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Scattering of HF Radio Waves from the Sea: A Review of the Theory Based upon a Simplified Model

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Abstract

A simplified, physically intuitive model of diffuse scattering of radio waves from a rough surface is used to present a self-contained derivation of first- and second-order cross sections, essentially in agreement with the standard expressions. Incoherent addition of the second-order contributions (electromagnetic and hydrodynamic) leads to a cross section which is slightly different from the more rigorously derived cross sections of Barrick (1972*b*) and Johnstone (1975). A surface current $\mathbf{v}$ has been incorporated in this model, with the main change to the cross section being a frequency shift of the entire spectrum by an amount $\Delta\omega = -2\mathbf{k}_0 \cdot \mathbf{v}$.

1. Introduction

The remote sensing of a distant object using a radar system relies on the user being able to relate the characteristics of the scattered signal to the properties of the target. The distance of the target can also be determined by examination of the time difference between transmitted and scattered signals. In the case of radar oceanography, the target is a remote area of sea surface and the properties we are interested in are the waves on that surface.

The scatter of electromagnetic radiation from the sea surface has been the subject of intensive investigation over the pass thirty-five years. By examining the temporal power spectrum of 13·56 MHz radio waves backscattered from the sea, Crombie (1955) correctly identified Bragg scatter as the simple physical interaction mechanism responsible for the dominant first-order peaks. Crombie suggested 'grazing incidence radio waves were coherently backscattered by sea surface waves with a wavelength one half the wavelength of the incident radio wave, propagating radially toward, or away from, the radar.'

Following these observations, Crombie and subsequently others, e.g. Barber (1959), Braude (1962), suggested that a variable frequency radar could be used to make detailed observations of the sea wave spectrum. For most applications, however, this technique has proved impractical at the longer ocean wavelengths (Barrick 1972*b*).

Observations of sea-induced Doppler shifts on ionospherically propagated radio signals were reported by Tveten (1967) and Ward (1969). They suggested the possibility of mapping sea state over a wide area using analysis of ionospherically propagated, Doppler-shifted, backscattered radio waves.

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About a decade after Crombie's experimental discovery, theoretical studies began to appear which confirmed his findings. Wait (1966) considered the scatter of electromagnetic radiation from a single sinusoidal wave train and found that a principal resonance occurs when the electromagnetic wavelength is double the wavelength of the sea wave. Barrick and Peake (1967, 1968) and Barrick (1972*a*) employed a boundary perturbation approach, initially developed by Rayleigh and generalised by Rice (1951), to quantitatively explain first-order sea echo from a random sea surface. They found that the sea echo power spectrum consists of two impulse functions whose amplitude was proportional to the ocean wave-height directional spectrum evaluated at the Bragg wavenumbers.

These 'first-order' theories predicted the dominant backscattered energy, however they failed to explain the broad echo continuum surrounding the first-order spikes evident in most records. Despite this shortcoming, a number of investigators including Peterson *et al.* (1970) and Teague *et al.* (1973) used the scatter of radio waves from the sea surface to remotely sense ocean wave spectra.

Conversations with Crombie led Stewart and Joy (1974) to display the potential for first-order backscattered radio waves to be used to measure ocean surface currents. Various investigators (Long and Trizna 1973; Tyler *et al.* 1974; Stewart and Barnum 1975) suggested methods for inferring dominant wind/wave directions from first-order sea echo by postulating models for ocean-wave directionality about the mean wind direction. Algorithms for extracting wind speed were also postulated; Stewart and Barnum (1975) related the wind speed to the breadth of the dominant first-order line at a level 10 dB below the maximum, while Ahearn *et al.* (1974) suggested the ratio of the first-order peaks to the continuum at zero Doppler shift as a wind speed parameter.

Hasselmann (1971) suggested that surface periodicities of one half the radar wavelength could be produced by higher order wave–wave interactions and that scatter of the incident radiation from these higher order waves would give rise to symmetric side-bands about the first-order Bragg lines which replicate the wave frequency spectrum. This suggestion that Doppler spectra contained structure directly related to the wind driven sea led to renewed interest, and a new direction, in the remote sensing of sea-state parameters.

Barrick (1972*b*) provided a quantitative link between the backscattered energy recorded by radar and the ocean wave-height spectrum of the sea surface from which the energy was scattered. By extending the boundary perturbation approach to second order he showed that a double interaction was responsible for the scattered energy evident in recorded sea echo spectra at frequencies other than those predicted by first-order scattering theory.

