Aust. J. Phys., 1990, **43**, 311–17

Nonlinear Instability of Circularly Polarised Waves in a Magnetised Relativistic Plasma

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Abstract

An intensity dependent nonlinear dispersion relation of a circularly polarised wave in a magnetised relativistic plasma is derived using a special Lorentz transformation and then the stability criteria of the wave are investigated. From numerical estimations it is observed that electromagnetic waves, having powers much below that for the occurrence of nonlinear phenomena due to self-action effects, are unstable in a magnetised relativistic dense plasma. The effects of a strong magnetic field on the group velocity and cut-off frequency in a dense plasma are also discussed.

1. Introduction

Investigations on nonlinear propagation of electromagnetic (EM) waves are important in the different contexts of laser–plasma interactions and astrophysical plasmas (Akhiezer *et al.* 1975; Chakraborty *et al.* 1984, 1987; Shukla *et al.* 1986; Chen 1987; Hora 1981; Sodha *et al.* 1976; Tsytovich 1970; ter Haar and Tsystovich 1981; Sur *et al.* 1988). In particular, nonlinear effects on the propagation of a circularly polarised wave are found to be very interesting, showing some fascinating characteristics of the wave. In fact, the circularly polarised wave has many useful properties and may be generated in the laboratory. For theoretical investigation, the problems become simpler because exact solutions of the field variables are obtained for a circularly polarised wave. In spite of these advantages, the problems of the nonlinear interaction of circularly polarised waves in plasma has not yet been exhaustively studied in the presence of a static magnetic field. However, Max and Perkins (1972), Steiger and Woods (1972), Max (1973), Goldstein and Salu (1973), Lee and Lerche (1978, 1979, 1980), Stenflo (1976, 1980), Paul *et al.* (1989) and others have studied several aspects of the interaction of circularly polarised waves under various physical situations in plasmas.

The motivation for the present study is to determine the effect of a strong magnetic field on the instability of circularly polarised waves propagating in a homogeneous relativistic plasma. To solve the problem in a simpler way, the field equations in the laboratory frame are transformed into a moving frame where the field variables become time dependent only (Winkles and Eldridge 1972; Decoster 1978; Paul and Chakraborty 1983*a*, 1983*b*). Then, following the usual mathematical technique, the nonlinear dispersion relations

0004-9506/90/030311$03.00

of circularly polarised waves have been derived. Analysing the dispersion relations, the stability criteria for the intensity of the EM waves have been obtained in terms of the static magnetic field, plasma frequency and wave frequency. The effect of a magnetic field on the cut-off frequency of the wave in a nonlinear plasma has also been determined.

From numerical estimates it is seen that laser beams having (i) power $\approx 10^{15-19}$ $\mathrm{W\,m^{-2}}$ and frequency $\approx 10^{14}$ Hz passing through a plasma having electron density $\approx 10^{20}\ \mathrm{cm^{-3}}$, static magnetic field $\approx 10^8$ G ($1\ \mathrm{G} \equiv 10^{-4}$ T), or (ii) power $\approx 10^{7-11}\ \mathrm{W\,m^{-2}}$ and frequency $\approx 10^{12}$ Hz passing through a plasma having electron density $\approx 10^{12}\ \mathrm{cm^{-3}}$, static magnetic field $\approx 10^7$ G, are both unstable. The cut-off frequency of an EM wave passing through a magnetised plasma (plasma frequency $\approx 10^6$ Hz) is found to be $7\cdot 9\times 10^7$ Hz.

2. Formulation

We assume that the plasma is cold and homogeneous. The collisions and gravitational forces are negligibly small and a strong static magnetic field is present in the direction of wave propagation. The plasma is subjected to high power laser radiations whose intensity is below a threshold value ($\approx 10^{24}\ \mathrm{W\,m^{-2}}$) and so stimulated Raman scattering, Brillouin scattering, self focussing, self-steepening etc. are absent (Kaw and Dawson 1970). The velocity of electrons is of relativistic order and ions are assumed to be at rest in comparison with the motion of electrons at high frequency radiations.

