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Effect of a Rotating Magnetic Field on the Tilting Instability of a Prolate Rotamak

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Abstract

The effect of a rotating magnetic field on the stability of a rotamak is investigated. Using a simple equilibrium model and the linearised equations of ideal magnetohydrodynamics, it is shown that a rotating field can be a contributing factor in the stabilisation of the tilting instability of a slightly prolate rotamak plasma.

1. Introduction

The possibility of using a rotating magnetic field (RMF) to drive steady currents in a plasma was originally demonstrated by Blevin and Thonemann (1962). In recent years this technique has been applied successfully in a series of rotamak experiments (Jones 1984; Durance *et al.* 1987). A rotamak essentially is a device in which a field reversed configuration (FRC) is maintained by means of an RMF. This RMF drives a steady toroidal (or azimuthal) current which, with the externally applied steady vertical field, confines the plasma. Most of these recent experiments have been concerned with spherical or near spherical configurations but some highly elongated plasmas have also been investigated (Jones and Knight 1985).

One feature common to all these experiments has been their remarkable reproducibility and the lack of any obvious signs of instability. This has led to the conjecture that, apart from driving the current, the RMF may also have a role in dynamically stabilising the plasma (Storer 1982; Jones 1984).

On the other hand conventional FRCs, which do not have imposed r.f. fields, also do not display the instabilities which are predicted by ideal MHD theory (Schwartzmeier *et al.* 1983). The reason for this has been attributed to kinetic stabilisation (Barnes *et al.* 1986). The effectiveness of kinetic stabilisation is associated with small values of the plasma parameter $\bar{s}$, which is a measure of the number of gyroradii within the separatrix (Slough *et al.* 1984). In rotamak experiments the values of $\bar{s}$ have always been small so that the apparent stability may well have been due to kinetic stabilisation. However, to enhance the thermal properties of FRCs (and also of rotamak plasmas), it is necessary to operate with much higher values of $\bar{s}$ for which kinetic stabilisation is supposed to lose its effectiveness. Therefore, it is of some interest to investigate the effect of the RMF on the stability of a rotamak

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plasma in an operating regime where $\bar{s}$ is large. In this paper we examine, in terms of a very simple model, the effect of an RMF on the internal tilting mode in a rotamak. It is well known (Clemente and Milovich 1981) that a spherical FRC is marginally stable and an oblate FRC stable to tilting. Therefore, we will confine the analysis to the case of a prolate rotamak. The linearised ideal MHD equations for small perturbations about an equilibrium are derived in Section 2. Using a specific perturbation to the equilibrium the equations for the perturbed fields are solved in Section 3. In Section 4 the energy principle is used to examine the stability of the configuration. A short summary and discussion in Section 5 concludes the paper.

2. Linearised Stability Equations

In the model of a rotamak adopted here we consider a plasma of low resistivity ($\eta \to 0$) surrounded by a thin, rigid, non-conducting shell in the shape of a prolate spheroid whose surface in cylindrical (r,ϕ,z) coordinates is given by $r^2/a^2 + z^2/b^2 = 1$. A transverse rotating magnetic field

$$\tilde{\boldsymbol{B}}_\omega = [B_\omega \cos(\phi - \omega t),\ -B_\omega \sin(\phi - \omega t),\ 0] \tag{1}$$

is applied which induces in the plasma a steady azimuthal current

$$\boldsymbol{J}_0 = (0,\ -ne\omega r,\ 0)\,. \tag{2}$$

Here B_ω is a constant and the electron number density n is assumed to be uniform. An externally applied steady magnetic field ensures that the separatrix of the steady field $\boldsymbol{B}_0$ lies on the plasma boundary, so that $\boldsymbol{B}_0$ corresponds to a Solov'ev (1976) equilibrium

$$\boldsymbol{B}_0 = B_0[rz/b^2,\ 0,\ 1 - 2r^2/a^2 - z^2/b^2]\,, \tag{3}$$

where

$$B_0 = \frac{a^2 b^2}{a^2 + 4b^2}\,\mu_0\, ne\omega\,. \tag{4}$$

The equations describing this system are those of Maxwell, the equation of motion

$$\rho \frac{\mathrm{d}\hat{\boldsymbol{V}}}{\mathrm{d}t} + \nabla \hat{\boldsymbol{P}} = \hat{\boldsymbol{J}} \times \hat{\boldsymbol{B}} \tag{5}$$