Barrick (1972*b*) and later Johnstone (1975) identified two separate mechanisms which contribute to the second-order backscattered energy: an electromagnetic contribution that takes account of double scatter of the incident radiation by two sea surface waves which results in coherent backscatter, and a hydrodynamic contribution which results from the backscatter of the incident radiation from second-order ocean waves which are the product of the nonlinear interaction of two sea surface waves.

Following the derivation of the backscatter cross section to second order, procedures for the remote sensing of sea-state conditions using portions of the sea echo spectrum other than the dominant first-order lines were suggested by Barrick *et al.* (1974) and Johnstone (1975). Closed form relationships for sea-state parameters based on the two-dimensional nonlinear integral expression for the second-order scattered energy spectrum were presented by Barrick (1977*a, b*). He developed techniques for extracting r.m.s. wave height, mean wave period and the nondirectional wave-height spectrum, which involved a weighting function to take account of the second-order coupling coefficient. Lipa (1977, 1978) presented a technique which determines the directional distribution of saturated ocean waves and the nondirectional wave-height spectrum of the non-saturated waves using integral inversion of the second-order radar echoes.

Maresca and Georges (1980) developed simple empirical relationships between characteristics of the Doppler spectrum and the r.m.s. wave height, the non-directional wave-height spectrum and the dominant wave period. These expressions were based on an extensive simulation in which cross sections were calculated for a wide variety of sea surface conditions.

Employing the closed form expressions presented by Barrick (1977*a, b*), together with relevant empirical relationships on the growth of a wind generated sea (Bretschneider 1970; Hasselmann *et al.* 1973), Dexter and Theodoridis (1982) produced an algorithm for the extraction of wind speed from recorded sea echo Doppler spectra. This algorithm, coupled with an appropriate wind direction technique, yields the complete sea surface wind vector from a backscatter Doppler spectrum.

One of the authors has been involved with analysis of data from James Cook University Coastal Ocean Surface Radar (COSRAD) which operates at 30 MHz in monostatic mode (McGann 1987). The standard method for analysis of such a spectrum is to use the theoretical cross sections developed by Barrick (1972*a, b*) for the first- and second-order scattering:

$$\sigma^{(1)}(\omega) = 2^6 \pi k_0^4 \sum_{m=\pm 1} S(-2m\mathbf{k}_0)\,\delta(\omega - m\omega_B), \tag{1·1}$$

where $\omega_B = \sqrt{2gk_0}$ and $S(\mathbf{k})$ is the directional wave-height spectrum, and

$$\sigma^{(2)}(\omega) = 2^6 \pi k_0^4 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{m_1=\pm 1}\sum_{m_2=\pm 1} |\Gamma_T|^2\, S(m_1\mathbf{k}_1)\, S(m_2\mathbf{k}_2)$$

$$\times\, \delta(\omega - m_1\sqrt{gk_1} - m_2\sqrt{gk_2})\, dp\, dq, \tag{1·2}$$

where $\mathbf{k}_1 = ((p-k_0), q)$ and $\mathbf{k}_2 = (-p-k_0, -q)$, respectively. The coupling coefficient Γ_T is given by

$$\Gamma_T = \Gamma_H + \Gamma_{EM},$$

where Γ_H and Γ_{EM} are the hydrodynamic and electromagnetic components:

$$\Gamma_H = -\tfrac{1}{2} i \left(k_1 + k_2 - \frac{(k_1 k_2 - \mathbf{k}_1 \cdot \mathbf{k}_2)(\omega^2 + \omega_B^2)}{m_1 m_2 \sqrt{k_1 k_2}(\omega^2 - \omega_B^2)} \right), \tag{1·3a}$$

$$\Gamma_{EM} = \tfrac{1}{2} \left(\frac{(\mathbf{k}_1 \cdot \mathbf{k}_0)(\mathbf{k}_2 \cdot \mathbf{k}_0)/k_0^2 - 2\mathbf{k}_1 \cdot \mathbf{k}_2}{\sqrt{\mathbf{k}_1 \cdot \mathbf{k}_2} + k_0 \Delta} \right), \tag{1·3b}$$

and Δ is the normalised surface impedance. This term accounts for ohmic losses experienced by the intermediate radio wave in the double scatter process. It arises from the finite conductivity of the sea water.

The Doppler spectrum contains two dominant peaks, corresponding to the delta functions in equation (1 · 1). Physically, these features may readily be understood in terms of Bragg resonance due to scattering by ocean waves of wavelength $\frac{1}{2}\lambda_0$ moving towards and away from the source/receiver respectively. An analogy is often made with the scattering of light from a ruled diffraction grating, but this model is incorrect, as pointed out by Naylor and Robson (1986), and the correct analogy is with a sinusoidal diffraction grating.

