and Dirac quantum numbers of mesonic $\bar{q}q$ states. In this way we obtained an approximate effective bilocal action for QCD. Herein we extend this procedure and proceed directly from QCD to an exact bilocal Bose field effective action for QCD, without approximation, in terms of the same finite set of bilocal fields. The vacuum configuration of $\mathrm{U_L}\otimes\mathrm{U_R}$ symmetric QCD is then characterised by this finite set of bilocal fields.

In Section 2 we present the derivation of the exact bilocal Bose field effective action for QCD. This effective action may be written as a sum of two terms, $S = S_0 + S_1$, where S_0 involves only the connected gluon two-point function, while S_1 involves all of the higher n-point connected gluon Green's functions. Thus, for example, glueball effects enter the bosonisation through S_1. We also establish the manner in which the bilocal fields transform under G and demonstate that the exact bilocal Bose field effective action is invariant under the global chiral group G.

In Section 3 we obtain an exact local Bose field bosonisation of QCD from the bilocal Bose field effective action derived in Section 2; that is, we demonstrate that the generating functional of QCD can be expressed as a functional integral over local meson-like variables where the exponent in the integrand is a local Bose field effective action for the theory.

In Section 4 we introduce the vacuum field configuration as it is defined in the functional formalism in connection with the bilocal Bose field effective action; that is, as the field configuration which minimises the action. This means that the vacuum field configuration must satisfy the Euler–Lagrange equations obtained from this action and hence these equations are called the vacuum field equations. All of the solutions to these equations are candidates for the vacuum field configuration. The actual vacuum field configuration is selected by requiring that it must minimise the action. This is the criterion of least action.

We consider the vacuum field equation in the light of the invariance of the bilocal field effective action under G and discuss two alternative solutions. The vacuum field equation is, in fact, a collection of coupled equations, one for each particular bilocal field, and the critical equation here is the one for the flavour and colour singlet scalar bilocal field $\beta_S^{00}(x, y)$. We demonstrate that this equation admits a solution $\beta_S^{00} \equiv 0$ which corresponds to a Wigner–Weyl realisation of the symmetry group G; a vacuum configuration which is itself invariant under G.

The alternative is to consider a non-trivial solution of this equation, that is, $\beta_S^{00} \neq 0$. This corresponds to a vacuum configuration which is invariant under $H = U_V(N_f)$, a subgroup of G. Given this assumption we find that the complex of equations necessarily admits solutions which may be obtained from the singlet scalar solution by an arbitrary chiral transformation in the coset space $G/H = U_A(N_f)$. Since the action is invariant under G then each of the vacua thus obtained is degenerate with the singlet scalar solution. The assumption that the equation for β_S^{00} admits a solution $\beta_S^{00} \neq 0$ corresponds then to the situation in which the vacuum manifold is the coset space G/H, in which case the global $U_L \otimes U_R$ symmetry of the theory is said to be dynamically broken in the vacuum.

To determine which of the two alternatives is dynamically selected by the theory one must apply the criterion of least action. In the present situation however we are unable to do this, primarily because we lack a gauge invariant approximation procedure which may be employed in connection with our Bose field effective action. Such approximation procedures are difficult to find (Marciano and Pagels 1978) and thus we are not in a position at present to determine which of these two realisations of chiral symmetry is dynamically favoured in $U_L \otimes U_R$ symmetric QCD.

In Section 5 we summarise our results and conclusions.

2. Bilocal Bose Field Effective Action in QCD

The generating functional for QCD in Euclidean metric is

$$Z[J_\mu^a, \bar{\eta}, \eta] = \int D\bar{q} Dq \prod_{a\mu} DA_\mu^a \Delta_f[A_\mu^a] \times \exp\left(-S[A_\mu^a, \bar{q}, q] + \int d^4x (\bar{\eta} q + \bar{q}\eta + J_\mu^a A_\mu^a) \right),$$

where

$$S[A_\mu^a, \bar{q}, q] = \int d^4x \left(\tfrac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2\xi}(\partial_\mu A_\mu^a)^2 + \bar{q}\{\gamma_\mu(\partial_\mu - i g \tfrac{1}{2}\lambda^a A_\mu^a) + M\} q \right),$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c, \qquad a = 1, 2, \ldots, N_c^2 - 1,$$

and where $J_\mu^a, \bar{\eta}, \eta$ are the external sources associated with the fields A_μ^a, q, $\bar{q}$; M is the quark mass matrix; and the algebra of the Euclidean Dirac matrices is

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}.$$

We have chosen to employ the covariant gauge fixing procedure which is the origin of the term in the action involving ξ; $\xi = 0$ corresponds to Landau gauge and $\xi = 1$

to Feynman gauge. In accordance with the Faddeev–Popov prescription for functional integration over gauge fields, the measure of the generating functional contains the Faddeev–Popov determinant $\Delta_f[A^a_\mu]$. In the covariant gauge fixing prescription we have chosen, this factor gives rise to ghost fields which appear only as internal loop contributions to the gluon propagator and vertex functions. Implicit in the integration measure there is also a normalisation factor defined such that $Z[0,0,0] = 1$.

In writing the action of QCD we have suppressed the flavour index. We consider general N_f flavour QCD and in most of what follows we treat the case of massless quarks: $M = 0$. In this limit QCD has global

$$\mathrm{G} = \mathrm{U_L}(N_f)\otimes\mathrm{U_R}(N_f) = \mathrm{U_L}(1)\otimes\mathrm{U_R}(1)\otimes\mathrm{SU_L}(N_f)\otimes\mathrm{SU_R}(N_f)$$

chiral symmetry; that is, the action of QCD is invariant under the following transformations of the quark fields:

$$\begin{array}{lll} \mathrm{U_L}(1): & q_\mathrm{L} \to e^{\mathrm{il}} q_\mathrm{L}, & q_\mathrm{R} \to q_\mathrm{R}, \\ \mathrm{U_R}(1): & q_\mathrm{L} \to q_\mathrm{L}, & q_\mathrm{R} \to e^{\mathrm{ir}} q_\mathrm{R}, \\ \mathrm{SU_L}(N_f): & q_\mathrm{L} \to e^{\mathrm{iL.T}} q_\mathrm{L}, & q_\mathrm{R} \to q_\mathrm{R}, \\ \mathrm{SU_R}(N_f): & q_\mathrm{L} \to q_\mathrm{L}, & q_\mathrm{R} \to e^{\mathrm{iR.T}} q_\mathrm{R}, \end{array} \tag{1}$$

where $q_{\mathrm{R,L}} = \frac{1}{2}(1\pm\gamma_5)q$, $\{T_i\}_{i=1,2,\ldots,N^2-1}$ are the generators of SU(N) and $\alpha \,.\, T = \Sigma_{i=1}^{N^2-1} \alpha_i T_i$. We expect that the results we obtain in the zero bare mass limit will be physically meaningful at least in the case $N_f = 2$ (and in some applications also for $N_f = 3$), since then the left–right symmetry which we obtain should reflect the physical situation very well.

We can write the generating functional of QCD as

$$Z[J^a_\mu, \bar\eta, \eta] = \int D\bar q Dq \exp\left(-\int d^4x(\bar q\gamma\,.\,\partial q - \bar\eta q - \bar q\eta)\right)$$
$$\times \exp\left[\int d^4x\left\{\left(\mathrm{i}g\bar q\frac{\lambda^a}{2}\gamma_\mu q\right)\frac{\delta}{\delta J^a_\mu}\right\}\right]\exp(W[J_\mu]),$$

where $W[J^a_\mu]$ is the generating functional of connected gluon Green's functions which have no internal quark loops:

$$\exp(W[J^a_\mu]) = \int \prod_{a\mu} DA^a_\mu \,\Delta_f[A^a_\mu]$$
$$\times \exp\left\{-\int d^4x\left(\tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 - J^a_\mu A^a_\mu\right)\right\}.$$

The generating functional of connected gluon Green's functions has the expansion

$$W[J^a_\mu] = \sum_{n=2}^{\infty}\int d^4x_1 \ldots d^4x_n$$
$$\times\frac{1}{n!}D^{a_1\ldots a_n}_{\mu_1\ldots\mu_n}(x_1,\ldots,x_n;\xi)\,J^{a_1}_{\mu_1}(x_1)\ldots J^{a_n}_{\mu_n}(x_n),$$

where $D^{a_1 \dots a_n}_{\mu_1 \dots \mu_n}(x_1, \dots, x_n; \xi)$ is the n-point connected gluon Green's function involving all internal gluon–gluon and gluon–ghost couplings.

