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Why Baryons are not Skyrmions

R. T. Cahill, C. D. Roberts and J. Praschifka

School of Physical Sciences, Flinders University of South Australia,
Bedford Park, S.A. 5042, Australia.

Abstract

We present a critical analysis of the Skyrmion model for baryons. Using model quantum field theories we determine the origin of meson chiral effective actions and show that Skyrmions, i.e. chiral topological solitons of these effective actions, cannot be related to baryons and that indeed Skyrmions in these models are unstable.

1. Introduction

In the study of the quark theory of hadrons much use is made of models which, it is claimed, are based on properties of quantum chromodynamics (QCD), the standard theory for hadrons. The chiral soliton model (CSM) (Witten 1983; Adkins *et al.* 1983) for baryons is one example which grew from the work of Skyrme (1961, 1962). In the CSM, static topological soliton solutions (i.e. Skyrmions) of the Euler–Lagrange (EL) equations for the meson chiral effective action are found, and those with unit winding number interpreted as baryons. The chiral effective action (Gasiorowicz and Geffen 1969) for mesons incorporates the effects of dynamically broken chiral symmetry and is assumed to follow from QCD. Here we critically examine this model for baryons. Our approach is to consider two model field theories for $N_C \otimes N_F$ spinor fields (quarks) with an interaction, shown in (1), describing spin-1 boson exchange. The two models differ in the number of colours; one has $N_C = 3$ and the other has $N_C = 1$, and both have N_F flavours. We comment later on the relevance of the $N_C = 3$ model to QCD.

Our criticism of the Skyrmion model for baryons is based on the following properties of these two models. First we show that both models have colour-singlet $\bar{q}q$ meson bound states and we determine the effective action for these mesons. An important point here is that spin-1 boson exchange between q and $\bar{q}$ is an attractive force for any number of colours. Further, it is shown that by a suitable choice of the coupling constant in the $N_C = 1$ model both models lead to the same effective action for colour-singlet mesons (up to an unimportant overall factor of N_C), and that this action is the same as that used in CSM calculations. Hence the two models are identical as far as colour-singlet mesons are concerned. Thus, if the EL equations of the meson effective action were to have solutions corresponding to Skyrmions, then these topological solitons must arise in both models. We will in fact argue that there

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are no such soliton solutions. The second thrust of our criticism involves qqq bound states, and here the two models greatly differ. In the $N_C = 1$ model, spin-1 boson exchange between q and q is well known to be repulsive. This may be established, for example, by considering the Bethe–Salpeter (BS) equation for bound states. Hence in the $N_C = 1$ model there can be no qqq bound states, i.e. no baryon-like states, since the quarks would be mutually repelled. However, in the $N_C = 3$ model the situation is very different. If we consider colour-singlet qqq states then it is well known that any two quarks are necessarily in a $\overline{3}$ colour state and in this colour state the exchange of spin-1 colour-octet bosons is attractive. This is also easily established by considering the appropriate BS equation. Hence the $N_C = 3$ model may have colour-singlet bound qqq states as the quarks are mutually attracted. These qqq bound states correspond to the baryons. Thus the colour states of the quarks play a critical role in the qqq sector but not in the $\bar{q}q$ sector.

Hence, because the two models have an identical colour-singlet meson sector but a completely different baryon sector, it is impossible for there to be any connection between any possible chiral solitons in these models and the existence of baryon-like states. The Skyrmion model for baryons is thus missing the key feature of QCD which is responsible for the binding of colour-singlet qqq states, namely the special dynamical role of the colour algebra in the formation of these baryon states. Further we will show that indeed there are no such soliton solutions to the meson EL equations, and thus the whole concept of baryons as Skyrmions is spurious.