with $\rho = nm_{\mathrm{i}}$, and Ohm's law

$$\hat{\boldsymbol{E}} + \hat{\boldsymbol{V}} \times \hat{\boldsymbol{B}} = \frac{1}{ne}\hat{\boldsymbol{J}} \times \hat{\boldsymbol{B}}\,, \tag{6}$$

where $\hat{\boldsymbol{V}}$ is the fluid velocity, $\hat{\boldsymbol{P}}$ the pressure, and $\hat{\boldsymbol{J}}$ and $\hat{\boldsymbol{B}}$ are the current density and magnetic field within the plasma. We shall also assume the fluid to be incompressible so that $\nabla \cdot \hat{\boldsymbol{V}} = 0$.

A quasi-static axisymmetric equilibrium can then be defined (Bertram 1989) as being a configuration in which each physical variable $\hat{Q}_0$ is of the form

$$\hat{Q}_0(r,\phi,z,t) = Q_0(r,z) + \tilde{Q}_0(r,\phi-\omega t,z), \tag{7}$$

and in which the average fluid velocity $\boldsymbol{V}_0$ is zero. Given such an equilibrium we can then consider the effects of small perturbations $\hat{Q}_1$ from their equilibrium values by assuming that the time dependence of $\hat{Q}_1$ is characterised by two different scales, one of which is associated with the growth rate (or frequency) λ of the perturbation, and the other with the frequency ω of the applied RMF. The value of ω is chosen to satisfy $\omega \gg \lambda$. The average of $\hat{Q}_1$ over a period of rotation, denoted by $\langle Q_1 \rangle = Q_1$, is a slowly varying function of time which suggests that we separate $\hat{Q}_1$ into slowly and rapidly varying components as

$$\hat{Q}_1(r,\phi,z,t) = Q_1(r,\phi,z,t) + \tilde{Q}_1(r,\phi,z,t), \tag{8}$$

with

$$\tilde{Q}_1(r,\phi,z,t) = \mathrm{Re}[q(r,\phi,z,t)\mathrm{e}^{\mathrm{i}\omega t}]. \tag{9}$$

Expansion of equations (5) and (6) to first order in the perturbed variables and separation of the rapidly and slowly varying time components yields the linearised stability equations

$$\rho\frac{\partial \boldsymbol{V}_1}{\partial t} + \rho\langle(\tilde{\boldsymbol{V}}_1 \cdot \nabla)\tilde{\boldsymbol{V}}_0 + (\tilde{\boldsymbol{V}}_0 \cdot \nabla)\tilde{\boldsymbol{V}}_1\rangle + \nabla P_1$$
$$= \boldsymbol{J}_0 \times \boldsymbol{B}_1 + \boldsymbol{J}_1 \times \boldsymbol{B}_0 + \langle\tilde{\boldsymbol{J}}_0 \times \tilde{\boldsymbol{B}}_1\rangle + \langle\tilde{\boldsymbol{J}}_1 \times \tilde{\boldsymbol{B}}_0\rangle, \tag{10}$$

$$\frac{\partial \boldsymbol{B}_1}{\partial t} = \nabla \times (\boldsymbol{V}_1 \times \boldsymbol{B}_0 + \langle\tilde{\boldsymbol{V}}_0 \times \tilde{\boldsymbol{B}}_1\rangle + \langle\tilde{\boldsymbol{V}}_1 \times \tilde{\boldsymbol{B}}_0\rangle)$$
$$- \frac{1}{ne}\nabla \times (\boldsymbol{J}_0 \times \boldsymbol{B}_1 + \boldsymbol{J}_1 \times \boldsymbol{B}_0 + \langle\tilde{\boldsymbol{J}}_1 \times \tilde{\boldsymbol{B}}_0\rangle + \langle\tilde{\boldsymbol{J}}_0 \times \tilde{\boldsymbol{B}}_1\rangle), \tag{11}$$