Using the properties of the functional derivative and the definition of the functional Taylor expansion it is not difficult to establish that

$$\exp\left(\int d^4x(\mathrm{i}\, g\bar{q}\tfrac{1}{2}\lambda^a\gamma_\mu\, q)\frac{\delta}{\delta J^a_\mu}\right)\exp(W[J^a_\mu]) = \exp(W[J^a_\mu + \mathrm{i}\, g\bar{q}\tfrac{1}{2}\lambda^a\gamma_\mu\, q]).$$

In the absence of an external source for the gluon field, that is with $J^a_\mu = 0$, the generating functional of QCD is then

$$Z[0, \bar{\eta}, \eta] = \int D\bar{q}Dq\, \exp\left(-\int d^4x(\bar{q}\gamma . \partial q - \bar{\eta}q - \bar{q}\eta)\right)$$
$$\times \exp(W[\mathrm{i}\, g\bar{q}\tfrac{1}{2}\lambda^a\gamma_\mu\, q]).$$

We can rewrite this expression as

$$Z[0, \bar{\eta}, \eta] = \exp\left\{W_1\left(\mathrm{i}\, g\frac{\delta}{\delta\eta(x)}\frac{\lambda^a}{2}\gamma_\mu\frac{\delta}{\delta\bar{\eta}(x)}\right)\right\}$$
$$\times \int D\bar{q}Dq\, \exp\left(-S[\bar{q}, q] + \int d^4x(\bar{\eta}q + \bar{q}\eta)\right), \tag{2}$$

where

$$S[\bar{q}, q] = \int d^4x\, d^4y\left(\bar{q}(x)\gamma . \partial\delta^4(x-y)\, q(y)\right.$$
$$\left. + \tfrac{1}{2}g^2\bar{q}(x)\frac{\lambda^a}{2}\gamma_\mu\, q(x)\, D^{ab}_{\mu\nu}(x-y)\,\bar{q}(y)\frac{\lambda^b}{2}\gamma_\nu\, q(y)\right), \tag{3}$$

$$W_1[J^a_\mu] = \sum_{n=3}^{\infty}\int d^4x_1 \dots d^4x_n \frac{1}{n!} D^{a_1 \dots a_n}_{\mu_1 \dots \mu_n}(x_1, \dots, x_n; \xi) \prod_{i=1}^{n} J^{a_i}_{\mu_i}(x_i).$$

The definition of W_1 involves the n ($\geqslant 3$) point connected gluon Green's functions. Little is known about the form of these functions, but in deriving the exact bilocal Bose field effective action it is necessary only to regard them as being formally defined. In the context of calculations which may proceed from this action it will be necessary to specify them more fully at the stage their contribution is to be considered. The difficult problem of calculating these higher n-point functions may involve the consideration of gluon condensates. However, these calculations are not critical to the general considerations regarding the realisation of chiral symmetry in QCD.

With the covariant gauge fixing procedure that we employ, the $n = 2$ point function $D^{ab}_{\mu\nu}(x; \xi)$ in (3) is expressed as follows:

$$D^{ab}_{\mu\nu}(x; \xi) = \delta^{ab}\int \frac{d^4k}{(2\pi)^4}\left(\frac{k^2\delta_{\mu\nu} - k_\mu k_\nu}{k^4} f(k^2; g^2, \xi) + \xi\frac{k_\mu k_\nu}{k^4}\right)e^{\mathrm{i}k.x}, \tag{4}$$

where $f(k^2; g^2, \xi) = 1/\{1 + \Pi(k^2; g^2, \xi)\}$ and where $\Pi(k^2; g^2, \xi)$ is the gluon self-energy (the vacuum polarisation) calculated in the absence of quark loops, which are included in our formalism at a later stage when the fermion integral is evaluated.

The action (3) is quartic in the quark fields and thus it is not possible to perform the functional integral in the generating functional (2) directly. To overcome this difficulty we first rearrange the quartic interaction term in (3) by using the Fierz identities for Euclidean Dirac matrices and for the colour and flavour matrices. We write $D^{ab}_{\mu\nu}(x;\xi) = \delta^{ab}D_{\mu\nu}(x;\xi)$ and the quartic interaction term becomes

$$
\begin{aligned}
-\tfrac{1}{2}g^2[\{ &\bar{q}(x)\, M^{\phi}_{\mathrm{S}}\, q(y)\, \bar{q}(y)\, M^{\phi}_{\mathrm{S}}\, q(x) + \bar{q}(x)\, M^{\phi}_{\mathrm{P}}\, q(y)\, \bar{q}(y)\, M^{\phi}_{\mathrm{P}}\, q(x)\}\, D(x-y;\xi) \\
&+ \{\bar{q}(x)\, M^{\mu\phi}_{\mathrm{V}}\, q(y)\, \bar{q}(y)\, M^{\nu\phi}_{\mathrm{V}}\, q(x) + \bar{q}(x)\, M^{\mu\phi}_{\mathrm{A}}\, q(y)\, \bar{q}(y)\, M^{\nu\phi}_{\mathrm{A}}\, q(x)\}\, D^{\mu\nu}(x-y;\xi) \\
&+ \{\bar{q}(x)\, M^{\rho\mu\phi}_{\mathrm{T}}\, q(y)\, \bar{q}(y)\, M^{\rho\nu\phi}_{\mathrm{T}}\, q(x) \\
&+ \bar{q}(x)\,\mathrm{i}\gamma_5\, M^{\rho\mu\phi}_{\mathrm{T}}\, q(y)\, \bar{q}(y)\,\mathrm{i}\gamma_5\, M^{\rho\nu\phi}_{\mathrm{T}}\, q(x)\}\, D^{\mu\nu}_{\mathrm{T}}(x-y;\xi)]\,,
\end{aligned}
$$

where the change of sign arises because of the Grassmann structure of the quark fields and where

$$
\begin{aligned}
M^{\phi}_{\mathrm{S}} &= 1\otimes C^A\otimes F^B, & \{\phi\} &= \{A, B\}, \\
M^{\phi}_{\mathrm{P}} &= \mathrm{i}\gamma_5\otimes C^A\otimes F^B, & M^{\mu\phi}_{\mathrm{V}} &= \mathrm{i}\gamma_\mu\otimes C^A\otimes F^B, \\
M^{\mu\phi}_{\mathrm{A}} &= \mathrm{i}\gamma_5\gamma_\mu\otimes C^A\otimes F^B, & M^{\rho\mu\phi}_{\mathrm{T}} &= \sigma_{\rho\mu}\otimes C^A\otimes F^B,
\end{aligned}
$$

$$
\{C^A; A=0,1,\dots,N_c^2-1\} = \left\{\left(\frac{C_2(R)}{N_c}\right)^{\frac{1}{2}} 1,\ \frac{\mathrm{i}}{2\sqrt{N_c}}\lambda^1,\ \dots,\ \frac{\mathrm{i}}{2\sqrt{N_c}}\lambda^{N_c^2-1}\right\},
$$

$$
C_2(R) = (N_c^2-1)/2N_c\,,
$$

$$
\{F^B; B=0,1,\dots,N_f^2-1\} = \left\{\frac{1}{\sqrt{N_f}}1, \sqrt{2}T^1, \dots, \sqrt{2}T^{N_f^2-1}\right\};
$$

and further, $\{\frac{1}{2}\lambda^a\}_{a=1,\dots,N_c^2-1}$ are the generators of $\mathrm{SU}(N_c)$ and $\{T^b\}_{b=1,\dots,N_f^2-1}$ are the generators of $\mathrm{SU}(N_f)$. Finally, we have

$$
D(x;\xi) = \tfrac{1}{4}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{1}{k^2}\{3f(k^2;g^2,\xi)+\xi\}\mathrm{e}^{\mathrm{i}k.x},
$$

$$
D^{\mu\nu}(x;\xi) = \tfrac{1}{4}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{1}{k^2}\left(\delta_{\mu\nu}\{f(k^2;g^2,\xi)+\xi\} + 2\frac{k_\mu k_\nu}{k^2}\{f(k^2;g^2,\xi)-\xi\}\right)\mathrm{e}^{\mathrm{i}k.x},
$$

$$
D^{\mu\nu}_{\mathrm{T}}(x;\xi) = -\tfrac{1}{16}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{1}{k^2}\left(\delta_{\mu\nu} - 4\frac{k_\mu k_\nu}{k^2}\right)\{f(k^2;g^2,\xi)-\xi\}\mathrm{e}^{\mathrm{i}k.x}.
$$

