# Improved Parton Distributions from the Quark Model 

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#### Abstract

There has recently been significant progress in the calculation of the twist-two piece of the quark/parton distributions corresponding to models like the MIT bag. However, in evaluating the required matrix elements of the quark field operators Signal and Thomas $(1988,1989)$ resorted to the Peierls-Yoccoz (1957) approximation-albeit with some cautionary remarks. We point out a problem with that approach which is solved by using the Peierls-Thouless (1962) approximation for the hadronic states. The very simple case of a nonrelativistic constant density quark wavefunction is solved in detail.


Over the past fifteen years there have been many attempts to relate the quark distributions measured in deep inelastic experiments to familiar quark models (Jaffe 1975; Le Yaouanc et al. 1975; Parisi and Petronzio 1976; Celenza and Shakin 1983, 1989; Ross and Jaffe 1980; Benesh and Miller 1987; Thomas 1988a). A major problem with many of these attempts was that the calculated distributions did not vanish for Bjorken- $x$ greater than one, whereas physical structure functions should vanish for $x$ outside the region $[0,1]$. This could be traced to the loss of translational invariance (in time and space) when model quark operators were used. Recently Signal and Thomas (1988) suggested that one could overcome this problem by guaranteeing energy-momentum conservation before making any model-dependent approximations. In this case the twist-two quark distribution is (Politzer 1980; Ellis et al. 1983; Jaffe 1983, 1985)

$$
\begin{equation*}
\left.q^{(2)}\left(x, \mu^{2}\right)=\frac{m}{(2 \pi)^{3}} \sum_{n} \int \mathrm{~d} \boldsymbol{p} \delta\left(m[1-x]-p_{n}^{+}\right)\left|\langle n \boldsymbol{p}| \psi_{+}(0)\right| N \mathbf{0}\right\rangle\left.\right|^{2}, \tag{1}
\end{equation*}
$$

where $\psi_{+}=\left(1+\alpha^{3}\right) \psi / 2, p_{n}^{+}$is the plus component of the momentum of the intermediate state $|n \boldsymbol{p}\rangle\left[p_{n}^{+}=\left(m_{n}^{2}+\boldsymbol{p}^{2}\right)^{1 / 2}+p_{z}\right]$ and we average over the spin of the nucleon target. The momentum scale for the quark field operators is $\mu^{2}$. While (1) is well known in formal derivations of the parton model, Signal and Thomas observed that it provides a practical starting point for calculations because a simple di-quark state $(n=2)$ dominates the sum over $n$ for $x$ beyond $0 \cdot 3$.

For a given model the input to (1) are the masses $m_{n}$ and the matrix elements $\langle n \boldsymbol{p}| \psi_{+}(0)|N 0\rangle$. The importance of the choice of masses was illustrated by Close and Thomas (1988) who showed that the well-known spin and flavour dependence of the parton distributions (e.g. the $d / u$ ratio) was related to the mass splitting of the spin- 0 and spin- 1 diquarks caused by one-gluon exchange. On the other hand, the matrix elements present a major problem because of the well-known difficulty of constructing eigenstates of momentum in all but the simplest nonrelativistic models. For the bag model, Signal and Thomas (1988, 1989) proposed using the Peierls-Yoccoz (PY) (1957) procedure, so that

$$
\begin{equation*}
|n \boldsymbol{p}\rangle=\phi_{n}^{-1}(p) \int \mathrm{d} \boldsymbol{x} \mathrm{e}^{\mathrm{i} \boldsymbol{p} \cdot \boldsymbol{x}}|n ; \boldsymbol{x}\rangle, \tag{2}
\end{equation*}
$$

where $|n ; \boldsymbol{x}\rangle$ is a bag state centred at $\boldsymbol{x}$, and $\phi_{n}(p)$ is chosen to ensure $\delta$-function normalisation. The coordinate representation is then

$$
\begin{align*}
\Psi_{n \boldsymbol{p}}^{\mathrm{PY}}\left(\boldsymbol{x}_{1}, \boldsymbol{x}_{2}\right) & \equiv\left\langle\boldsymbol{x}_{1} \boldsymbol{x}_{2} \mid n \boldsymbol{p}\right\rangle \\
& =\phi_{n}^{-1}(\boldsymbol{p}) \int \mathrm{d} \boldsymbol{x} \mathrm{e}^{\mathrm{i} \boldsymbol{p} \cdot \boldsymbol{x}} \Phi_{n_{1}}\left(\boldsymbol{x}_{1}-\boldsymbol{x}\right) \Phi_{n_{2}}\left(\boldsymbol{x}_{2}-\boldsymbol{x}\right), \tag{3}
\end{align*}
$$