In any case, the first-order features are easily understood and explained theoretically in terms of (1 · 1). In contrast, the theory behind (1 · 2) is readily accessible only to the specialist. It is an aim of this article to present a simplified theory of the scattering process which circumvents much of the lengthy mathematical analysis usually associated with the derivation of (1 · 2) (Johnstone 1975). We shall also include the all-important terms accounting for surface currents in both the first- and second-order analysis. A calculation along these lines was presented by Robson (1984), but so-called second-order hydrodynamic effects were not included. The present paper shows explicit details of these effects and therefore generalises the earlier work.

In Section 2 we give details of the model and produce the first-order scattering cross section (1 · 1) to within an arbitrary multiplication constant. Section 3 extends these ideas to account for multiple scattering (electromagnetic effects), while Section 4 contains details of the calculation including second-order hydrodynamic effects. The essential features of (1 · 2) are reproduced, with the added bonus of terms accounting for surface currents.

Finally, we emphasise the theoretical character of this paper. Radar oceanographers more interested in the applied side of this project can find full details in the thesis of McGann (1987).

2. Diffuse Scattering Model and First-order Cross Section

(a) Diffuse Scattering Model

In this work we shall make use of incremental radar cross sections. The cross section for radiation scattered at angular frequency ω per unit frequency interval, per unit area of scattering region is (Johnstone 1975):

$$\sigma(\omega) = \frac{4\pi R^2 \Theta(\omega)}{A\,|\,E_0|^2}, \tag{2·1}$$

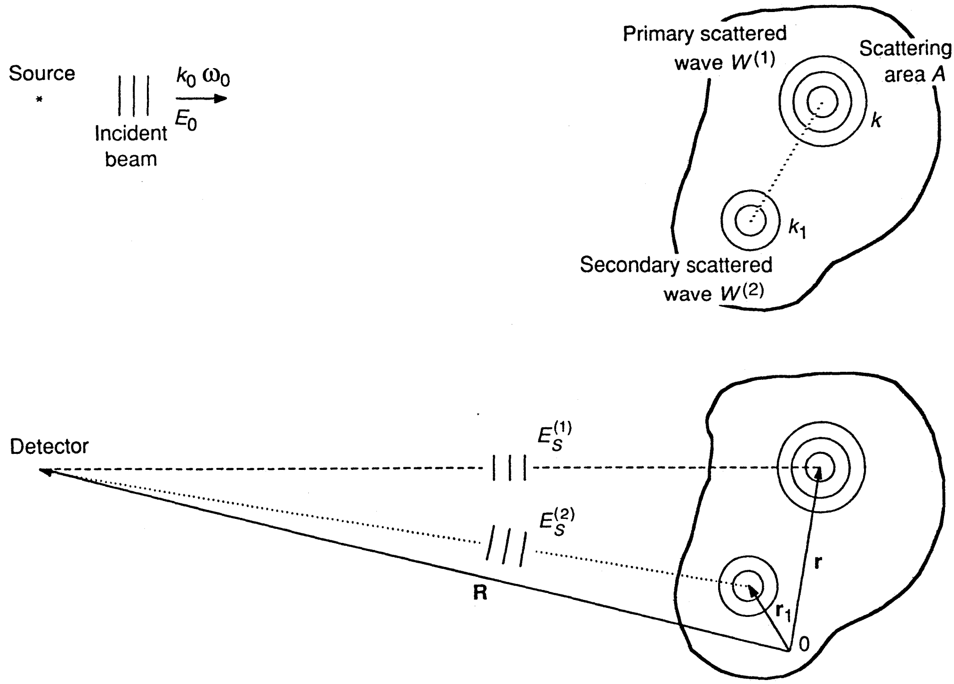

Fig. 1. Schematic representation of possible first- and second-order scatter of the incident beam of vertically polarised electromagnetic radiation.

where $\Theta(\omega)$ is the spectral power density of the scattered signal,

$$\Theta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\tau} B_S(\tau) d\tau \tag{2·2}$$

and

$$B_S(\tau) = B_S(t - t') = \langle E_S(t)\, E_S^*(t') \rangle \tag{2·3}$$

is the autocorrelation function for the scattered field, the angle brackets denoting an appropriate averaging process.