In writing the Fierz transformed quartic interaction term we have made use of the fact that

$$
(\mathrm{i}\gamma_5\sigma^{\rho\mu})_{rs}\, D^{\mu\nu}_{\mathrm{T}}(x;\xi)(\mathrm{i}\gamma_5\sigma^{\rho\nu})_{tu} = (\sigma^{\rho\mu})_{rs}\, D^{\mu\nu}_{\mathrm{T}}(x;\xi)(\sigma^{\rho\nu})_{tu},
$$

so that the manifest invariance of the action under G is preserved.

We now introduce bilocal auxiliary fields (Kleinert 1976; Schrauner 1977) into the generating functional (2) by multiplying, with the appropriate change in normalisation, by the constant factor $\int D\boldsymbol{\beta}\,\exp(-I[\boldsymbol{\beta}])$, where

$$\begin{aligned} I[\boldsymbol{\beta}] = \tfrac{1}{2}g^2 \int & \mathrm{d}^4x\,\mathrm{d}^4y\{\beta_S^\phi(y,x)\,\beta_S^\phi(x,y) + \beta_P^\phi(y,x)\,\beta_P^\phi(x,y)\}\,D(x-y) \\ & + \{\beta_V^{\mu\phi}(y,x)\,\beta_V^{\nu\phi}(x,y) + \beta_A^{\mu\phi}(y,x)\,\beta_A^{\nu\phi}(x,y)\}\,D^{\mu\nu}(x-y) \\ & + \{\beta_T^{\rho\mu\phi}(y,x)\,\beta_T^{\rho\nu\phi}(x,y) - \tfrac{1}{4}\epsilon^{\rho\mu\alpha\beta}\beta_T^{\alpha\beta\phi}(y,x)\,\epsilon^{\rho\nu\gamma\delta}\beta_T^{\gamma\delta\phi}(x,y)\}\,D_T^{\mu\nu}(x-y), \end{aligned} \tag{5}$$

and

$$\boldsymbol{\beta} = \{\beta_S^\phi, \beta_P^\phi, \beta_V^{\mu\phi}, \beta_A^{\mu\phi}, \beta_T^{\rho\mu\phi}\},$$

$$D\boldsymbol{\beta} = \prod_\phi D\beta_S^\phi\, D\beta_P^\phi \left\{ \prod_\mu D\beta_V^{\mu\phi}\, D\beta_A^{\mu\phi} \left(\prod_{\rho>\mu} D\beta_T^{\rho\mu\phi} \right) \right\}.$$

The bilocal fields are hermitian, that is $\{\beta(x,y)\}^\dagger = \beta(y,x)$ for each bilocal field. In (5) we have deliberately neglected to write the gauge parameter explicitly and we shall continue to do this in the following.

The shift in the bilocal fields

$$\beta_S^\phi(x,y) \to \beta_S^\phi(x,y) + \bar{q}(y)\,M_S^\phi\,q(x) \qquad \text{etc.}$$

has unit Jacobian and effects the reduction of the exponent in the integrand of the functional integral over the quark fields to second order. The generating functional becomes

$$\begin{aligned} Z[\bar\eta,\eta] = {} & \exp\left\{ W_1\left(\mathrm{i}\,g \frac{\delta}{\delta\eta(x)} \frac{\lambda^a}{2}\,\gamma_\mu \frac{\delta}{\delta\bar\eta(x)} \right) \right\} \\ & \times \int D\bar{q} D q D\boldsymbol{\beta}\, \exp\left(-S[\bar{q},q,\boldsymbol{\beta}] + \int \mathrm{d}^4x(\bar\eta q + \bar{q}\eta) \right), \end{aligned}$$

where

$$S[\bar{q},q,\boldsymbol{\beta}] = \int \mathrm{d}^4x\,\mathrm{d}^4y[\bar{q}(x)\{\gamma\,.\,\partial\delta^4(x-y) + \Sigma(x,y;[\boldsymbol{\beta}])\}\,q(y)] + I[\boldsymbol{\beta}], \tag{6}$$

and

$$\begin{aligned} \Sigma(x,y;[\boldsymbol{\beta}]) = {} & \{M_S^\phi\,\beta_S^\phi(x,y) + M_P^\phi\,\beta_P^\phi(x,y)\}\,g^2 D(x-y) \\ & + \{M_V^{\mu\phi}\,\beta_V^{\nu\phi}(x,y) + M_A^{\mu\phi}\,\beta_A^{\nu\phi}(x,y)\}\,g^2 D^{\mu\nu}(x-y) \\ & + M_T^{\rho\mu\phi}\{\beta_T^{\rho\nu\phi}(x,y) - \tfrac{1}{2}\gamma_5\,\epsilon^{\rho\nu\alpha\beta}\,\beta_T^{\alpha\beta\phi}(x,y)\}\,g^2 D_T^{\mu\nu}(x-y). \end{aligned} \tag{7}$$

The action in (6) is invariant under the chiral group G acting in the following way: the right and left combinations q_R and q_L transform according to (1) while the bilocal fields transform according to

$$F^B(\beta_S^{AB}+i\beta_P^{AB}) \to U_L\, F^B(\beta_S^{AB}+i\beta_P^{AB})U_R^\dagger\,,$$

$$F^B(\beta_S^{AB}-i\beta_P^{AB}) \to U_R\, F^B(\beta_S^{AB}-i\beta_P^{AB})U_L^\dagger\,,$$

$$F^B(\beta_V^{\mu AB}-\beta_A^{\mu AB}) \to U_R\, F^B(\beta_V^{\mu AB}-\beta_A^{\mu AB})U_R^\dagger\,,$$

$$F^B(\beta_V^{\mu AB}+\beta_A^{\mu AB}) \to U_L\, F^B(\beta_V^{\mu AB}+\beta_A^{\mu AB})U_L^\dagger\,,$$

$$F^B(\beta_T^{\rho\mu AB}-\tfrac{1}{2}\epsilon^{\rho\mu\alpha\beta}\beta_T^{\alpha\beta AB}) \to U_L\, F^B(\beta_T^{\rho\mu AB}-\tfrac{1}{2}\epsilon^{\rho\mu\alpha\beta}\beta_T^{\alpha\beta AB})U_R^\dagger\,,$$

$$F^B(\beta_T^{\rho\mu AB}+\tfrac{1}{2}\epsilon^{\rho\mu\alpha\beta}\beta_T^{\alpha\beta AB}) \to U_R\, F^B(\beta_T^{\rho\mu AB}+\tfrac{1}{2}\epsilon^{\rho\mu\alpha\beta}\beta_T^{\alpha\beta AB})U_L^\dagger\,, \tag{8}$$

where $U_L = e^{i(l+L.T)}$ and $U_R = e^{i(r+R.T)}$.