where $\Phi_{n_{i}}\left(\boldsymbol{x}_{i}-\boldsymbol{x}\right)$ are the bag wavefunctions for a particle at $\boldsymbol{x}_{i}$, in state $n_{i}$ [ $n=\left(n_{1}, n_{2}\right)$ ], with the bag centred at $\boldsymbol{x}$. For the nucleon we make the simplest approximation, namely that all quarks are in the ls bag state $\Phi$, so that

$$
\begin{equation*}
\Psi_{N \mathbf{0}}^{\mathrm{PY}}\left(\mathbf{x}_{1}, \boldsymbol{x}_{2}, \mathbf{x}_{3}\right)=\left\langle\mathbf{x}_{1} \mathbf{x}_{2} \mathbf{x}_{3} \mid N \mathbf{0}\right\rangle=\phi_{N}^{-1}(\mathbf{0}) \int \mathrm{d} \boldsymbol{x}^{\prime} \Phi\left(\mathbf{x}_{1}-\mathbf{x}^{\prime}\right) \Phi\left(\mathbf{x}_{2}-\mathbf{x}^{\prime}\right) \Phi\left(\mathbf{x}_{3}-\mathbf{x}^{\prime}\right) . \tag{4}
\end{equation*}
$$

Finally, we can evaluate the matrix element of $\psi_{+}(\mathbf{0})$ needed in (1):

$$
\begin{align*}
\langle n \boldsymbol{p}| \Psi_{+}(\mathbf{0})|N \mathbf{0}\rangle= & \phi_{n}^{-1}(\boldsymbol{p}) \phi_{N}^{-1}(\mathbf{0}) \int \mathrm{d} \boldsymbol{x} \mathrm{~d} \boldsymbol{x}^{\prime} \mathrm{d} \boldsymbol{x}_{1} \mathrm{~d} \boldsymbol{x}_{2} \mathrm{e}^{-\mathrm{i} \boldsymbol{p} \cdot \boldsymbol{x}} \Phi_{n 1}^{\dagger}\left(\boldsymbol{x}_{1}-\boldsymbol{x}\right) \phi\left(\boldsymbol{x}_{1}-\boldsymbol{x}^{\prime}\right) \\
& \times \Phi_{n 2}^{\dagger}\left(\boldsymbol{x}_{2}-\boldsymbol{x}\right) \Phi\left(\boldsymbol{x}_{2}-\boldsymbol{x}^{\prime}\right) \Phi_{+}\left(-\boldsymbol{x}^{\prime}\right) \tag{5}
\end{align*}
$$

While the quark distributions calculated for $n=2$ using the PY approximation [i.e. (5) with $n_{1}=n_{2}=1 \mathrm{~s}$ ] seemed quite reasonable, the calculation was not without ambiguity. In particular, the momentum dependence of the matrix elements $\langle n \boldsymbol{p}| \psi_{+}(0)|N \mathbf{0}\rangle$ and $\langle n \mathbf{0}| \psi_{+}(0)\left|N-\boldsymbol{p}^{\prime}\right\rangle$ [where nonrelativistically $\left.\boldsymbol{p}^{\prime}=\left(m / m_{n}\right) \boldsymbol{p}\right]$ is not the same. This difficulty can be traced to the momentum dependence of the normalisation constant $\phi_{n}(p)$, and the related fact that the PY approximation yields internal wavefunctions that depend to some extent on the total momentum. On physical grounds one may prefer one form, but this is clearly unsatisfactory. Here we shall show that by using the Peierls-Thouless (PT) (1962) approximation,* which does yield internal intrinsic wavefunctions independent of the total momentum, one restores Galilean invariance to the matrix elements of $\psi_{+}$. Rather than attempt to use the full bag model

[^0]we consider a simple model where the quarks are nonrelativistic and have constant density. Even this case is quite complicated to calculate using the PT approximation. Nevertheless, this model is simple enough that we can obtain analytic results for the matrix elements in both approximations.