$$\rho\frac{\partial \tilde{\boldsymbol{V}}_1}{\partial t} + \rho(\boldsymbol{V}_1 \cdot \nabla)\tilde{\boldsymbol{V}}_0 + \rho(\tilde{\boldsymbol{V}}_0 \cdot \nabla)\boldsymbol{V}_1 + \nabla\tilde{P}_1$$
$$= \boldsymbol{J}_0 \times \tilde{\boldsymbol{B}}_1 + \boldsymbol{J}_1 \times \tilde{\boldsymbol{B}}_0 + \tilde{\boldsymbol{J}}_0 \times \boldsymbol{B}_1 + \tilde{\boldsymbol{J}}_1 \times \boldsymbol{B}_0 + \{F_1\}, \tag{12}$$

$$\frac{\partial \tilde{\boldsymbol{B}}_1}{\partial t} = \nabla \times (\tilde{\boldsymbol{V}}_1 \times \boldsymbol{B}_0 + \boldsymbol{V}_1 \times \tilde{\boldsymbol{B}}_0 + \tilde{\boldsymbol{V}}_0 \times \boldsymbol{B}_1)$$
$$- \frac{1}{ne}\nabla \times (\boldsymbol{J}_0 \times \tilde{\boldsymbol{B}}_1 + \boldsymbol{J}_1 \times \tilde{\boldsymbol{B}}_0 + \tilde{\boldsymbol{J}}_0 \times \boldsymbol{B}_1 + \tilde{\boldsymbol{J}}_1 \times \boldsymbol{B}_0) + \nabla \times \{\boldsymbol{G}_1\}, \tag{13}$$

where $\{Q\}$ denotes the quantity $Q - \langle Q \rangle$ and

$$\boldsymbol{F}_1 = \tilde{\boldsymbol{J}}_0 \times \tilde{\boldsymbol{B}}_1 + \tilde{\boldsymbol{J}}_1 \times \tilde{\boldsymbol{B}}_0 - \rho(\tilde{\boldsymbol{V}}_1 \cdot \nabla)\tilde{\boldsymbol{V}}_0 - \rho(\tilde{\boldsymbol{V}}_0 \cdot \nabla)\tilde{\boldsymbol{V}}_1, \tag{14}$$

$$\boldsymbol{G}_1 = \tilde{\boldsymbol{V}}_1 \times \tilde{\boldsymbol{B}}_0 + \tilde{\boldsymbol{V}}_0 \times \tilde{\boldsymbol{B}}_1 - \frac{1}{ne}(\tilde{\boldsymbol{J}}_1 \times \tilde{\boldsymbol{B}}_0 + \tilde{\boldsymbol{J}}_0 \times \tilde{\boldsymbol{B}}_1). \tag{15}$$

For a given small displacement $\boldsymbol{\zeta}$ with $\boldsymbol{V}_1 = \delta\boldsymbol{\zeta}/\delta t$, it is possible in principle to solve equations (11)–(13) for the perturbed fields $\boldsymbol{B}_1$, $\tilde{\boldsymbol{B}}_1$ and $\tilde{\boldsymbol{V}}_1$ so that the force terms in the equation of motion (10) can be evaluated. To test a given equilibrium configuration for stability to a particular mode, we will use a simplified form of the energy principle (Freidberg 1982) based on the fact that if $\boldsymbol{\zeta}$ represents an eigenmode of the system, or a good approximation to the exact eigenmode, the stability of the system to that particular mode can be determined from the sign of the quantity

$$\delta W = -\tfrac{1}{2}\int \boldsymbol{\zeta}.\boldsymbol{F}\,\mathrm{d}V, \tag{16}$$

where $\boldsymbol{F}$ represents the collection of force terms in equation (10).

3. Determination of Perturbed Fields

To determine the effect of the RMF on the $m = 1$ tilting mode in a slightly prolate spheroidal plasma (m is the azimuthal mode number), we use a simple generalisation of the equilibrium derived in Bertram (1989). For the case $\omega \gg \omega_0$ and $\omega \gg \omega_{\mathrm{ci}}$, where $\omega_0 = eB_0/m_{\mathrm{i}}$ and $\omega_{\mathrm{ci}} = eB_\omega/m_{\mathrm{i}}$, the rotating field fully penetrates the plasma and the steady driven current corresponds to the electron fluid in synchronous rotation with the rotating field. The steady fields are therefore given by equations (2) and (3) and the oscillating equilibrium fields are $\tilde{\boldsymbol{B}}_0 = \tilde{\boldsymbol{B}}_\omega + \mathrm{O}(\omega_{\mathrm{ci}}/\omega)$, $\tilde{\boldsymbol{J}}_0/J_0 = \mathrm{O}(\omega_{\mathrm{ci}}/\omega)$, and

$$\tilde{\boldsymbol{V}}_0 = \tfrac{1}{2}\omega_{\mathrm{ci}}[z\sin(\phi-\omega t),\, z\cos(\phi-\omega t),\, -(rb^2/a^2)\sin(\phi-\omega t)]\,. \tag{17}$$