The transformation properties of both the quark fields and the bilocal fields under vector $U_V(N_f)$ are obtained from (1) and (8) respectively by setting $l = r$ and $\boldsymbol{L} = \boldsymbol{R}$ while their transformation properties under axial $U_A(N_f)$ are obtained by setting $-l = r$ and $-\boldsymbol{L} = \boldsymbol{R}$.

We now evaluate the functional integral over the quark fields and obtain

$$Z[\bar\eta,\eta] = \int D\beta\, \exp(-S[\bar\eta,\eta,\beta])\,, \tag{9}$$

where

$$S[\bar\eta,\eta,\beta] = S_0[\beta]+S_1[\bar\eta,\eta,\beta]\,,$$

and where

$$S_0[\beta] = -\mathrm{Tr\,Ln}\{(G^{-1})(x,y;[\beta])\} + I[\beta]\,, \tag{10}$$

and $S_1[\bar\eta,\eta,\beta]$ is defined by

$$\exp(-S_1[\bar\eta,\eta,\beta]) = \exp\left\{W_1\left(i\,g\frac{\delta}{\delta\eta(x)}\frac{\lambda^a}{2}\gamma_\mu\frac{\delta}{\delta\bar\eta(x)}\right)\right\}$$
$$\times\exp\left(\int d^4x\, d^4y\, \bar\eta(x)\, G(x,y;[\beta])\,\eta(y)\right).$$

The action $S[\bar\eta,\eta,\beta]$ is an exact bilocal Bose field representation of the action for $U_L \otimes U_R$ symmetric QCD. Further, $G(x,y;[\beta])$ is the Green's function of the operator

$$(G^{-1})(x,y;[\beta]) = \gamma\,.\,\partial\delta^4(x-y)+\Sigma(x,y;[\beta])\,,$$

that is,

$$\int d^4w\,(G^{-1})(x,w;[\beta])\,G(w,y;[\beta]) = 1_{S,C,F}\,\delta^4(x-y)\,,$$

$$\int d^4w\, G(x,w;[\beta])(G^{-1})(w,y;[\beta]) = 1_{S,C,F}\,\delta^4(x-y)\,.$$

We remark that if we had retained the quark mass matrix M then the derivation of the bilocal bosonisation would proceed exactly as before, the only change in the result being that, in $G^{-1}(x, y; [\beta])$, $\gamma \,.\, \partial$ is replaced by $\gamma \,.\, \partial + M$.

It is clear in the above that $G(x, y; [\beta])$ is the Euclidean propagator for a relativistic quark in an external non-local potential $\Sigma(x, y; [\beta])$. It should be understood however that in general this is not equivalent to the complete quark propagator in QCD which is defined in the functional integral formulation through

$$\langle q(x)\, \bar{q}(y) \rangle = \int D\beta \; G(x, y; [\beta]) \exp(-S[0, 0, \beta]) \,.$$

In dealing with the vacuum configuration we are interested primarily in the effective action $S[\bar{\eta}, \eta, \beta]$ in the absence of external source terms, and thus we shall concentrate our attention on $S[0, 0, \beta] \equiv S[\beta]$. The structure of $S_0[\beta]$ is described explicitly in (10), however $S_1[0, 0, \beta] \equiv S_1[\beta]$ is not expressible in such a simple closed form, instead, $S_1[\beta]$ is a sum of all of the admissible connected vacuum diagrams for the theory which involve the n ($\geqslant 3$) point connected gluon Green's functions explicitly. To illustrate this we write S_1, showing the lowest order terms explicitly, as

$$\begin{aligned} S_1[\beta] = & \int \mathrm{d}^4 x \, \mathrm{d}^4 y \, \mathrm{d}^4 z \; \mathrm{i} \frac{g^3}{3!} \, D^{abc}_{\mu\nu\rho}(x, y, z; \xi) \\ & \times \Bigg\{ 2 \mathrm{Tr}\bigg(\frac{\lambda}{2}^{a} \gamma_\mu \, G(x, y; [\beta]) \frac{\lambda}{2}^{b} \gamma_\nu \, G(y, z; [\beta]) \frac{\lambda}{2}^{c} \gamma_\rho \, G(z, x; [\beta]) \bigg) \\ & - 3 \mathrm{Tr}\bigg(\frac{\lambda}{2}^{a} \gamma_\mu \, G(x, y; [\beta]) \frac{\lambda}{2}^{b} \gamma_\nu \, G(y, x; [\beta]) \bigg) \mathrm{Tr}\bigg(\frac{\lambda}{2}^{c} \gamma_\rho \, G(z, z; [\beta]) \bigg) \\ & + \mathrm{Tr}\bigg(\frac{\lambda}{2}^{a} \gamma_\mu \, G(x, x; [\beta]) \bigg) \mathrm{Tr}\bigg(\frac{\lambda}{2}^{b} \gamma_\nu \, G(y, y; [\beta]) \bigg) \mathrm{Tr}\bigg(\frac{\lambda}{2}^{c} \gamma_\rho \, G(z, z; [\beta]) \bigg) \Bigg\} \\ & + \ldots . \end{aligned}$$

In general, as illustrated above, $S_1[\beta]$ is a functional of

$$D^{a_i \ldots a_{i+k}}_{\mu_i \ldots \mu_{i+k}}(x_i, \ldots, x_{i+k})$$

and

$$\mathrm{Tr}\bigg(\frac{\lambda}{2}^{a_j} \gamma_{\mu_j} \, G(x_j, x_{j+1}; [\beta]) \ldots \frac{\lambda}{2}^{a_{j+m}} \gamma_{\mu_{j+m}} \, G(x_{j+m}, x_j; [\beta]) \bigg), \tag{11}$$

each term in this part of the action being constructed from appropriate numbers of these elements contracted in such a way as to form connected vacuum diagrams. This observation allows us to demonstrate that, having performed the fermionic integration to obtain the exact bilocal Bose field effective action for QCD in (9), we have preserved the global chiral symmetry which was manifest in (6).

Obviously $I[\beta]$ is unchanged after we integrate out the fermions and so it remains invariant under the transformations in (8). To establish the invariance of the remaining terms in the action $S[\beta]$ it is convenient to define

$$S_{\pm}(x,y) = g^2 D(x-y) C^A F^B \{\beta_S^{AB}(x,y) \pm i\beta_P^{AB}(x,y)\},$$

$$V_{\pm}^{\mu}(x,y) = i g^2 D^{\mu\nu}(x-y) C^A F^B \{\beta_V^{\nu AB}(x,y) \mp \beta_A^{\nu AB}(x,y)\},$$

$$T_{\pm}(x,y) = g^2 D_T^{\mu\nu}(x,y) C^A F^B \sigma^{\rho\mu} \{\beta_T^{\rho\nu AB}(x,y) \mp \tfrac{1}{2}\epsilon^{\rho\nu\alpha\beta}\beta_T^{\alpha\beta AB}(x,y)\},$$

and $P_{R,L} = \frac{1}{2}(1 \pm \gamma_5)$, since then we may write

$$\Sigma(x,y;[\beta]) = \{S_+(x,y) + \gamma \cdot V_+(x,y) + T_+(x,y)\} P_R$$
$$+ \{S_-(x,y) + \gamma \cdot V_-(x,y) + T_-(x,y)\} P_L .$$

With this representation of $\Sigma[\beta]$ it is not difficult to establish that under the transformations in (8)

$$G^{-1} \to (G^{-1})' = (U_R P_L + U_L P_R) G^{-1} (U_R^\dagger P_R + U_L^\dagger P_L), \tag{12a}$$

$$G \to G' = (U_R P_R + U_L P_L) G (U_R^\dagger P_L + U_L^\dagger P_R). \tag{12b}$$

Using an obvious abbreviated notation, then under G

$$\mathrm{Tr\,Ln}\{G^{-1}\} \to \mathrm{Tr\,Ln}\{(U_R P_L + U_L P_R) G^{-1} (U_R^\dagger P_R + U_L^\dagger P_L)\}$$
$$= \mathrm{Tr\,Ln}\{(\gamma \cdot \partial\delta^4)(U_R P_R + U_L P_L)(\delta^4 + (\gamma \cdot \partial\delta^4)^{-1}\Sigma)(U_R^\dagger P_R + U_L^\dagger P_L)\}$$
$$= \mathrm{Tr\,Ln}\{(\gamma \cdot \partial\delta^4 + \Sigma)(U_R^\dagger P_R + U_L^\dagger P_L)(U_R P_R + U_L P_L)\}$$
$$= \mathrm{Tr\,Ln}\{G^{-1}\}.$$

The remaining part of the action, S_1, involves the fermion vacuum loops in (11) and using equation (12b) it is a simple matter to prove that such terms are invariant under G.