With the above assumptions, the plasma equations in the laboratory inertial frame S can be written as follows:

$$\frac{\partial \boldsymbol{p}}{\partial t} + (\boldsymbol{v}.\nabla)\boldsymbol{p} = -e\boldsymbol{E} - \frac{e}{c}(\boldsymbol{v}\times\boldsymbol{H}), \tag{1}$$

$$\frac{\partial N}{\partial t} + \nabla.(N\boldsymbol{v}) = 0, \tag{2}$$

$$\nabla\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t}, \tag{3}$$

$$\nabla\times\boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} - \frac{4\pi e N\boldsymbol{v}}{c}, \tag{4}$$

$$\nabla.\boldsymbol{E} = 4\pi e(N_{\mathrm{i}} - N), \tag{5}$$

$$\nabla.\boldsymbol{H} = 0, \tag{6}$$

$$\boldsymbol{P} = m_0\,\boldsymbol{v}/(1 - v^2/c^2)^{1/2}, \tag{7}$$

where N and N_{i} are the number densities of electrons and ions, m_0 is the rest mass, e is the charge of the electron, v is the velocity of electrons, c is the velocity of light, E and H are the electric field and magnetic fields, and the other symbols have their usual meaning.

Now, using a special Lorentz transformation (Winkles and Eldridge 1972; Decoster 1978) the nonlinear Maxwell–Lorentz equations of the plasma can be written in a moving frame S′ as (Paul and Chakraborty 1983*a*, 1983*b*; Paul 1983):

$$p'_x = -eE'_x - \frac{eH'_y V}{c(\gamma_0 V - V_0)}\left(V_0(\gamma_0 - 1) - \frac{Vv'_z}{\gamma_0^3\{\gamma_0 V - V_0 + (\gamma_0 - 1)v'_z\}}\right)$$

$$- \frac{eVv'_y H'_z}{\gamma_0 c\{\gamma_0 V - V_0 + (\gamma_o - 1)v'_z\}}, \tag{8}$$

$$p'_y = -eE'_y + \frac{eH'_x V}{c(\gamma_0 V - V_0)}\left(V_0(\gamma_0 - 1) - \frac{Vv'_z}{\gamma_0^3\{\gamma_0 V - V_0 + (\gamma_0 - 1)v'_z\}}\right)$$

$$+ \frac{eVv'_x H'_z}{\gamma_0 c\{\gamma_0 V - V_0 + (\gamma_0 - 1)v'_z\}}, \tag{9}$$

$$p'_z = -\frac{eVE'_z}{\gamma_0\{\gamma_0 V - V_0 + (\gamma_0 - 1)v'_z\}} + \frac{eVp'_z E'_z}{m_0 \gamma_0^3 \gamma' V_0\{\gamma_0 V - V_0 + (\gamma_0 - 1)v'_z\}}$$

$$+ \frac{e(p'_x E'_x + p'_y E'_y)[V(\gamma_0^2 V + \gamma_0 V_0 - V_0) + v'_z\{V(\gamma_0^2 - \gamma_0 + 1) - V_0\}]}{m_0 \gamma_0 \gamma' V_0 (V + v'_z)\{\gamma_0 V - V_0 + (\gamma_0 - 1)v'_z\}}$$

$$+ \frac{e(\boldsymbol{p}' \times \boldsymbol{H}')_z V\{v'_z V(V + V\gamma_0^6 - V_0\gamma_0^5) + \gamma_0^4 V(\gamma_0 V - V_0)^2\}}{cm_0 \gamma_0^5 \gamma' V_0 (V + v'_z)(\gamma_0 V - V_0)\{\gamma_0 V - V_0 + (\gamma_0 - 1)v'_z\}} \tag{10}$$

$$N' = \text{constant} = N_0, \tag{11}$$

$$(H'_x, H'_y) = 0, \qquad H'_z = \text{constant} = H_0 \text{ (say)}, \tag{12}$$