This last equation is not exact but is correct to order $\delta = (b^2 - a^2)/a^2$. Since $\tilde{\boldsymbol{V}}_0$ enters the stability equations only through products with perturbed quantities which are of order ϵ say, the error introduced by (17) is of order $\epsilon\delta$ and is negligible provided $\delta \ll 1$.

Consider a displacement of the form

$$\boldsymbol{\zeta}_1 = \epsilon[(z/b^2)\sin\phi,\, (z/b^2)\cos\phi,\, -(r/a^2)\sin\phi]\,. \tag{18}$$

When $a = b$ and $B_\omega = 0$ this is an exact eigenfunction of equations (10) and (11) corresponding to a zero eigenvalue. Therefore, provided δ and B_ω are small, equation (18) represents a good approximation to the exact eigenfunction and can be used in (16) to test for stability. This same analysis has been carried out for the case $B_\omega = 0$ and $a \neq b$ by Clemente and Milovich (1981) to investigate the instability of a prolate ($b^2 > a^2$) FRC.

Equations (10)–(15) can be simplified considerably if we use the properties of the equilibrium fields. Ignoring terms of order $\omega_{\mathrm{ci}}/\omega$, the terms involving $\tilde{\boldsymbol{J}}_0$ can be dropped from (10)–(15). We shall also ignore the terms involving the operators $\tilde{\boldsymbol{V}}_1.\nabla$, $\tilde{\boldsymbol{V}}_0.\nabla$ and $\boldsymbol{V}_1.\nabla$. The justification for this is not obvious at this stage but it will be shown later that for the solutions we obtain these terms are indeed negligible.

As a result of these simplifications (10)–(13) can be written as

$$\rho\frac{\partial \boldsymbol{V}_1}{\partial t} = \boldsymbol{J}_0 \times \boldsymbol{B}_1 + \boldsymbol{J}_1 \times \boldsymbol{B}_0 + \langle \tilde{\boldsymbol{J}}_1 \times \tilde{\boldsymbol{B}}_\omega \rangle - \nabla P_1 , \tag{19}$$

$$\rho\frac{\partial \tilde{\boldsymbol{V}}_1}{\partial t} = \boldsymbol{J}_0 \times \tilde{\boldsymbol{B}}_1 + \boldsymbol{J}_1 \times \tilde{\boldsymbol{B}}_\omega + \tilde{\boldsymbol{J}}_1 \times \boldsymbol{B}_0 + \{\tilde{\boldsymbol{J}}_1 \times \tilde{\boldsymbol{B}}_\omega\} - \nabla \tilde{P}_1 , \tag{20}$$

$$\frac{\partial \boldsymbol{B}_1}{\partial t} = \nabla \times \left(\boldsymbol{V}_1 \times \boldsymbol{B}_0 + \langle \tilde{\boldsymbol{V}}_0 \times \tilde{\boldsymbol{B}}_1 \rangle + \langle \tilde{\boldsymbol{V}}_1 \times \tilde{\boldsymbol{B}}_\omega \rangle - \frac{\rho}{ne}\frac{\partial \boldsymbol{V}_1}{\partial t} \right), \tag{21}$$

$$\frac{\partial \tilde{\boldsymbol{B}}_1}{\partial t} = \nabla \times \left(\tilde{\boldsymbol{V}}_1 \times \boldsymbol{B}_0 + \boldsymbol{V}_1 \times \tilde{\boldsymbol{B}}_\omega + \tilde{\boldsymbol{V}}_0 \times \boldsymbol{B}_1 + \{\tilde{\boldsymbol{V}}_1 \times \tilde{\boldsymbol{B}}_\omega\} + \{\tilde{\boldsymbol{V}}_0 \times \tilde{\boldsymbol{B}}_1\} - \frac{\rho}{ne}\frac{\partial \tilde{\boldsymbol{V}}_1}{\partial t} \right). \tag{22}$$