Combining the above results it is clear that the exact bilocal Bose field effective action for QCD, $S[\beta] = S_0[\beta] + S_1[\beta]$, is invariant under the transformations of the global chiral group G.

3. Local Bose Field Effective Action for QCD

In (9) we have an exact bilocal Bose field representation of the QCD generating functional. In some applications of the formulation we present herein it is useful to proceed one step further and obtain a local Bose field representation of the generating functional. This also helps to reveal the physical content of the bilocal fields.

We may write

$$\beta_S^{\phi}(x,y) = B_S^{\phi}\left(\frac{x+y}{2}, x-y\right) \equiv B_S^{\phi}(w,z).$$

Introducing a complete orthonormal (c.o.n.) set of functions $\{\Gamma_{Sk}^{\phi}(z)\}$, we may write

$$B_S^{\phi}(w,z) = \tilde{\Gamma}_S^{\phi}(z) + \sum_k \Phi_{Sk}^{\phi}(w)\,\Gamma_{Sk}^{\phi}(z), \tag{13}$$

where $\tilde{\Gamma}^{\phi}_{S}(z)$ is the translationally invariant vacuum function associated with $\beta^{\phi}_{S}(x, y)$, about which we shall say more in Section 4. We may perform expansions analogous to (13) for each of the bilocal fields β^{ϕ}_{P} etc. by introducing c.o.n. sets $\{\Gamma^{\phi}_{Pk}(z)\}$ etc. In this way the bilocal fields are each represented in terms of an infinite set of local Bose fields, with the appropriate quantum numbers, each of which describes a particular meson-like fluctuation about the vacuum configuration. For a particular colour, flavour and spinor bilocal field then the associated local Bose fields can be interpreted as describing the complete range of quantum excitations in this channel.

The choice of the c.o.n. sets $\{\Gamma_k\}$ is, in principle, arbitrary. In practice, however, a truncation of these sets is necessary to facilitate calculations. Then, in particular channels, optimal sets may be chosen such that the mass functionals of the associated local Bose fields are minimised. The elements of the set can be related to the form factors of the particular meson excitations with which they are associated. This connection emerges from the structure of the meson mass and coupling functionals, which we consider in detail elsewhere.

Consider now the problem of defining a functional integral over a local field $\Theta(x)$: $\int D\,\Theta(x)\,F[\Theta]$. Now suppose that $\{\Gamma_k(x)\}$ is a c.o.n. set in the space in which $\Theta(x)$ takes its values, then this functional integral, if it exists, may be rigorously defined as follows:

$$\int D\Theta(x)\,F[\Theta] \equiv \lim_{N\to\infty} \int \prod_{k=-N}^{N} \frac{\mathrm{d}c_k}{\sqrt{\pi}}\, F\Big[\sum_{k=-N}^{N} c_k\,\Gamma_k(x)\Big], \tag{14}$$

since we may write

$$\Theta(x) = \lim_{N\to\infty} \sum_{k=-N}^{N} C_k\,\Gamma_k(x).$$

As an example one might consider $F[\Theta] = \exp(-\langle\Theta|H|\Theta\rangle)$ with H hermitian and Γ_k defined such that $H\Gamma_k = \lambda_k^2\,\Gamma_k$.

In complete analogy with this procedure, using (13) and its analogues, the bilocal functional integral in (9) may be written in terms of functional integrals over local Bose fields, that is,

$$\int D\beta(x, y)\,\exp(-S[\bar{\eta}, \eta, \beta]) = \int \prod_{k} D\boldsymbol{\Phi}_k(w)\,\exp(-S[\bar{\eta}, \eta, \boldsymbol{\Phi}_k]), \tag{15}$$

up to a constant which is cancelled with appropriate normalisation, where

$$\int \prod_{k} D\boldsymbol{\Phi}_k(w)\,\exp(-S[\bar{\eta}, \eta, \boldsymbol{\Phi}]) \equiv \lim_{N\to\infty} \int \{D\boldsymbol{\Phi}(w)\}_N\,\exp(-S[\bar{\eta}, \eta, \{\boldsymbol{\Phi}\}_N]), \tag{16}$$

and where

$$\{D\boldsymbol{\Phi}(w)\}_N = \prod_{k=-N}^{N}\left[\prod_{\phi} D\Phi^{\phi}_{Sk}\,D\Phi^{\phi}_{Pk}\left\{\prod_{\mu} D\Phi^{\mu\phi}_{Vk}\,D\Phi^{\mu\phi}_{Ak}\left(\prod_{\rho>\mu} D\Phi^{\rho\mu\phi}_{Tk}\right)\right\}\right],$$

$$\{\boldsymbol{\Phi}\}_N = \{\Phi^{\phi}_{Sk}, \Phi^{\phi}_{Pk}, \Phi^{\mu\phi}_{Vk}, \Phi^{\mu\phi}_{Ak}, \Phi^{\rho\mu\phi}_{Tk}\}; \quad k = -N, \ldots, -1, 0, 1, \ldots, N,$$

and Φ^{ϕ}_{Pk} etc. are of course the analogues of Φ^{ϕ}_{Sk} in (13) associated with their own particular c.o.n. sets $\{\Gamma^{\phi}_{Pk}\}$ etc. In (15) we have an exact local Bose field representation of the QCD generating functional in four dimensional Euclidean space which may be understood rigorously through (16) and (14). This exact local field bosonisation is one of the major results of this paper. It provides, as we show elsewhere, a practical and quantitative connection between QCD and hadron phenomena.

We discuss in detail elsewhere (Cahill and Roberts 1985; Praschifka *et al.* 1986*a*, 1986*b*) the manner in which the local Bose field effective action may be expanded to determine 'bare' masses of and couplings between the Bose fields described by the local fields $\boldsymbol{\Phi}_k$, which are identified with the mesons of the physical hadron spectrum. Here we remark that the form taken by this expansion of the action, which involves the fermionic determinant, depends on the vacuum functions $\tilde{\Gamma}^{\phi}_{S}(z)$ etc. and is self-regularising because the functions $\{\Gamma^{\phi}_{Sk}\}$ etc. provide natural cutoffs in the integrals which arise. This is an important property of the transition through the bilocal bosonisation to the local bosonisation in (15). The effective action so obtained for these local meson fields reproduces the standard meson phenomenology including, for example, the anomalous interactions and many of the results of current algebra such as mass formulae for the Nambu–Goldstone bosons and $\pi\pi \rightarrow \pi\pi$ scattering lengths (Weinberg 1966).

4. Realisations of Chiral Symmetry in $\mathrm{U_L}\otimes\mathrm{U_R}$ Symmetric QCD

The generating functional in (9) with $\bar{\eta} = 0 = \eta$ is the expression, in the functional formalism, for the vacuum to vacuum transition amplitude. It is obvious that the greatest contribution to this functional integral is provided by that set of bilocal fields $\{\boldsymbol{\beta}\}$ for which the action $S[\boldsymbol{\beta}]$ in (9) assumes its minimum value. This set of bilocal fields is the one about which any perturbative expansion of functional integrals in the theory should be performed and we call this set the dynamically selected vacuum field configuration. These are the vacuum functions referred to in Section 3.