In general the PT wavefunction has the form

$$
\begin{equation*}
\Psi_{N \boldsymbol{k}}^{\mathrm{PT}}\left(\left\{\boldsymbol{x}_{i}\right\}\right)=\mathrm{e}^{\mathrm{i} \boldsymbol{k} \cdot\langle\boldsymbol{x}\rangle} \int \mathrm{d} \boldsymbol{k}^{\prime} F\left(\boldsymbol{k}^{\prime}\right) \int \mathrm{d} \boldsymbol{r} \mathrm{e}^{\mathrm{i} \boldsymbol{k}^{\prime} \cdot(\boldsymbol{r}-\langle\boldsymbol{x}\rangle)} \prod_{i=1}^{3} \Phi\left(\boldsymbol{x}_{i}-\boldsymbol{r}\right) \tag{6}
\end{equation*}
$$

where $\langle\boldsymbol{x}\rangle$ is the c.m. coordinate of the particles located at $\left\{\boldsymbol{x}_{i}\right\}$. There are a number of ways by which one could choose the weight function $F\left(\boldsymbol{k}^{\prime}\right)$; for example, by minimising the expectation value of the Hamiltonian. We shall choose the simplest scheme, namely $F\left(\boldsymbol{k}^{\prime}\right)$ constant, in which case the wavefunction becomes

$$
\begin{equation*}
\varphi_{N k}^{\mathrm{PT}}\left(\boldsymbol{x}_{1}, \boldsymbol{x}_{2}, \boldsymbol{x}_{3}\right)=\mathrm{e}^{\mathrm{i} \boldsymbol{k} \cdot\langle\boldsymbol{x}\rangle} N_{3} \Phi\left(\mathbf{x}_{1}-\langle\boldsymbol{x}\rangle\right) \Phi\left(\mathbf{x}_{2}-\langle\boldsymbol{x}\rangle\right) \Phi\left(\mathrm{x}_{3}-\langle\boldsymbol{x}\rangle\right) . \tag{7}
\end{equation*}
$$

Note that physically this wavefunction is very attractive because the centre of the bag is now defined by the position of the three quarks, rather than being an independent variable as in the case of equations (2)-(4).

The calculation of the matrix elements is much more complicated in the PT case than in the PY because of the complicated mixing of the position variables appearing in (7). This is the reason why we only attempt to do the calculation with constant density wavefunctions:

$$
\begin{equation*}
\Phi(\boldsymbol{r})=(4 \pi)^{-1 / 2} N\binom{g(r)}{0} \chi, \tag{8}
\end{equation*}
$$

where $g(r)$ is $\theta(R-r), N^{-2}$ is $R^{3} / 3$ and $\chi$ is a Pauli spinor. However, regardless of the choice of wavefunction, one can show formally that the matrix elements required in (1) are well defined, Galilean invariant and vanish as the momentum transfer becomes large. For example, for the $3 \rightarrow 2$ matrix element, using the fact that $\psi(\boldsymbol{x})$ destroys a quark at position $\boldsymbol{x}$, one readily finds that

$$
\begin{align*}
\langle\boldsymbol{p} n=2| \psi(\boldsymbol{x})\left|\boldsymbol{p}^{\prime} N=3\right\rangle= & \left(N_{2}^{\mathrm{pT}}\right)\left(N_{3}^{\mathrm{pT}}\right) 6^{3}(4 \pi)^{-3} \mathrm{e}^{\mathrm{i}\left(\boldsymbol{p}^{\prime}-\boldsymbol{p}\right) \cdot \boldsymbol{x}} \int \mathrm{d} \boldsymbol{\alpha} \mathrm{~d} \boldsymbol{\beta} \\
& \times \mathrm{e}^{\mathrm{i} \boldsymbol{\beta} \cdot\left(3 \boldsymbol{p}-2 \boldsymbol{p}^{\prime}\right)} \phi^{+}(\boldsymbol{\alpha}) \phi(\boldsymbol{\alpha}+\boldsymbol{\beta}) \phi^{+}(-\boldsymbol{\alpha}) \phi(\boldsymbol{\beta}-\boldsymbol{\alpha}) \phi(-2 \boldsymbol{\beta}) . \tag{9}
\end{align*}
$$

We note the factor of $\exp \left\{\mathrm{i}\left(\boldsymbol{p}^{\prime}-\boldsymbol{p}\right) . \boldsymbol{x}\right\}$ which ensures translational invariance. By inspection of (9) it is obvious that the ambiguity mentioned earlier has been resolved.

Although we evaluate (9) for the simple, constant density wavefunction given in (6), we are eventually interested in calculating $q^{(2)}(x)$ for the MIT bag model (Chodos et al. 1974). Therefore we choose the di-quark mass to be 700 MeV and take the radius to be 1 fm (Thomas 1988b). With these parameters we


Fig. 1. Twist-two valence-quark momentum distributions corresponding to only di-quark intermediate states (cf. equation 1) for a bag of radius: (a) 1.0 fm and (b) 0.5 fm . See text for details.
compare, in Fig. la, the $3 \rightarrow 2$ contribution to the twist-two quark distribution calculated in the PY and the PT approximations. In both cases the matrix elements were obtained analytically and then $q^{(2)}\left(x, \mu^{2}\right)$ evaluated by numerical integration over the transverse momentum [ $p_{z}$ being fixed by the $\delta$-function in (1)].