Furthermore, in equation (21) $(\rho/ne)\partial \boldsymbol{V}_1/\partial t$ is of order λ/ω_0 smaller than the term $\boldsymbol{V}_1 \times \boldsymbol{B}_0$ and can be ignored. Similarly, the right-hand side of (22) is dominated by $\partial \tilde{\boldsymbol{V}}_1/\partial t$ which is of order ω/ω_0 larger than the other terms, so that (20)–(22) can be replaced by

$$\rho\nabla \times \frac{\partial \tilde{\boldsymbol{V}}_1}{\partial t} = \nabla \times (\boldsymbol{J}_0 \times \tilde{\boldsymbol{B}}_1 + \boldsymbol{J}_1 \times \tilde{\boldsymbol{B}}_\omega + \tilde{\boldsymbol{J}}_1 \times \boldsymbol{B}_0 + \{\tilde{\boldsymbol{J}}_1 \times \tilde{\boldsymbol{B}}_\omega\}), \tag{23}$$

$$\frac{\partial \boldsymbol{B}_1}{\partial t} = \nabla \times (\boldsymbol{V}_1 \times \boldsymbol{B}_0 + \langle \tilde{\boldsymbol{V}}_0 \times \tilde{\boldsymbol{B}}_1 \rangle + \langle \tilde{\boldsymbol{V}}_1 \times \tilde{\boldsymbol{B}}_\omega \rangle), \tag{24}$$

$$\tilde{\boldsymbol{B}}_1 = -\frac{\rho}{ne}\nabla \times \tilde{\boldsymbol{V}}_1 . \tag{25}$$

To solve these equations when $\boldsymbol{\zeta}_1$ is given by (18), we note that the term $\boldsymbol{J}_1 \times \tilde{\boldsymbol{B}}_\omega$ in (23) generally will give rise to both $m=0$ and 2 harmonics in $\tilde{\boldsymbol{V}}_1$. For the $m=0$ component let us try a rigid body oscillation which must be of the form

$$\tilde{\boldsymbol{V}}_1 = (0, \alpha r \sin\omega t, 0). \tag{26}$$

After evaluating $\boldsymbol{B}_1$ and $\tilde{\boldsymbol{B}}_1$ using equations (24) and (25) and substituting the results into (23) we find that (26) in fact represents an *exact* solution of the equations, provided the spatially constant α is given by

$$\alpha = \tfrac{5}{2}\frac{\omega_{\mathrm{ci}}\,\epsilon}{a^2+4b^2}. \tag{27}$$

Remarkably, there are no contributions from the higher harmonics. The full solution of (23)–(25) for the perturbed fields can then be expressed as

$$\boldsymbol{B}_1 = -(B_0\,\epsilon/b^2)\left[\left(1-\frac{r^2}{a^2}-\frac{2z^2}{b^2}\right)\sin\phi,\ \left(1-\frac{2r^2}{a^2}-\frac{2z^2}{b^2}\right)\cos\phi,\ \frac{rz}{a^2}\sin\phi\right], \tag{28}$$

$$\tilde{\boldsymbol{B}}_1 = \left(0, 0, -\frac{5B_\omega\,\epsilon}{4b^2+a^2}\sin\omega t\right). \tag{29}$$

The currents corresponding to these fields are

$$J_1 = -\frac{B_0 \epsilon}{\mu_0 b^2}\left[\left(\frac{1}{a^2}+\frac{4}{b^2}\right)z\cos\phi,\ -\left(\frac{1}{a^2}+\frac{4}{b^2}\right)z\sin\phi,\ -5\frac{r}{a^2}\cos\phi\right], \tag{30}$$

$$\tilde{J}_1 = 0. \tag{31}$$

It is now relatively easy to justify the approximations that were made earlier to simplify the equations. As an example, using (17) and (26) to evaluate the terms involving $\tilde{V}_0$ omitted from (10), we find that

$$T_1 = \rho\langle(\tilde{V}_1 . V)\tilde{V}_0 + (\tilde{V}_0 . V)\tilde{V}_1\rangle \sim \rho\omega_{\rm ci}^2 \epsilon$$

which, using equation (4), can be written as

$$T_1 \sim \mu_0^{-1} B_0 B_\omega \epsilon(\omega_{\rm ci}/\omega).$$

On the other hand, evaluation of the term $T_2 = J_0 \times B_1$ yields $T_2 \sim \mu_0^{-1} B_0^2 \epsilon$ so that

$$\frac{T_1}{T_2} \sim \frac{B_\omega}{B_0}\frac{\omega_{\rm ci}}{\omega} \ll 1.$$