The first step in determining the vacuum configuration is to find the solutions of the Euler–Lagrange equations obtained from the action $S[\boldsymbol{\beta}]$ since these field configurations will furnish extremals of this action. These equations may be written as

$$\frac{\delta S[\boldsymbol{\beta}]}{\delta\boldsymbol{\beta}(x,y)} = \frac{\delta S_0[\boldsymbol{\beta}]}{\delta\boldsymbol{\beta}(x,y)} + \frac{\delta S_1[\boldsymbol{\beta}]}{\delta\boldsymbol{\beta}(x,y)} = 0,$$

and we shall refer to them as the vacuum field equations.

In general these equations will admit a number of sets of bilocal fields which extremise the action and so, to determine the vacuum field configuration, the set which actually minimises the action must be found. This is simply done by calculating the differences between the action $S[\boldsymbol{\beta}]$ evaluated at the various extremals.

We are able to make some general statements about the solutions of the vacuum field equations based on the invariance of the action under G. Firstly, we notice that under the transformation $\mathrm{U_L} = -1$, $\mathrm{U_R} = 1$ the bilocal fields transform as follows:

$$\beta^{\phi}_{\mathrm{S,P}} \rightarrow -\beta^{\phi}_{\mathrm{S,P}}, \quad \beta^{\mu\phi}_{\mathrm{V,A}} \rightarrow \beta^{\mu\phi}_{\mathrm{V,A}}, \quad \beta^{\rho\mu\phi}_{\mathrm{T}} \rightarrow -\beta^{\rho\mu\phi}_{\mathrm{T}},$$

and therefore

$$S[\beta] = S[\beta_S^\phi, \beta_P^\phi, \beta_V^{\mu\phi}, \beta_A^{\mu\phi}, \beta_T^{\rho\mu\phi}] = S[-\beta_S^\phi, -\beta_P^\phi, \beta_V^{\mu\phi}, \beta_A^{\mu\phi}, -\beta_T^{\rho\mu\phi}].$$

It follows then that the vacuum field equations admit the solution

$$\tilde{\beta}_S^\phi = 0, \quad \tilde{\beta}_P^\phi = 0, \quad \tilde{\beta}_T^{\rho\mu\phi} = 0, \tag{17}$$

with $\tilde{\beta}_{V,A}^{\mu\phi}$ the solutions of

$$\left.\frac{\delta S}{\delta \beta_{V,A}^{\mu\phi}}\right|_{\tilde{\beta}} = 0. \tag{18}$$

This may be seen for the scalar field as follows: the action is invariant under G and so

$$\frac{\delta S[\beta_S^\phi, \ldots]}{\delta \beta_S^\phi} = \frac{\delta S[-\beta_S^\phi, \ldots]}{\delta \beta_S^\phi},$$

and therefore

$$\left.\frac{\delta S}{\delta B^\phi}\right|_{B^\phi = \beta_S^\phi} = -\left.\frac{\delta S}{\delta B^\phi}\right|_{B^\phi = -\beta_S^\phi} \Rightarrow \left.\frac{\delta S}{\delta \beta_S^\phi}\right|_{\beta_S^\phi = 0} = 0. \tag{19}$$

That is, an even functional necessarily has an extremal at the zero value of its argument provided that the functional derivative is defined at that configuration. The proofs in the case of the pseudoscalar and tensor fields are logically identical and so we shall not present them here.

If in addition we suppose that the vector and axial vector vacuum field equations (18) admit the solution

$$\tilde{\beta}_V^{\mu 00} \neq 0, \quad \tilde{\beta}_V^{\mu 0b} = 0, \quad \tilde{\beta}_V^{\mu a0} = 0, \quad \tilde{\beta}_V^{\mu ab} = 0, \quad \tilde{\beta}_A^{\mu\phi} = 0, \tag{20}$$

where $a, b = 1, \ldots, N_f^2 - 1$ as in Section 2, then this configuration (equations 17 and 20) corresponds to a Wigner–Weyl realisation of the symmetry group G. This means that this vacuum configuration is invariant under G and therefore that it is not related to any other configuration by a chiral transformation.

An alternative to this consists in supposing that the colour and flavour singlet scalar field vacuum field equation admits a non-zero solution, $\tilde{\beta}_S^{00} = \bar{\beta} \neq 0$, in addition to the zero solution which we have shown to exist above (in equation 19). We suppose then that the complex of vacuum field equations has the solution

$$\tilde{\beta}_S^{00} = \bar{\beta} \neq 0, \quad \tilde{\beta}_S^{0b} = 0, \quad \tilde{\beta}_S^{a0} = 0, \quad \tilde{\beta}_S^{ab} = 0, \quad \tilde{\beta}_P^\phi = 0, \quad \tilde{\beta}_T^{\rho\mu\phi} = 0,$$

$$\tilde{\beta}_V^{\mu 00} = V_\mu \neq 0, \quad \tilde{\beta}_V^{\mu 0b} = 0, \quad \tilde{\beta}_V^{\mu a0} = 0, \quad \tilde{\beta}_V^{\mu ab} = 0, \quad \tilde{\beta}_A^{\mu\phi} = 0. \tag{21}$$

Given that this is a solution to the vacuum field equation then, because of the invariance of S under G, the complex of equations necessarily also has the solution

$$\tilde{\beta}_S^{0B} = \frac{1}{\sqrt{N_f}} \mathrm{Tr}_F\left[\frac{\mathrm{U}+\mathrm{U}^\dagger}{2} F^B\right]\bar{\beta}, \qquad \tilde{\beta}_P^{0B} = -\frac{i}{\sqrt{N_f}} \mathrm{Tr}_F\left[\frac{\mathrm{U}-\mathrm{U}^\dagger}{2} F^B\right]\bar{\beta}, \tag{22}$$

where $U = U_L U_R^\dagger$ is an arbitrary element of the coset space $G/H = U_A(N_f)$ (any element of H is mapped onto the identity) and the remaining fields in (21) are unchanged. To see this we first note, using (8), that under an arbitrary transformation $U \in G/H$ the fields in (21) transform as follows:

$$\beta_S^{0B} \to \frac{1}{\sqrt{N_f}} \mathrm{Tr}_F\left[\frac{U+U^\dagger}{2} F^B\right]\bar{\beta} \equiv \bar{\sigma}_B,$$

$$\beta_P^{0B} \to -\frac{i}{\sqrt{N_f}} \mathrm{Tr}_F\left[\frac{U-U^\dagger}{2} F^B\right]\bar{\beta} \equiv \bar{\pi}_B,$$

with the remaining fields unaffected by the transformation. Since the action is invariant under G then

$$S[\bar{\beta}, 0, 0, V, \ldots] = S[\bar{\sigma}_0, \bar{\sigma}_b, \bar{\pi}_B, V, \ldots].$$

We again only consider the scalar case since the pseudoscalar case is similar and we have then

$$\left.\frac{\delta S}{\delta \beta_S^{0B}}\right|_{\beta_S^{0B}=\bar{\sigma}_B,\, \beta_P^{0B}=\bar{\pi}_B,\, \ldots} = \frac{\delta S[\bar{\sigma}_0, \bar{\sigma}_b, \bar{\pi}_B, V, \ldots]}{\delta \bar{\sigma}_B} = \frac{\delta S[\bar{\beta}, 0, 0, V, \ldots]}{\delta \bar{\sigma}_B}$$

$$= \sqrt{N_f}\, \mathrm{Tr}_F\left[\frac{U+U^\dagger}{2} F^B\right] \left.\frac{\delta S}{\delta \beta_S^{00}}\right|_{\beta_S^{00}=\bar{\beta},\, \beta_S^{0b}=0,\, \beta_P^{\phi}=0,\, \ldots}$$

$$= 0, \quad \text{by assumption}.$$

This result is a manifestation of the fact that given a functional which is invariant under a global symmetry group and a particular solution of the Euler–Lagrange equation for this functional, then all of the configurations which may be obtained from the particular solution as the result of an application of a transformation from the group must necessarily also satisfy the Euler–Lagrange equation.