2. Two Model Quantum Field Theories

The two models are defined by the action, in Euclidean metric,

$$S[\bar{q}, q] = \int d^4x\, \bar{q}(\partial\!\!\!/ + m)q + \int d^4x\, d^4y\, \bar{q}(x)\frac{\gamma^\mu \lambda^a}{2} q(x)\, D(x-y)\, \bar{q}(y)\frac{\gamma^\mu \lambda^a}{2} q(y)\,. \tag{1}$$

For $N_C = 3$, $\{\frac{1}{2}\lambda^a; a = 1\dots8\}$ are the generators of $\mathrm{SU}(N_C)$, while we set $\frac{1}{2}\lambda^a \to 2/\sqrt{3}$, with $a = 1$, for $N_C = 1$. The interaction is characterised by the same $D(x)$ in both models, and is taken to have a form that models gluon exchange. The action has a global $\mathrm{G} = \mathrm{U_L}(N_F) \otimes \mathrm{U_R}(N_F)$ chiral symmetry in the limit $m = 0$. The key difference between the models is that the $N_C = 3$ model has the nonabelian colour algebra of QCD, while the $N_C = 1$ model has no colour algebra. A conventional analysis of these two models using the BS bound state equations shows that for $N_C = 3$ there can be baryon-like states, as discussed above, but that for $N_C = 1$ there are no qqq bound states because in this case the boson exchange is repulsive.

We now determine the effective action for the meson sector of both models. The spectrum of the quantum field theories (QFTs) may be formally defined by

$$\mathrm{Tr}\exp(-HT) = \int D\bar{q}\, Dq \exp(-S[\bar{q}, q])\,, \tag{2}$$

where the usual prescriptions for Grassmann integration apply and T is the Euclidean time interval. We now express (2) in terms of integrations over a set of local Bose fields. The effective action that arises for these fields will be shown to contain the usual meson chiral action. We anticommute the quark fields to obtain

$$\mathrm{Tr}\exp(-HT) = \int D\bar{q}\,Dq\,\exp\biggl(-\int \bar{q}(\not{\partial}+m)q + \int \bar{q}(x)\frac{M^\theta}{2}q(y)\,D(x-y)\bar{q}(y)\frac{M^\theta}{2}q(x)\biggr), \quad (3)$$

in which the following Fierz rearrangements were used:

$$\gamma^\mu_{rs}\gamma^\mu_{tu} = K^a_{ru}K^a_{ts}; \qquad \{K^a\} = \left\{\mathbf{1},\ \mathrm{i}\gamma_5,\ \frac{\mathrm{i}}{\sqrt{2}}\gamma^\mu,\ \frac{\mathrm{i}}{\sqrt{2}}\gamma^\mu\gamma_5\right\},$$

$$\lambda^a_{\alpha\beta}\lambda^a_{\gamma\delta} = C^b_{\alpha\delta}C^b_{\gamma\beta}; \qquad \{C^b\} = \begin{cases}\left\{\frac{4}{3}\mathbf{1},\ \frac{\mathrm{i}}{\sqrt{3}}\lambda^a;\ a=1,\ldots,8\right\}, & \text{for } N_C = 3\\ 4/\sqrt{3}, & \text{for } N_C = 1,\end{cases}$$

$$\delta_{ij}\delta_{kl} = F^c_{il}F^c_{kj}; \qquad \{F^c\} = \left\{\frac{1}{\sqrt{N_F}}\mathbf{1},\ \sqrt{2}T^1,\ldots,\sqrt{2}T^{N_F^2-1}\right\},$$

where $\{T^a\}$ are the generators of $\mathrm{SU}(N_F)$ and $M^\theta = K^a \otimes C^b \otimes F^c$. We now introduce an infinite set of real local Bose fields $\{\phi^{\theta i}(X),\ X \in R^4\}$ and a complete set $\{\Gamma^{\theta i}(Y);\ \Gamma^{\theta i}(Y)^* = \Gamma^{\theta i}(-Y),\ Y \in R^4\}$ (in the limit $T \to \infty$) for each value of θ so that any 'hermitian' bilocal Bose field on $R^4 \otimes R^4$ has the expansion

$$B^\theta(x,y) = \sum_i \phi^{\theta i}\left(\frac{x+y}{2}\right)\Gamma^{\theta i}(x-y). \quad (4)$$