4. Stability Criterion

From the expressions for the perturbed fields and currents it is now possible to evaluate the force terms on the right-hand side of (19) and to calculate out the energy integral (16). The pressure does not contribute since

$$\int \boldsymbol{\zeta} . \nabla P \, dV = \int_S P\boldsymbol{\zeta} . \boldsymbol{n} \, dS = 0,$$

where S denotes the surface of the plasma. Since $\tilde{J}_1 = 0$, (16) becomes

$$\delta W_V = -\tfrac{1}{2}\int \boldsymbol{\zeta} . (J_1 \times B_0 + J_0 \times B_1)\, dV. \tag{32}$$

This integral, which now contains no contributions from the oscillating fields, has been evaluated by Clemente and Milovich (1981) as

$$\delta W_V = -\frac{4\pi}{15}\frac{B_0^2 \epsilon^2}{\mu_0}\frac{b^2 - a^2}{b^3}. \tag{33}$$

However, we must also take into account any surface contribution to δW. If $[B_1]$ is the change in the perturbed steady or oscillating fields across the boundary of the plasma there is a surface contribution of the form

$$\begin{aligned}\delta W_S &= -\frac{1}{2\mu_0}\int_S \boldsymbol{\zeta} . [(\boldsymbol{n}\times[B_1])\times B_0]\, dS \\ &= \frac{1}{2\mu_0}\int_S (B_0 . [B_1])\boldsymbol{\zeta} . \boldsymbol{n}\, dS - \frac{1}{2\mu_0}\int_S (\boldsymbol{n} . B_0)(\boldsymbol{\zeta} . [B_1])\, dS.\end{aligned} \tag{34}$$

For a fixed boundary the first term on the right-hand side of (34) is zero. The second term vanishes for the static equilibrium field if the separatrix coincides with the plasma boundary, but does not necessarily vanish for the oscillating field since in general $\boldsymbol{n}.\tilde{\boldsymbol{B}}_0$ is nonzero. Therefore, we must evaluate the term

$$\delta W_S = -\frac{1}{2\mu_0}\int_S \langle(\boldsymbol{n}.\tilde{\boldsymbol{B}}_\omega)(\boldsymbol{\zeta}.[\tilde{\boldsymbol{B}}_1])\rangle \, \mathrm{d}S. \tag{35}$$

The quantity $[\tilde{\boldsymbol{B}}_1]$ can be obtained by solving for the perturbed oscillating vacuum field $\tilde{\boldsymbol{B}}_1^{\mathrm{v}}$ subject to the boundary conditions $\boldsymbol{n}.\tilde{\boldsymbol{B}}_1 = \boldsymbol{n}.\tilde{\boldsymbol{B}}_1^{\mathrm{v}}$ on S and $\tilde{\boldsymbol{B}}_1^{\mathrm{v}} \to 0$ when $r^2+z^2 \to \infty$. In cylindrical coordinates this is a difficult problem which is more easily solved in a spheroidal coordinate system. The evaluation of (35), which is carried out in the Appendix, yields

$$\delta W_S = \frac{5\pi}{8}\,\frac{b}{a^2+4b^2}\,\frac{B_\omega^2\epsilon^2}{\mu_0} \approx \frac{\pi}{8\mu_0}\,\frac{B_\omega^2\epsilon^2}{b}. \tag{36}$$

From (33) and (36) it follows that the plasma is stable ($\delta W_V + \delta W_S > 0$) when

$$B_\omega^2 > \frac{32}{15}B_0^2\left(1-\frac{a^2}{b^2}\right). \tag{37}$$

That this stability criterion does not depend explicitly on the frequency ω of the rotating field is a consequence of the fact that in RMF current drive ω is related to $\boldsymbol{J}_0$ and $\boldsymbol{B}_0$ through (2) and (4).