We remark that since the action is invariant under G then all of the vacua described by equations (22), which includes the one in (21) as a special case ($U = 1$), are degenerate. This configuration corresponds then to the situation in which the vacuum manifold is the coset space $G/H = U_A(N_f)$ and thus to a mixed, partial Nambu–Goldstone realisation of the chiral symmetry.

As we have mentioned, the vacuum field equations are the Euler–Lagrange equations obtained from the action in (9), which are

$$g^2 D(x-y)\, \beta_{S,P}^{\phi}(x,y) = g^2 D(x-y)\, \mathrm{Tr}\{M_{S,P}^{\phi}\, G(x,y;[\beta])\} - \frac{\delta S_1[\beta]}{\delta \beta_{S,P}^{\phi}(y,x)}, \tag{23}$$

$$g^2 D^{\mu\nu}(x-y)\, \beta_{V,A}^{\nu\phi}(x,y) = g^2 D^{\mu\nu}(x-y) \mathrm{Tr}\{M_{V,A}^{\nu\phi}\, G(x,y;[\beta])\} - \frac{\delta S_1[\beta]}{\delta \beta_{V,A}^{\mu\phi}(y,x)},$$

$$g^2 D_T^{\mu\nu}(x-y)\, \beta_T^{\rho\nu\phi}(x,y) = g^2 D_T^{\mu\nu}(x-y)\, \mathrm{Tr}\{M_T^{\rho\nu\phi}\, G(x,y;[\beta])\} - \tfrac{1}{2} \frac{\delta S_1[\beta]}{\delta \beta_T^{\rho\mu\phi}(y,x)}.$$

Multiplying each equation in (23) by the matrix tensor product associated with its particular bilocal field and forming the sum of the equations thus obtained, we arrive at the matrix equation

$$\begin{aligned}\Sigma(x,y;[\beta]) = {} & M_{\mathrm{S}}^{\phi}\left(g^2 D(x-y)\,\mathrm{Tr}\{M_{\mathrm{S}}^{\phi} G(x,y;[\beta])\} - \frac{\delta S_1}{\delta\beta_{\mathrm{S}}^{\phi}}\right) \\ & + M_{\mathrm{P}}^{\phi}\left(g^2 D(x-y)\,\mathrm{Tr}\{M_{\mathrm{P}}^{\phi} G(x,y;[\beta])\} - \frac{\delta S_1}{\delta\beta_{\mathrm{P}}^{\phi}}\right) \\ & + M_{\mathrm{V}}^{\mu\phi}\left(g^2 D^{\mu\nu}(x-y)\,\mathrm{Tr}\{M_{\mathrm{V}}^{\nu\phi} G(x,y;[\beta])\} - \frac{\delta S_1}{\delta\beta_{\mathrm{V}}^{\mu\phi}}\right) \\ & + M_{\mathrm{A}}^{\mu\phi}\left(g^2 D^{\mu\nu}(x-y)\,\mathrm{Tr}\{M_{\mathrm{A}}^{\nu\phi} G(x,y;[\beta])\} - \frac{\delta S_1}{\delta\beta_{\mathrm{A}}^{\mu\phi}}\right) \\ & + g^2 D_{\mathrm{T}}^{\mu\nu}(x-y)\big(M_{\mathrm{T}}^{\rho\mu\phi}\mathrm{Tr}\{M_{\mathrm{T}}^{\rho\nu\phi} G(x,y;[\beta])\} + \mathrm{i}\gamma_5 M_{\mathrm{T}}^{\rho\mu\phi}\mathrm{Tr}\{\mathrm{i}\gamma_5 M_{\mathrm{T}}^{\rho\nu\phi} G(x,y;[\beta])\}\big) \\ & - M_{\mathrm{T}}^{\rho\mu\phi}\frac{\delta S_1}{\delta\beta_{\mathrm{T}}^{\rho\mu\phi}},\end{aligned}$$

where $\Sigma(x,y;[\beta])$ is as defined in (7). We perform an inverse Fierz transformation in the first part of each term in this equation and obtain the matrix vacuum field equation

$$\Sigma(x,y;[\beta]) = g^2 D_{\mu\nu}(x-y)\frac{\lambda^a}{2}\gamma_\mu\, G(x,y;[\beta])\frac{\lambda^a}{2}\gamma_\nu - H(x,y;[\beta]), \tag{24}$$

where

$$\begin{aligned}H(x,y;[\beta]) = {} & M_{\mathrm{S}}^{\phi}\frac{\delta S_1}{\delta\beta_{\mathrm{S}}^{\phi}(y,x)} + M_{\mathrm{P}}^{\phi}\frac{\delta S_1}{\delta\beta_{\mathrm{P}}^{\phi}(y,x)} \\ & + M_{\mathrm{V}}^{\mu\phi}\frac{\delta S_1}{\delta\beta_{\mathrm{V}}^{\mu\phi}(y,x)} + M_{\mathrm{A}}^{\mu\phi}\frac{\delta S_1}{\delta\beta_{\mathrm{A}}^{\mu\phi}(y,x)} + M_{\mathrm{T}}^{\rho\mu\phi}\frac{\delta S_1}{\delta\beta^{\rho\mu\phi}(y,x)}.\end{aligned}$$

The solution of (24) that we have discussed above in connection with a Wigner–Weyl realisation of the symmetry group G is given by

$$\Sigma^{\mathrm{W}}(x,y) = \mathrm{i}\gamma\,.\,C^{\mathrm{W}}(x,y);\quad C_\mu^{\mathrm{W}}(x,y) = \left(\frac{C_2(R)}{N_c N_f}\right)^{\frac{1}{2}} g^2 D^{\mu\nu}(x-y)\,\beta_{\mathrm{V}}^{\nu 00}(x,y). \tag{25}$$

The alternative solution corresponding to the situation in which the vacuum manifold is the coset space G/H is given by

$$\Sigma(x,y) = \mathrm{i}\gamma\,.\,C(x,y) + V B(x,y), \tag{26}$$

where

$$V = P_R\,U + P_L\,U^\dagger; \qquad U \in G/H, \quad P_{R,L} = \tfrac{1}{2}(1 \pm \gamma_5)$$
$$= e^{i\gamma_5(\eta + T.\pi)}; \qquad \eta,\ |\pi| \in [0,\, 2\pi], \tag{27}$$

and $B(x, y)$ is related to $\bar{\beta}$ in (21) via

$$B(x, y) = \left(\frac{C_2(R)}{N_c\, N_f}\right)^{\frac{1}{2}} g^2 D(x-y)\,\bar{\beta}(x, y).$$

To obtain the explicit forms of C^{W}, C and B we must substitute (25) and (26) alternately into (24) and solve for these functions. However, since we do not have a simple closed form for $S_1[\beta]$ and the gluon n-point functions are not well known, this is a very difficult problem. Also, as has been observed by Marciano and Pagels (1978), it is difficult to find approximation techniques in QCD which respect the requirement of gauge invariance. We have thus far been unable to develop a demonstrably gauge invariant approximation procedure which may be employed in connection with our Bose field effective action. As a consequence of these problems we are not able to solve for the vacuum functions discussed above and so we are unable to apply the criterion of least action to determine which of the alternatives discussed above is realised in QCD.

We may, however, speculate on the consequences entailed in our bosonisation if the mixed, partial Nambu–Goldstone realisation of chiral symmetry is dynamically favoured in the theory. That this is the dynamically favoured realisation is in accordance with the existence of an octet of low mass pseudoscalar mesons which, if we suppose that $N_f = 3$ chiral symmetry is a good symmetry of QCD, may be identified with the eight broken generators of $SU_A(N_f{=}3)$. However, as we have shown above, the $U_A(1)$ symmetry is dynamically broken in the same manner as the $SU_A(N_f)$ symmetry and the vacuum manifold is thus necessarily $U_A(N_f)$. One might then expect there to be another low mass pseudoscalar meson associated with the broken generator of $U_A(1)$. No such meson has been observed and this fact is referred to as the $U_A(1)$ problem.