$$(\dot{E}'_x, \dot{E}'_y) = \frac{4\pi e N_0 (v'_x, v'_y)}{\gamma_0 (1 + \beta_0 \beta'_z)}, \qquad \dot{E}'_z = \frac{4\pi e N_0 (v'_z + V_0)}{\gamma_0 (1 + \beta_0 \beta'_z)}, \tag{13}$$

where a derivative with respect to t' is denoted by a dot. Further,

$$\gamma_0 = (1 - V_0^2/c^2)^{-1/2}, \qquad \gamma = (1 - v'^2/c^2)^{-1/2},$$

where $V = \omega/k$ is the phase velocity of the wave, $\beta'_z = v'_z/c$ and $\beta_0 = V_0/c$, V_0 $(=kc^2/\omega)$ is the velocity of a moving frame relative to the laboratory frame, and ω and k are the frequency and wave number in the laboratory frame S.

In the S′ frame we consider the left circularly polarised wave (LCP):

$$E' = (a \cos\theta_L, a \sin\theta_L, 0), \tag{14}$$

and the right circularly polarised wave (RCP):

$$E' = (a \cos\theta_R, a \sin\theta_R, 0), \tag{15}$$

where $\theta_L = \omega'_+ t'$, $\theta_R = \omega'_- t'$, ω'_+ and ω'_- are wave frequencies of the LCP and RCP components, and a is the constant amplitude of the waves.

Then, using (14) and (15) in (8) to (13), the intensity-dependent dispersion relations for the LCP and RCP waves in the S′ frame are

$$\omega'_\pm (1 - e^2 a^2 \omega'^2_\pm / \omega_p^4 m_0^2 c^2)^{1/2} - \omega_p^2 \mp V\Omega_0 \omega'_\pm / \gamma_0^3 (V - V_0) = 0, \tag{16}$$

where $\Omega_0 = eH_0/m_0 c$ and the plasma frequency $\omega_p = (4\pi N_0 e^2/m_0)^{1/2}$. Equation (16) is also referred to as a nonlinear dispersion relation because it includes some nonlinear effects. The dispersion relation in the laboratory S frame can be obtained by substituting $\omega'_\pm = \omega_\pm/\gamma_0$, where $\gamma_0^2 = \omega_\pm^2/(\omega_\pm^2 - k_\pm^2 c^2)$.

Table 1. Limiting values of the amplitudes of the stable electromagnetic waves under different conditions

ω/ω_p	Ω_0/ω	Limiting values of α	Limiting values of α_0
5	10	0·006	0·013
	15	0·1	0·02
	20	0·133	0·026
	25	0·166	0·033
	30	0·2	0·04
10	10	0·0166	0·00166
	15	0·025	0·0025
	20	0·033	0·0033
	25	0·042	0·0042
	30	0·05	0·005
15	10	0·0076	0·0005
	15	0·011	0·0007
	20	0·015	0·001
	25	0·018	0·0012
	30	0·022	0·0015
20	10	0·0042	0·0002
	15	0·00625	0·000312
	20	0·00833	0·000416
	25	0·0104	0·00052
	30	0·0125	0·00062

3. Results and Discussion

After some mathematical manipulation equation (16) yields the dispersion relations for the LCP and RCP waves,

$$k_\pm^2 c^2 = \frac{\omega_p^2}{\alpha}[Y_\pm \pm \{Y_\pm^2 - 2\alpha^2 X_\pm^{-1}(Y_\pm - X_\pm)\}^{1/2}] \tag{17}$$

in the laboratory frame S, where

$$X_\pm = \omega_p^2/\omega_\pm^2, \qquad Y_\pm = 1 + \frac{\alpha^2 \omega_\pm^2}{2\omega_p^2} \mp \frac{\Omega_0}{\omega_\pm},$$

and where α $(= ea/m_0 \omega_p c)$ is the dimensionless constant wave amplitude, while the subscripts + and − represent the LCP and RCP waves respectively.