In the experiment of Durance *et al.* (1987), a typical prolate rotamak configuration had the parameters $a/b = 0{\cdot}8$ and $B_0 = 30$ G. Equation (37) predicts that this configuration is stable when $B_\omega = 26$ G which is not very different from the value $B_\omega = 20$ G used in the experiment. However, it must be remembered that the value of $\bar{s}$ in this experiment is rather small and this may well be the major factor in the apparent stability of the configuration.

5. Conclusions

By examining the internal $m=1$ tilting mode in a simplified model of a rotamak we have shown that the rotating magnetic field can be a contributing factor to the stabilisation of a rotamak plasma. This supports a previous conjecture by Storer (1982). The analysis does not necessarily offer a full explanation for the lack of experimental evidence of instabilities in the rotamak. Other mechanisms may well be responsible for this (Barnes *et al.* 1986). To simplify the analysis it was necessary to adopt the incompressible fluid model and the frequency of the rotating field ω was assumed to be much greater than the ion cyclotron frequency ω_0. In the absence of an r.f. field the tilting instability is a current driven mode in which pressure effects are negligible so that, in the context of ideal MHD, the tilting instability can be described using the incompressible fluid model. In the case of the rotamak the inclusion of the RMF complicates the problem considerably. The interaction between the RMF and the steady plasma fields most likely results in pressure effects. However, the analysis for a compressible plasma does not appear to be tractable, particularly in view of the fact that there is no simple way to describe the equilibrium configuration (Bertram 1989). Nevertheless, even though the analysis was restricted to incompressible plasmas in near spherical equilibrium configurations, the results indicate that when the elongation is increased, the relative amplitude of the RMF, B_ω/B_0, must also be increased to maintain stability. This does seem to put a limit on the effectiveness of RMF stabilisation of more highly elongated plasmas.

The analysis can also be carried out for a spherical plasma where instead of $\omega \gg \omega_0$, the equilibrium fields (Bertram 1989) satisfy the more realistic condition $\omega_0 \gg \omega \gg \omega_{ci}$. This analysis, the details of which we have not presented, yields the result $\delta W = 0$. This is an obvious result since for this equilibrium both the steady and oscillating fields correspond to spherical Hill's vortices (Bertram 1989). At any instant the combined field is also a spherical Hill's vortex which is well known to be marginally stable to tilting.

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Appendix

To evaluate the surface integral

$$\delta W_S = -\tfrac{1}{2}\int_S \langle(\boldsymbol{n}\cdot\tilde{\boldsymbol{B}}_\omega)\boldsymbol{\zeta}\cdot[\tilde{\boldsymbol{B}}_1]\rangle\,\mathrm{d}S\,, \tag{A1}$$

we make use of prolate spheroidal coordinates (u,θ,ϕ) (Morse and Feshbach 1953) which are related to the cylindrical coordinates (r,ϕ,z) by

$$r = l(u^2-1)^{\frac{1}{2}}\sin\theta\,, \qquad z = lu\cos\theta\,, \tag{A2}$$

where $l^2 = b^2 - a^2$. The plasma boundary is at $u = u_B = b/l$. The spheroidal components of a vector $\boldsymbol{P}$ are related to the cylindrical components by the transformation

$$P_u = \Delta^{-\frac{1}{2}}(uP_r \sin\theta + (u^2-1)^{\frac{1}{2}} P_z \cos\theta),$$

$$P_\theta = \Delta^{-\frac{1}{2}}((u^2-1)^{\frac{1}{2}} P_r \cos\theta - uP_z \sin\theta), \tag{A3}$$

with $\Delta = u^2 - \cos^2\theta$.

The perturbed field $\tilde{\boldsymbol{B}}_1$ given by equation (31) can be expressed in the new coordinate system as

$$\tilde{\boldsymbol{B}}_1 = -B_z \Delta^{-\frac{1}{2}} \sin\omega t((u^2-1)^{\frac{1}{2}} \cos\theta, -u\sin\theta, 0), \tag{A4}$$

where $B_z = 5B_\omega \epsilon/(a^2 + 4b^2)$.