The shape of the PY and PT quark distributions are very similar, both peaking at $x \approx 0.35$. (This is of course primarily determined by the choice of di-quark mass.) However, there is almost a factor of 2 difference in the magnitude of the two calculations. This was quite a surprise to us, and we do not know how much of this reduction is specific to the constant density case with its sharp edge. (For example, for a nonrelativistic harmonic oscillator the PY and PT procedures give identical results.)

As explained in detail by Jaffe (1983, 1985), equation (1) does not give the complete quark momentum distribution for $x<0$. In particular, one would need to include semi-disconnected and even (for $x<-1$ ) disconnected diagrams to
obtain the full result. Nevertheless, if we retain just the connected diagrams (1) has the convenient normalisation condition

$$
\begin{equation*}
\int_{-\infty}^{+1} \mathrm{~d} x q^{(2)}\left(x, \mu^{2}\right)=1 \tag{10}
\end{equation*}
$$

For the PY approximation to the bag (or constant density model) it is easy to show that this sum rule is satisfied exactly by the $1 \mathrm{~s}^{2}$ configuration, provided the same radius $R$ is used. On the other hand, explicit calculation for the PT case gives $0 \cdot 504$-again using the same radius for the intermediate state. This suggests that excited configurations play a more important role in the PT case (at least for the constant density case). The weakness of the constant density model is that we do not have a Hamiltonian and cannot improve the first term by adding excited states. However, if we were to use the closure approximation, that is to replace $m_{n}$ by $\bar{m}$ in (1), the sum over $n$ can be formally carried out. One then obtains a distribution, $q_{\mathrm{CL}}^{(2)}\left(x, \mu^{2}\right)$, which automatically satisfies (7):

$$
\begin{equation*}
q_{\mathrm{CL}}^{(2)}\left(x, \mu^{2}\right)=\frac{m}{(2 \pi)^{3}} \int \mathrm{~d} \boldsymbol{p} \delta\left(m[1-x]-p^{+}\right) \rho(\boldsymbol{p}), \tag{11}
\end{equation*}
$$

with

$$
\begin{gather*}
p^{+}=\left(\boldsymbol{p}^{2}+\bar{m}^{2}\right)^{1 / 2}+p_{z},  \tag{12}\\
\rho(\boldsymbol{p})=(2 \pi)^{9} 6^{6}\left(N_{3}^{\mathrm{pT}}\right)^{2}\left(\frac{4 \pi R^{3}}{3}\right)^{-3} \\
\times \int \mathrm{d} \boldsymbol{\alpha}\left|\int \mathrm{~d} \boldsymbol{\beta} \mathrm{e}^{3 \mathrm{\beta} \cdot \boldsymbol{p}} \theta(R-|\boldsymbol{\alpha}+\boldsymbol{\beta}|) \theta(R-|\boldsymbol{\alpha}-\boldsymbol{\beta}|) \theta(R-2 \beta)\right|^{2} . \tag{13}
\end{gather*}
$$

[It is clear from (11) that no matter what choice we make for $\bar{m}$ the quark distribution still vanishes for $x \geq 1$.]

Equation (13) has been evaluated by expanding the two $\theta$-functions in terms of Legendre polynomials. The results for $\bar{m}=700 \mathrm{MeV}$ and $R=1 \mathrm{fm}$ are also shown in Fig. la. Clearly there is a quite respectable semi-quantitative agreement between this calculation and the PY result. A similar level of agreement is obtained for a much smaller bag radius, $R=0.5 \mathrm{fm}$, as shown in Fig. lb. Of course, if one were really attempting a precision comparison with data there are significant differences. In particular, the PT result does extend to significantly larger values of $x$. For the present, however, we are not in a position to determine $m_{2}$ (or $\bar{m}$ ) to better than a few hundred MeV . For many purposes then the PY approximation should provide an acceptable level of precision.

It is obviously of considerable interest to see whether the unexpectedly large difference between the $1 \mathrm{~s}^{2}$ contribution and the closure result persists with a smoother quark wavefunction. This is actively under investigation, but the algebraic difficulties are such that for most applications the PY approximation will continue to be preferred.

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[^0]:    * We note that the PT method is technically much more complicated than PY. The advantage of PT is that the internal state is independent of the overall momentum of the system-a property not respected by PY.