Now we multiply (3) by the constant

$$\int D\phi^{\theta i}\exp\left(-\tfrac{1}{2}\int \frac{B^\theta(x,y)\,B^\theta(y,x)}{D(x-y)}\right),$$

change orders of integration, and finally change variables $\phi^{\theta i}(X) \to \phi^{\theta i}(X)'$, such that $B^\theta(x,y) \to B^\theta(x,y) + D(x-y)\,\bar{q}(y)\,\frac{1}{2}M^\theta\,q(x)$. It may be shown that the Jacobian for this transformation is $J = 1$. The choice of the multiplicative constant ensures that the terms quartic in the Grassmann variables disappear. Now the Grassmann integration may be done yielding the result, up to an (infinite) multiplicative constant,

$$\mathrm{Tr}\exp(-HT) = \int D\phi\,\exp(-S[\phi]), \quad (5)$$

where the Bose local-field effective action is

$$S[\phi] = -\mathrm{Tr}\,\mathrm{Ln}\left((\not{\partial}+m)\delta^{(4)}(x-y) + \frac{M^\theta}{2}B^\theta(x,y)\right) + \tfrac{1}{2}\int \frac{B^\theta(x,y)\,B^\theta(y,x)}{D(x-y)}, \quad (6)$$

with $B^{\theta}(x, y)$ defined by (4). The representation (5), in terms of local Bose fields, of (2) which was in terms of the fundamental fermion fields, illustrates a technique for bosonising fermion field theories. At this stage the effective action is somewhat formal as it involves the functional Tr Ln. To properly complete the bosonisation it is necessary to obtain an explicit form for $S[\phi]$. To do this we expand $S[\phi]$ about its minimum in powers of $\phi^{\theta i}$ and $\partial_{\mu}\phi^{\theta i}$, corresponding to a long-wavelength expansion. In this way we obtain the low energy meson effective action of the spinor theory.

The minimum is determined by

$$\delta S[\phi^{\theta i}]/\delta\phi^{\theta i} = 0\,, \tag{7}$$

which gives

$$B^{\theta}(x, y) = D(x-y)\,\mathrm{Tr}\left(G(x, y)\,\frac{M^{\theta}}{2}\right), \tag{8}$$

where $G^{-1}(x, y) = (\not{\partial} + m)\delta^{(4)}(x-y) + \frac{1}{2}M^{\theta}B^{\theta}(x, y)$. Then (7) becomes, after using inverse Fierz transformations, and assuming a colour-singlet minimum,

$$(\not{\partial} + m)\,G(x, z) - \tfrac{4}{3}\int \mathrm{d}^4 y\; D(x-y)\gamma_{\mu}\,G(x, y)\,\gamma_{\mu}\,G(y, z) = \delta^{(4)}(x-z)\,, \tag{9}$$

where we have used $\frac{1}{2}\lambda^{a}\frac{1}{2}\lambda^{a} = \frac{4}{3}$. Hence both models have the same EL equations. It is important to note that (9) is the exact EL equation of the Bose effective action.

Assuming translation invariant solutions of (9) we define

$$\Sigma(p) = \int \mathrm{d}^4 x\,\frac{M^{\theta}}{2}\,B^{\theta}(x)\exp(\mathrm{i}\,p\,.\,x)\,,$$

and (9) becomes

$$\Sigma(p) = m + \tfrac{4}{3}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\,D(p-q)\gamma_{\mu}\,\frac{1}{\mathrm{i}\,\not{q} + \Sigma(q)}\,\gamma_{\mu}\,. \tag{10}$$

The solution of this equation has the form, in the chiral limit $m = 0$,

$$\Sigma(q) = \mathrm{i}\{A(q^2) - 1\}\not{q} + V B(q^2)\,,$$

where V is the matrix $V = \exp(\mathrm{i}\sqrt{2}\pi^{a}F^{a}\gamma_5)$, with $\{\pi^{a}\}$ arbitrary real constants, and the scalar functions $A(q^2)$ and $B(q^2)$ satisfying the usual coupled integral equations associated with (10).