To calculate the vacuum field $\tilde{\boldsymbol{B}}^{\mathrm{V}}$ we write

$$\tilde{B}_u^{\mathrm{V}} = -(l^2\sqrt{\Delta}(u^2-1)^{\frac{1}{2}} \sin\theta)^{-1} \sin\omega t \frac{\partial\psi}{\partial\theta},$$

$$\tilde{B}_\theta^{\mathrm{V}} = (l^2\sqrt{\Delta}\sin\theta)^{-1} \sin\omega t \frac{\partial\psi}{\partial u}. \tag{A5}$$

This ensures that $\nabla . \tilde{\boldsymbol{B}}^{\mathrm{V}} = 0$. The condition $\nabla \times \tilde{\boldsymbol{B}}^{\mathrm{V}} = 0$ then leads to

$$(u^2-1)\frac{\partial^2\psi}{\partial u^2} + \sin\theta \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = 0, \tag{A6}$$

which can be solved by separation of variables to yield

$$\psi = (u^2-1)^{\frac{1}{2}} \sin^2\phi \sum_{n=0}^{\infty} [a_n P_{n+1}^1(u) + C_n Q_{n+1}^1(u)] P_n^{(1,1)}(\cos\theta). \tag{A7}$$

Here P_n^1 and Q_n^1 are the Legendre functions and $P_n^{(1,1)}$ are Jacobi polynominals (Abramowitz and Stegun 1965). The required vacuum field is obtained from (A5) and (A7) by selecting the $n = 0$ term which gives

$$\tilde{B}_u^{\mathrm{V}} = -\frac{2c_0}{l^2\sqrt{\Delta}} Q_1^1 \cos\theta \sin\omega t, \qquad \tilde{B}_\theta^{\mathrm{V}} = \frac{2c_0}{l^2\sqrt{\Delta}} Q_1^0 \sin\theta \sin\omega t. \tag{A8}$$

On the plasma boundary where $u_B^2 = b^2/(b^2 - a^2) \gg 1$ we can use the approximate forms $Q_1^0 = (3u^2)^{-1}$ and $Q_1^1 = -\frac{2}{3}(u^2-1)^{\frac{1}{2}}/u^3$ to write $\tilde{\boldsymbol{B}}^{\mathrm{V}}$ as

$$\tilde{\boldsymbol{B}}^{\mathrm{V}} = \frac{2c_0 \sin\omega t}{3l^2\sqrt{\Delta_B}}\left(2u_B^{-3}(u_B^2-1)^{\frac{1}{2}} \cos\theta, u_B^{-2}\sin\theta, 0\right). \tag{A9}$$

The boundary condition $\boldsymbol{n}.\tilde{\boldsymbol{B}}_1 = \boldsymbol{n}.\tilde{\boldsymbol{B}}^{\mathrm{V}}$ at $u = u_B$ determines the value of c_0 as

$$c_0 = -\tfrac{3}{4} B_z b^3/l, \tag{A10}$$

so that we obtain for $[\tilde{\boldsymbol{B}}_1] = \tilde{\boldsymbol{B}}^{\mathrm{V}} - \tilde{\boldsymbol{B}}_1$,

$$[\tilde{\boldsymbol{B}}_1] = [0, -\tfrac{3}{2}B_z(u_{\mathrm{B}}/\surd\Delta_{\mathrm{B}})\sin\theta\sin\omega t, 0]. \tag{A11}$$

The values of $\tilde{\boldsymbol{B}}_\omega$ and ζ at $u = u_{\mathrm{B}}$ can be expressed in terms of the spheroidal coordinates as

$$\tilde{\boldsymbol{B}}_\omega = B_\omega[(u_{\mathrm{B}}/\surd\Delta_{\mathrm{B}})\sin\theta\cos(\phi-\omega t), (au_{\mathrm{B}}/b\surd\Delta_{\mathrm{B}})\cos\theta\cos(\phi-\omega t), - \sin(\phi-\omega t)], \tag{A12}$$

$$\zeta = \frac{\epsilon}{b}[0, (l/a)\surd\Delta_{\mathrm{B}}\sin\phi, \cos\theta\cos\phi], \tag{A13}$$

which, when substituted into (A1) and using $\mathrm{d}S = l^2(u_{\mathrm{B}}^2-1)^{\frac{1}{2}}\surd\Delta_{\mathrm{B}}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ yields the result

$$\delta W_S = \frac{3b}{8}B_\omega B_z\epsilon\int_0^\pi \sin^3\theta\,\mathrm{d}\theta\int_0^{2\pi}\sin^2\phi\,\mathrm{d}\phi$$

$$= \frac{\pi}{2}bB_\omega B_z\epsilon. \tag{A14}$$

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