A simplified theory of sea echo backscatter based on the work of Robson (1984) is utilised. An incident plane wave with electric field

$$E_0\, e^{i(\mathbf{k}_0 \cdot \mathbf{r} - \omega_0 t)} \tag{2·4}$$

is assumed to be diffusely scattered from an element of the ocean surface of area d^2r centred on a point $\mathbf{r}$, into a spherical wave $W^{(1)}$ of wavenumber k (Fig. 1). This spherical wave may in turn be scattered into a second spherical wave $W^{(2)}$ by an element of ocean surface d^2r_1 centred on $\mathbf{r}_1$. We assume that the amplitude of the electric field of the scattered wave is proportional to:

1. The amplitude of the incident wave
2. The amplitude of the scattering water wave
3. The area from which the scattering takes place.

(b) First-order Radar Cross Section

According to our model the amplitude of the field at the detector at R due to first-order scattering from an element of surface d^2r centred on $\mathbf{r}$ is

$$c\,E_0\,e^{i(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)}\,f(\mathbf{r},t)\,\frac{e^{ik|\mathbf{R}-\mathbf{r}|}}{|\mathbf{R}-\mathbf{r}|}\,d^2r, \tag{2·5}$$

where $f(\mathbf{r},t)$ is related to the amplitude of the water wave as follows:

$$f(\mathbf{r},t)=\int d^2K\int d\Omega\,(\mathbf{k}_0\cdot\mathbf{K})\,\tilde{f}(\mathbf{K},\Omega)\,e^{i(\mathbf{K}\cdot\mathbf{r}-\Omega t)}. \tag{2·6}$$

Here $\tilde{f}(\mathbf{K},\Omega)$ is the Fourier transform of the wave height in space–time, $(\mathbf{k}_0\cdot\mathbf{K})$ is a 'projection factor' to account for other than normal incidence, and c is a dimensionless constant of proportionality.* We assume the receiver distance R is large compared with the dimensions of the scattering region, i.e. $R\gg A^{\frac{1}{2}}$. It follows therefore that $R\gg r$, and consequently

$$k|\mathbf{R}-\mathbf{r}|\approx k\left(R-\frac{\mathbf{r}\cdot\mathbf{R}}{R}\right)=kR-\mathbf{k}\cdot\mathbf{r},$$

where $\mathbf{k}=k\mathbf{R}/R$. This approximation allows us to rewrite (2·5) as

$$c\,E_0\,e^{i(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)}\,f(\mathbf{r},t)\,\frac{e^{ikR}}{R}\,e^{-i\mathbf{k}\cdot\mathbf{r}}\,d^2r, \tag{2·7}$$

and integrating over the entire scattering region yields for the scattered amplitude

$$E_S^{(1)}(t)=cE_0\frac{e^{ikR}}{R}\int_A f(\mathbf{r},t)\,e^{[-i(\Delta\mathbf{k}\cdot\mathbf{r}+\omega_0 t)]}\,d^2r, \tag{2·8}$$

where $\Delta\mathbf{k}=\mathbf{k}-\mathbf{k}_0$. Forming the autocorrelation function for the scattered field gives

$$\langle E_S^{(1)}(t)\,E_S^{(1)*}(t')\rangle=\frac{|c|^2|E_0|^2}{R^2}\int_A\int_A\langle f(\mathbf{r},t)f^*(\mathbf{r}',t')\rangle$$
$$\times e^{[-i\Delta\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-i\omega_0(t-t')]}\,d^2r'\,d^2r. \tag{2·9}$$

Here we have assumed that the electric fields associated with different order scattering† are uncorrelated, i.e.

$$\langle E_S^{(i)}(t)\,E_S^{(j)*}(t')\rangle=0\qquad i\neq j. \tag{2·10}$$

* Where integration limits are not displayed the integration is from $-\infty$ to ∞.
† $E_S^{(j)}(t)$ is the electric field associated with j^{th} order scattering.

This allows us to calculate the cross section for different order scattering independently. We then simply add the cross sections to give the effective total radar cross section at a particular frequency. If we assume that the wave-height field is both stationary and homogeneous, then

$$\langle f(\mathbf{r},t)\, f^*(\mathbf{r}',t')\rangle = B_f(\mathbf{r}-\mathbf{r}',t-t'), \tag{2·11}$$

i.e. the space–time autocorrelation function depends only on the relative displacement $\mathbf{r}-\mathbf{r}'$ and the time difference $\tau = t-t'$. The autocorrelation function for the scattered field is

$$B_S(\tau) = \frac{|c|^2|E_0|^2}{R^2}\int_A\int_A B_f(\mathbf{r}-\mathbf{r}',\tau)\, e^{[-i\Delta\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-i\omega_0\tau]}\, d^2r'\, d^2r. \tag{2·12}$$