We have shown (Cahill and Roberts 1985; Roberts and Cahill 1987; Roberts *et al.* 1987; Praschifka *et al.* 1987) by numerical calculations that a choice for $D(x)$ that models the infrared slavery and asymptotic freedom properties of the gluon propagator leads to $B(q^2) \neq 0$ and that this solution of (10) gives an absolute minimum for S. The matrix V indicates that the manifold of minima of $S[\phi]$ is the coset space $\mathrm{G/H} = \mathrm{U_A}(N_F)$, where $\mathrm{H} = \mathrm{U_V}(N_F) \subset \mathrm{G}$. Hence the chiral symmetry becomes a hidden symmetry. To properly account for this in expanding $S[\phi]$ it is convenient to make V a matrix valued field defined by $\{\pi^{a}(x)\}$ now space–time dependent, and with the $\phi^{\theta i}$ redefined implicitly by writing (4) in the form

$$\sum_{\theta} \frac{M^{\theta}}{2} B^{\theta}(x, y) = V\left(\frac{x+y}{2}\right) B(x-y) + \sum_{\theta i} \phi^{\theta i}\left(\frac{x+y}{2}\right) \Gamma^{\theta i}(x-y) \frac{M^{\theta}}{2}. \quad (11)$$

Here $B(x)$ is the Fourier transform of $B(q)$ and is one member of the pseudoscalar set $\{\Gamma\}$ which is now implicitly excluded in the sum in (11). The bosonisation is now in terms of the local fields $\{\pi^a(x)\}$ and $\{\phi^{\theta i}(x)\}$, and the functional measure in (5) now includes a Haar measure for $U(x)$ where $U(x) = \exp\{i\sqrt{2}\pi^a(x)F^a\}$ and $V(x) = P_L\, U(x)^\dagger + P_R\, U(x)$, where $P_L = \frac{1}{2}(1-\gamma_5)$ and $P_R = \frac{1}{2}(1+\gamma_5)$. The degenerate minimum of S is now: $U(x) =$ constant field, $\phi^{\theta i} = 0$.

If $m \neq 0$ then the solution of (9) is, for small m, $i\{A_m(q^2)-1\}\not{q}+B_m(q^2)$; $B_m \approx m+B$ and $A_m \approx A$, i.e. $U = 1$. In this case (11) is still useful.

The explicit form of $S[U,\phi]$ is now obtained by expanding the Tr Ln in (5). Here we consider only the $U(x)$ dependence, which describes the Nambu–Goldstone (NG) bosons. The $\{\phi\}$ describe the massive boson states, such as $\rho, \omega, a_1, \ldots$. The expansion gives

$$\begin{aligned} S[U,\phi] = & \left\{\int d^4x \left(\frac{f_\pi^2}{4} \mathrm{Tr}(\partial_\mu U \partial_\mu U^\dagger) + \kappa_1 N_C \mathrm{Tr}(\partial^2 U \partial^2 U^\dagger)\right.\right. \\ & \left.\left. + \kappa_2 N_C \mathrm{Tr}([\partial_\mu U \partial_\mu U^\dagger]^2) + \kappa_3 N_C \mathrm{Tr}(\partial_\mu U \partial_\nu U^\dagger \partial_\mu U \partial_\nu U^\dagger)\right) + \ldots\right\}_R \quad (12) \\ & + i\left\{\frac{\lambda\sqrt{2}N_C}{60\pi^2} \epsilon_{\mu\nu\rho\sigma} \int d^4x\, \mathrm{Tr}(\pi . F\partial_\mu \pi . F\partial_\nu \pi . F\partial_\rho \pi . F\partial_\sigma \pi . F) + \ldots\right\}_I , \end{aligned}$$