In deference to this experimental fact, most phenomenological models of hadrons are constructed on the premise that the vacuum manifold in massless QCD is $SU_A(N_f)$; however, as we have demonstrated above, the vacuum manifold cannot simply be $SU_A(N_f)$ but must be $U_A(N_f)$.

The standard theoretical argument for the non-existence of the isoscalar low mass pseudoscalar meson is based on the Abelian $U_A(1)$ anomaly in QCD. It is argued from this that in the presence of a non-trivial instanton vacuum the Goldstone boson corresponding to the dynamically broken $U_A(1)$ symmetry does not appear as a zero mass pole of gauge invariant Green's functions, but only of gauge variant ones. Consequently it cannot be detected in any experiment (Pascual and Tarrach 1984).

In our work we have integrated out the gauge fields and we are thus left with the apparent contradiction manifest in the $U_A(1)$ problem. However, when we consider the expansion of the Bose field effective action for massless QCD in terms of fluctuations about the vacuum using (13), we find that there are anomalous interaction terms involving the isoscalar pseudoscalar field which are not matched by like terms for the isovector pseudoscalar fields. This provides a mechanism for removing the naively expected mass degeneracy of the isovector pseudoscalar fields and the isosinglet pseudoscalar field. We will discuss this in detail elsewhere.

A local space–time dependent form of the matrix U in (27) has been shown (Cahill and Roberts 1985; Cahill *et al.* 1985) to play an important role in the understanding of QCD based hadron phenomenology. The local matrix $\mathrm{U}(x)$ is a coordinatisation of the G/H group manifold, that is, it is a matrix valued field which takes values in the coset space G/H. It is understood as the matrix valued field which describes the Goldstone fluctuations about the vacuum configuration. Using (12) we find easily that, under the global chiral group G, U transforms as follows: $\mathrm{U} \rightarrow \mathrm{U_L}\,\mathrm{U}\mathrm{U}_\mathrm{R}^\dagger$. This transformation law and the fact that U takes values in the coset space G/H are determined by the exact bilocal formulation of QCD developed herein and help to establish a connection between U and the matrix valued field used in the construction of phenomenological chiral Lagrangians.

In the latter connection we observe that if the dynamics of the field $\mathrm{U}(x)$, which is obtained in our bilocal formulation by expanding the action in terms of $\partial_\mu \mathrm{U}(x)$ about $\partial_\mu \mathrm{U}(x) = 0$, is such that finite energy stationary configurations satisfy

$$\mathrm{U}(x) \rightarrow 1 \quad \text{as} \quad |x| \rightarrow \infty\,,$$

then $\mathrm{U}(x)$ may be considered as a mapping from (R^3 compactified onto) S^3 into the $\mathrm{U_A}(N_f)$ group space. A dynamical theory involving a matrix $\mathrm{U}(x)$ with these properties has an associated homotopy group which in this case is

$$\begin{aligned}\pi_3\{\mathrm{U}(N_f)\} &= \pi_3\{\mathrm{SU}(N_f)\otimes\mathrm{U}(1)\} = \pi_3\{\mathrm{SU}(N_f)\}\\ &= Z, \quad N_f \geqslant 2;\\ &= 0, \quad \text{otherwise}.\end{aligned}$$

The theory would thus admit topological soliton solutions of its associated field equations characterised by a conserved integer charge if $N_f \geqslant 2$.

We see then that in the bilocal formulation of QCD a matrix field arises naturally, which has all of the properties ascribed to the matrix field that is used in constructing phenomenological Lagrangians which describe low energy hadron phenomena. Thus, in general, the exact bosonisation relates QCD to a generalised non-linear sigma model. Importantly, the expansion of the action indicated above also contains the anomalous couplings characteristic of the Wess–Zumino term (Cahill *et al.* 1985). However, because of the bilocal structure of the fields in our theory, the couplings we obtain necessarily include the effects of finite meson size.

We remark in closing this section that the situation is much simpler in massless quantum electrodynamics (QED) where, because this is an Abelian gauge theory, there is no S_1 contribution in the bilocal Bose field effective action and the vacuum polarisation in the photon two-point function (4) is identically zero in the absence of fermion loops. In this theory the vacuum field equation is given by (24) with $H = 0$, $\frac{1}{2}\lambda^a$ replaced by the number 1, no flavour degree of freedom and, as pointed out above, the vacuum polarisation in the photon two-point function identically zero.

Massless QED has global $\mathrm{U_L}(1)\otimes\mathrm{U_R}(1)$ chiral symmetry and so the analysis of this section can be applied in this situation (Roberts and Cahill 1986) with the result again that there are two possible alternative realisations of the chiral symmetry: a Wigner–Weyl realisation and a mixed, partial Nambu–Goldstone realisation; the analogues of those discussed herein. The effective action in this theory is much simpler and so the criterion of least action can be applied without approximation to reveal that the vacuum manifold in massless QED is the coset space $U_A(1) = U_L(1) \otimes U_R(1)/U_V(1)$.

5. Summary

Beginning with chirally symmetric QCD we have derived an exact bilocal Bose field representation of the generating functional in which the action is expressed in terms of bilocal boson-type fields and preserves the invariance of the original action, which involved quark and gluon fields, under the global chiral group $G = U_L \otimes U_R$. A local Bose field representation of the generating functional was then obtained by expressing the bilocal fields in terms of an infinite set of local fields. This last step is particularly significant since it provides a direct connection between QCD and meson phenomenology. The effective action is expressed as a sum of two terms, $S = \mathcal{S}_0 + \mathcal{S}_1$, where we have a closed form for $\mathcal{S}_0$ but no such simple expression for $\mathcal{S}_1$. Nevertheless, it is easily established that, in the absence of external sources, $\mathcal{S}_1$ simply contains all of the connected vacuum diagrams for the theory which involve n ($\geqslant 3$) point connected gluon Green's functions. Thus, all of the peculiarities of the gluonic sector, such as glueballs, enter the formalism through $\mathcal{S}_1$ in a systematic way.

This exact bosonisation provides an ideal framework in which to analyse the physical content of QCD using variables which are closely related to the observables of hadron phenomena. As a first step in this analysis it is necessary to determine the vacuum manifold in the Bose field formulation of QCD. We have shown that one allowed vacuum manifold corresponds to a Wigner–Weyl realisation of the chiral symmetry in which the vacuum configuration is invariant under the global chiral group G. The alternative vacuum manifold is the coset space $G/H = U_A(N_f)$ [where $H = U_V(N_f)$ is a subgroup of G] which corresponds to a mixed, partial Nambu–Goldstone realisation of the chiral symmetry. The bilocal field configurations corresponding to these manifolds are the solutions of the Euler–Lagrange equations obtained from the effective action and so we call these equations the vacuum field equations. The realisation that is dynamically favoured by the theory is determined by applying a stability criterion, that is, the criterion of least action.

In order to apply the criterion of least action it is first necessary to determine the solutions of the vacuum field equations. This is a very difficult problem because it requires a knowledge of all of the gluon n-point functions. Furthermore, because of the unsurprising complexity of the Bose field effective action, it is in practice necessary to determine an approximate form of this action to employ in calculations such as the evaluation of the difference between the action calculated at its various extremals. We are then obstructed by the difficult problem of finding a gauge invariant approximation procedure which may be employed in connection with our Bose field effective action. As a consequence of this we have not been able to determine which of the two alternatives discussed herein is dynamically favoured in the theory.

As we have shown, the bosonisation of QCD that we have obtained is ideally suited to the study of the realisation of chiral symmetry in QCD. It is also particularly well suited to the study of meson dynamics. In this context we remark finally that elsewhere (Praschifka *et al.* 1986*a*, 1986*b*) we have applied this formalism to the decay $\rho \longrightarrow \pi\pi$, as one example of a non-anomalous meson coupling, and also to the decay $\omega \longrightarrow 3\pi$, which is an example of an anomalous meson coupling.

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