Let us assume that the static magnetic field is strong enough so that the gyrofrequency Ω_0 becomes much higher than the wave frequency ω. Under this condition $k_\mp^2$ is real, i.e. the LCP wave will remain stable. But when the intensity of the wave satisfies the condition

$$\alpha > (\Omega_0 \omega_p^2/6\omega^3)^{1/2} \tag{18}$$

k_-^2 is complex, i.e. the RCP wave becomes unstable. It is important to mention here that α is usually much less than unity. Anyway, the condition stated in (18) gives some idea about the limits of the RCP wave intensity for the occurrence of instability at different values of the static magnetic field. If we write $\Omega_0 = \mu\omega$ and $\omega = \nu\omega_p$, where μ and ν are integers much greater than 1, then (18) gives

$$\alpha > (\mu/6\nu^2)^{1/2}. \tag{19}$$

The limiting value of the power of the unstable RCP wave can be obtained from the relation (Chakraborty 1977)

$$P = 6 \cdot 7\alpha_0^2\,\omega^2 \times 10^{-4}\ \mathrm{erg\,cm^{-2}} \tag{20}$$

(1 erg $\equiv 10^{-7}$ J) where $\alpha = (\omega/\omega_p)\alpha_0$, and α_0 $(=ea/m_0\,\omega c)$ is the dimensionless amplitude of the EM wave.

For different values of Ω_0, ω and ω_p, the critical values of the dimensionless amplitude α_0 of the stable EM waves are estimated numerically and these are given in Table 1. In a dense plasma having an electron density $\approx 10^{20}$ cm^{-3}, the powers of the unstable EM waves will be (i) $5 \cdot 8 \times 10^{19}$, (ii) $5 \cdot 3 \times 10^{19}$, (iii) $6 \cdot 9 \times 10^{15}$ and (iv) $7 \cdot 6 \times 10^{16}$ W m^{-2}, when the wave frequency, plasma frequency and gyrofrequency satisfy the conditions (i) $\omega/\omega_p = 5$, $\Omega_0/\omega = 20$, (ii) $\omega/\omega_p = 10$, $\Omega_0/\omega = 30$, (iii) $\omega/\omega_p = 15$, $\Omega_0/\omega = 25$ and (iv) $\omega/\omega_p = 20$, $\Omega_0/\omega = 10$, respectively. From numerical estimates it is also found that under the conditions stated in (i) to (iv) for wave frequency ω, plasma frequency ω_p and gyrofrequency Ω_0, the EM waves (RCP) having powers $5 \cdot 8 \times 10^{11}$, $5 \cdot 3 \times 10^{11}$, $6 \cdot 9 \times 10^{7}$ and $7 \cdot 6 \times 10^{8}$ W m^{-2} become unstable in a plasma having an electron density $\approx 10^{12}$ cm^{-3}.

Now, from (17), the cut-off frequency of the wave may be obtained as

$$\frac{\omega_c^4\,\alpha^2}{2\omega_p^2} + \omega_c^2 \mp \omega_c\,\Omega_0 - \omega_p^2 = 0. \tag{21}$$

Therefore, when $\Omega_0 \gg \omega_p$ and $\omega \gg \omega_p$, the frequency of the non-propagating wave becomes

$$\omega_c = (2\Omega_0\,\omega_p^2/\alpha^2)^{1/2}. \tag{22}$$

From (22) it is observed that due to nonlinear interaction in a strong magnetised plasma, the cut-off frequency of an EM wave is much higher than the plasma frequency of the medium. For an EM wave having $\alpha \approx 0 \cdot 2$, ω_c becomes $7 \cdot 93 \times 10^7$ Hz if ω_p is 10^6 Hz and Ω_0 is of the order of 10^{10} Hz.

From (17), the expression for the phase velocity of a strong EM wave in a nonlinear plasma can also be easily obtained. In a plasma having an electron density $\approx 5 \times 10^{20}$ cm^{-3}, the phase velocity of an Nd–glass laser beam (having power $\approx 10^{20}$ W m^{-2}, frequency $1 \cdot 78 \times 10^{15}$ Hz and wavelength $1 \cdot 06\ \mu$m) will be $1 \cdot 27 \times 10^{10}$ cm s^{-1}, if the static magnetic field is of the order of 10^6 G.