where $\{\ldots\}_R$ and $\{\ldots\}_I$ indicate the real and imaginary parts of S, and

$$f_\pi^2 = 2N_C \int \frac{d^4q}{(2\pi)^4} \frac{B^2}{(sA^2+B^2)^2} \left\{3A^2 + 2B\frac{dB}{ds} + 3s\left(\frac{dB}{ds}\right)^2 - \frac{s(A^2+2B\, dB/ds)^2}{sA^2+B^2} + sB\frac{d^2B}{ds^2}\right\} \ldots,$$

where dA/ds terms are not shown, $s = q^2$, and where explicit expressions for $\{\kappa_i\}$ are given in Roberts *et al.* (1987).

In the $N_C = 3$ model there are also colour-octet fields in (12) (not shown) which are a manifestation of the nonabelian structure of this model, and which are coupled to the colour-singlet fields. In the CSM these colour-octet fields are neglected.

The imaginary part of $S[U,\phi]$ in (12), which includes all the chiral anomalies, is evaluated for small π only. The coefficient $\lambda[A, B]$ may be evaluated exactly in these model QFTs as

$$\lambda = 12 \int_0^\infty s\, ds \frac{A^4B^6 + 2sA^3B^6 dA/ds - 2sA^4B^5 dB/ds}{(sA^2+B^2)^5} \qquad (s = q^2). \quad (13)$$

Following Witten (1983) $\lambda[A, B]$ must be an integer to avoid $\exp(-S[U])$ in (5) becoming multivalued for topologically non-trivial configurations of $U(x)$. The λ in (13) appears to depend on A and B and thus on the choice of $D(x)$. But changing variables to $c(s) = B^2(s)/sA(s)^2$, we obtain

$$\lambda[A, B] = 12 \int_0^\infty \mathrm{d}c \, \frac{c^2}{(1+c)^5} = 1 .$$

The exact bosonisation of the model QFTs has led to the effective action $S[U, \phi]$. The ϕ parts of this action and their couplings to $U(x)$ may be explicitly calculated by expansion. The usual chiral effective action is included in this expansion. If only the low mass states are retained then an optimal choice for the corresponding $\Gamma^{\theta i}(x)$ needs to be made. One choice is to determine these $\Gamma^{\theta i}$ by minimising the mass functionals $m_\phi[\Gamma]$ which arise in the expansion. We have shown (Cahill *et al.* 1987*a*) that this is equivalent to solving the BS bound state equations. An important property of the bosonisation is that there exists a class of $D(x)$ such that the vacuum has hidden chiral symmetry and the consequent masses and coupling constants arising in the effective action are given by divergence free integrals involving $A(q)$, $B(q)$ and $\{\Gamma(q)\}$. The $\Gamma(q)$ play the role of meson form factors, while $B(q)$ is automatically the NG boson form factor and, as expected, $m_{\mathrm{NG}}[B] = 0$.

In the chiral limit $m = 0$ the NG fields $\{\pi\}$ are massless, but if we consider $N_F = 3$ and a small diagonal quark mass matrix with elements m_{u}, m_{d} and m_{s}, with $m_{\mathrm{u}} = m_{\mathrm{d}}$, then the above meson effective action acquires mass terms for the NG pseudoscalars. For the octet $\{\pi\}$ we identify the members as the three pions, four kaons and one eta. Their masses are $m_\pi^2 = \mu m_{\mathrm{u}}$, $m_{\mathrm{K}}^2 = \mu(m_{\mathrm{u}} + m_{\mathrm{s}})/2$ and $m_\eta^2 = \mu(m_{\mathrm{u}} + 2m_{\mathrm{s}})/2$, where

$$\mu = \frac{\langle \bar{q}q \rangle}{f_\pi^2}, \qquad \langle \bar{q}q \rangle = 8N_C \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \, \frac{B}{A^2 q^2 + B^2},$$

with f_π^2 given above. Specifying $D(x)$ and the m allows these masses to be calculated. It is easy to check that these masses satisfy the Gell-Mann–Okubo mass formula $4m_{\mathrm{K}}^2 = 3m_\eta^2 + m_\pi^2$. The good agreement with experimental data for these and other results from the $N_C = 3$ model suggests that conclusions of the model analysis here are relevant to QCD.

We have shown that, in both models, the low energy colour-singlet meson sector is described by the familiar chiral effective action. If we were to now invoke the usual CSM argument for topological solitons we would consider finite energy static configurations for which $U(x) \to \mathbf{1}$, as $|x| \to \infty$. This $U(x)$ may be considered as a mapping from S^3 (compactified R^3) into G/H. The homotopy group in this case is

$$\pi_3(\mathrm{G/H}) = \pi_3(\mathrm{SU}(N_F) \otimes \mathrm{U}(1)) = \pi_3(\mathrm{SU}(N_F)) = \begin{cases} Z, & N_F \geqslant 2 \\ 0, & \text{otherwise}. \end{cases}$$

Thus, for $N_F \geqslant 2$ there exists the possibility of topological soliton solutions of the EL equations, $\delta S[U]/\delta U(x) = 0$, which may be characterised by a conserved integer charge Z. Of course this homotopy group analysis in no way guarantees that any such solitons actually exist—it only raises a possibility. We will now show that no such solitons exist, i.e. the Skyrmions fail the stability test.

We consider the EL equations for the meson effective action,

$$\frac{\delta S[U,\phi]}{\delta U(x)} = 0, \qquad \frac{\delta S[U,\phi]}{\delta\phi(x)} = 0, \qquad U(x) \in \mathrm{G/H}. \tag{14}$$

In the extensive Skyrmion literature (see Zahed and Brown 1986 for a recent review) many truncated versions of these equations are used. The truncation always amounts to keeping only some of the long-wavelength parts. Even then the terms in $S[U,\phi]$ are carefully selected as some of the terms shown in (12) are known to de-stabilise the soliton. Soliton solutions of these equations are then found, usually by using the hedgehog ansatz for $U(x)$. The fundamental question is whether or not we get solitons when all the terms are retained. This might appear to be an impossible question to answer. However, we have an explicit expression for the equations in (14), because (14) is nothing more than (7) (we have changed variables since equation 7), and for colour-singlet states (7) has the form in (9). Thus the equations of motion which are supposed to have soliton solutions are, without the various drastic truncations that are usually employed, nothing more than the Dyson–Schwinger equation (9), which is a matrix equation in spin and flavour space. Of course no soliton solutions of this equation are known.

3. Conclusions

Hence we reach the conclusion, from several lines of argument, that the Skyrmion model for baryons is a spurious model. First we showed, using our two models, that the colour-singlet meson effective action 'forgets' whether or not the original quark action had a nonabelian colour algebra. This we did by constructing the meson effective action for the two models and demonstrating that they were identical (up to an overall factor of N_C), and also identical to that used in phenomenology. In contrast we showed that the nonabelian colour algebra is essential for baryon states, with the $N_C = 1$ model not capable of producing baryon states. Hence there can be no connection between the meson effective action and baryon states. Finally, we showed that Skyrmions are spurious solutions whose stability and thus existence depends critically on which terms are retained in the meson effective action and which are discarded. Keeping all the terms was shown to lead to the conclusion that Skyrmions, if they exist, would have to be soliton solutions to a Dyson–Schwinger equation, which itself has also 'forgotten' whether or not the quarks carried a colour index, and no such solutions are known.

A proper modelling of baryons clearly demands a careful treatment of the colour algebra in such a way as to retain a unique feature of QCD, namely that in a colour-singlet baryon any two quarks are in a $\bar{3}$ colour state and that for these states colour-octet gluon exchange leads to mutual attraction between the quarks. Progress towards such a model, one which is covariant and preserves the consequences of hidden chiral symmetry, has been reported by Cahill *et al.* (1987*a*, 1987*b*).

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