4. Some Concluding Remarks

In plasmas, nonlinear phenomena like self-focussing, self-trapping, self-phase modulation etc., occur when the power of the EM wave is higher than threshold values which depend mainly on the electron density, pulse duration or frequency of the laser beam etc. Kaw and Dawson (1970) have shown that the threshold power of an EM wave for the occurrence of self-action phenomena in a dense plasma (electron density $\approx 10^{21}$ cm^{-3}) is 10^{23} W m^{-2}. In the present investigation it is seen that for a CO_2 laser beam, with $\lambda = 10 \cdot 6\,\mu$m and frequency $1 \cdot 78 \times 10^{14}$ Hz, the EM waves of power 10^{15-19} W m^{-2} are unstable in a plasma having an electron density $\approx 10^{20}$ cm^{-3} and static magnetic field $\approx 10^{8}$ G. Moreover, it is observed that in the presence of a static magnetic field of order 10^{7} G, a ruby-laser having a power $\approx 10^{7-11}$ W m^{-2} and frequency $\approx 10^{12}$ Hz becomes unstable in a plasma of electron density $\approx 10^{12}$ cm^{-3}. So, in strong magnetised plasmas there is every possibility of the wave becoming unstable before the occurrence of nonlinear phenomena due to self-action effects. The instability of the electromagnetic wave discussed in the previous section is not due to self-focussing or filamentation etc., but is of the modulational instability type. It is important to mention that the precessional rotation (PR) and its complementary nonlinear effects (i.e. shifts of wave parameters) in magnetised plasmas should be determined for EM waves having powers within our present estimated limit. Beyond these limits the waves are unstable and so the results obtained for the PR and its related effects, namely induced magnetisation and synchrotron radiation, will have no physical significance (Chakraborty *et al.* 1984, 1986). It is also to be noted that a d.c. magnetic field would be generated by the circular motion of the plasma electrons and an inverse Faraday effect would arise in the plasma. Pomeau and Quamada (1967), Steiger and Woods (1972) and Talin *et al.* (1975) have theoretically discussed this effect in plasmas. Deschamps *et al.* (1970) experimentally observed uniform magnetisation ($\sim 10^{-2}$ G) in plasmas using pulsed microwave signals having a frequency 3000 MHz with a repetition frequency of 10 Hz.

In our present investigation we have neglected the inverse Faraday effect generated in the plasma as the static magnetic field has been assumed to be very strong. For numerical estimates we have used the data of laser induced dense plasmas. However, numerical estimates can also be made for the stability of the wave in the plasmas of magnetic stars, pulsars, the Crab Nebula etc., and interesting results can be obtained which may help to understand the physical processes in these astrophysical bodies.

We now make suggestions for further investigations involving the present work: (i) The motion of the electrons has only been considered in the present analysis. Consideration of both electrons and ions (Stenflo and Tsintsadze 1979; Bhattacharyya 1983) would surely give new information on the stability of the wave. In multicomponent plasmas particularly, the presence of negative ions would have a significant role on the propagation of electromagnetic waves. (ii) Instability of waves due to interaction of two or three waves (Das and Sihi 1979, 1980; Brodin and Stenflo 1989) in a strong magnetised plasma can be studied as it gives some interesting results which are very important in physical situations. It is to be noted that the special Lorentz transformation cannot be used for deriving the nonlinear dispersion relation of the interacting waves, because the condition required for space-independence, $V_0 = kc^2/w$ (V_0 is the velocity of the moving frame relative to the inertial frame, k the wave number, w the wave frequency and c the velocity of light) is not satisfied for two or more interacting waves. (iii) Consideration of thermal effects on the propagation of waves in nonlinear plasmas gives important results. In warm or hot plasmas, the stability of the wave in the presence of a strong magnetic field can be investigated using the special Lorentz transformation (Chakraborty *et al.* 1984).

Acknowledgments

The author is grateful to Prof. B. Chakraborty, Plasma Physics Group, Department of Mathematics, for his constant encouragement in the preparation of the paper. The author is also thankful to the referee for his valuable comments which helped to bring the manuscript to its present form.

References

Akhiezer, A. I., Akhiezer, I. A., Polovin, R. V., Sitenko, A. G., and Stepanov, K. N. (1975). 'Plasma Electrodynamics', Vol. 2 (Pergamon: Oxford).
Bhattacharyya, B. (1983). *Phys. Rev.* A **27**, 568.
Brodin, G., and Stenflo, L. (1989). *J. Plasma Phys.* **42**, 187.
Chakraborty, B. (1977). *Phys. Rev.* A **16**, 1297.
Chakraborty, B., Khan, M., and Bhattacharyya, B. (1986). *J. Appl. Phys.* **59**, 1473.
Chakraborty, B., Paul, S. N., Khan, M., and Bhattacharyya, B. (1984). *Phys. Reports* **141**, 182.
Chakraborty, B., Sur, A. K., and Paul, S. N. (1987). *Aust. J. Phys.* **40**, 675.
Chen, L. (1987). 'Waves and Instabilities in Plasmas' (World Scientific: Singapore).
Das, K. P., and Sihi, S. (1979). *Plasma Phys.* **21**, 183.
Das, K. P., and Sihi, S. (1980). *Beitrage Plasma Phys.* **20**, 215.
Decoster, A. (1978). *Phys. Reports* **47**, 285.
Deschamps, J., Fitaine, M., and Lageutle, M. (1970). *Phys. Rev. Lett.* **25**, 1330.
Goldstein, S., and Salu, Y. (1973). *J. Plasma Phys.* **10**, 231.
Hora, H. (1981). 'Physics of Laser Driven Plasmas' (Wiley: New York).
Kaw, P. K., and Dawson, J. (1970). *Phys. Fluids* **17**, 472.
Lee, M. A., and Lerche, I. (1978). *J. Plasma Phys.* **20**, 313.
Lee, M. A., and Lerche, I. (1979). *J. Plasma Phys.* **21**, 43.
Lee, M. A., and Lerche, I. (1980). *J. Plasma Phys.* **24**, 89.
Max, C. E. (1973). *Phys. Fluids* **16**, 1480.
Max, C. E., and Perkins, F. (1972). *Phys. Rev. Lett.* **29**, 1731.
Paul, S. N. (1983). Ph.D. Thesis, Jadavpur University, Calcutta.
Paul, S. N., and Chakraborty, B. (1983*a*). *Aust. J. Phys.* **36**, 67.
Paul, S. N., and Chakraborty, B. (1983*b*). *Physica* C **122**, 139.
Paul, S. N., Sur, A. K., and Pakira, G. (1989). *J. Plasma Phys.* **41**, 37.
Pomeau, Y., and Quamada, D. (1967). *Compt. Rend.* B **264**, 517.
Shukla, P. K., Rao, N. N., Yu, Y. Y., and Tsintsadge, N. L. (1986). *Phys. Reports* **138**, 1.
Sodha, M. S., Ghatak, A. K., and Tripathi, V. K. (1976). *In* 'Progress in Optics', Vol. XIII (Ed. E. Wolf), p. 175 (North-Holland: Amsterdam).
Steiger, A. D., and Woods, C. M. (1972). *Phys. Rev.* A **5**, 1467.
Stenflo, L. (1976). *Phys. Scripta* **14**, 320.
Stenflo, L. (1980). *Phys. Scripta* **21**, 831.
Stenflo, L., and Tsintsadze, N. L. (1979). *Astrophys. Space Sci.* **64**, 513.
Sur, A. K., Pakira, G., and Paul, S. N. (1988). *Ind. J. Phys.* B **62**, 459.
Talin, B., Kaftandjian, V. P., and Klein, L. (1975). *Phys. Rev.* A **11**, 648.
ter Haar, D., and Tsytovich, V. N. (1981). *Phys. Reports* **73**, 177.
Tsytovich, V. N. (1970). 'Nonlinear Effects in Plasmas' (Plenum: New York).
Winkles, B. B., and Eldridge, O. (1972). *Phys. Fluids* **15**, 1790.

Manuscript received 26 October 1989, accepted 3 April 1990