If we transform to a new set of variables

$$\mathbf{u} = \mathbf{r}-\mathbf{r}' \quad \text{and} \quad \mathbf{w} = \tfrac{1}{2}(\mathbf{r}+\mathbf{r}'), \tag{2·13}$$

then $B_f(\mathbf{u},\tau)$ will only depend upon $\mathbf{u}$, allowing us to integrate with respect to $\mathbf{w}$ immediately, bringing out a factor of A, the scattering area,

$$B_S(\tau) = \frac{|c|^2|E_0|^2 A}{R^2}\int_A B_f(\mathbf{u},\tau)\, e^{[-i\Delta\mathbf{k}\cdot\mathbf{u}-i\omega_0\tau]}\, d^2u. \tag{2·14}$$

[Note that the Jacobian for the transformation (2·13) is unity.] We can now determine the spectral power density $\Theta(\omega)$ using expression (2·2)

$$\begin{aligned}\Theta(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau} B_S(\tau)\, d\tau \\ &= \frac{|c|^2|E_0|^2 A}{R^2}\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_A B_f(\mathbf{u},\tau)\, e^{[-i\Delta\mathbf{k}\cdot\mathbf{u}+i\Delta\omega\tau]}\, d^2u\, d\tau,\end{aligned} \tag{2·15}$$

where $\Delta\omega = \omega-\omega_0$. By definition, the ocean wave-height variance density spectrum $S(\mathbf{K},\Omega)$ is given by

$$S(\mathbf{K},\Omega) = \left(\frac{1}{2\pi}\right)^3 \int d^2u \int d\tau\, e^{[-i\mathbf{K}\cdot\mathbf{u}+i\Omega\tau]}\, B_\eta(\mathbf{u},\tau), \tag{2·16}$$

where $\eta(\mathbf{r},t)$ is the water displacement from equilibrium. Thus the first-order spectral power density is given by

$$\Theta^{(1)}(\omega) = (2\pi)^2\frac{|c|^2|E_0|^2 A}{R^2}\, S(\Delta\mathbf{k},\Delta\omega)\,|\mathbf{k}_0\cdot\Delta\mathbf{k}|^2 \tag{2·17}$$

and substitution into (2·1) gives

$$\sigma^{(1)}(\omega) = 2^4\pi^3\,|c|^2\, S(\Delta\mathbf{k},\Delta\omega)\,|\mathbf{k}_0\cdot\Delta\mathbf{k}|^2\,. \tag{2·18}$$

Since $f(\mathbf{r},t)$ is real, we require that $\tilde{f}(\mathbf{K},\Omega)$ be of the form

$$\begin{aligned}\tilde{f}(\mathbf{K},\Omega) &= f_{\mathbf{K}}\,\delta(\Omega-\Omega_{\mathbf{K}})+f^*_{-\mathbf{K}}\,\delta(\Omega+\Omega_{-\mathbf{K}})\\ &= \tilde{f}^*(-\mathbf{K},-\Omega)\,. \end{aligned} \tag{2·19}$$

From equations (2·6) and (2·11) we have

$$B_f(\mathbf{u},\tau)=\int d^2K\int d\Omega\int d^2K'\int d\Omega\,|\,\mathbf{k}_0\cdot\mathbf{K}||\,\mathbf{k}_0\cdot\mathbf{K}'|\,\langle\tilde{f}(\mathbf{K},\Omega)\,\tilde{f}^*(\mathbf{K}',\Omega')\rangle$$

$$\times\, e^{[i(\mathbf{K}\cdot\mathbf{r}-\mathbf{K}'\cdot\mathbf{r}')-i(\Omega t-\Omega' t')]} \tag{2·20}$$

and taking the inverse Fourier transform we have

$$|\,\mathbf{k}_0\cdot\mathbf{K}||\,\mathbf{k}_0\cdot\mathbf{K}'|\,\langle\tilde{f}(\mathbf{K},\Omega)\,\tilde{f}^*(\mathbf{K}',\Omega')\rangle=\frac{1}{(2\pi)^6}\int d^2r\int dt\int d^2r'\int dt'\,B_f(\mathbf{u},\tau)$$

$$\times\, e^{[-i(\mathbf{K}\cdot\mathbf{r}-\mathbf{K}'\cdot\mathbf{r}')+i(\Omega t-\Omega' t')]}\,, \tag{2·21}$$

which can be rewritten in terms of $\mathbf{u}$ and τ as

$$|\,\mathbf{k}_0\cdot\mathbf{K}||\,\mathbf{k}_0\cdot\mathbf{K}'|\,\langle\tilde{f}(\mathbf{K},\Omega)\,\tilde{f}^*(\mathbf{K}',\Omega')\rangle=\frac{1}{(2\pi)^6}\int d^2u\int d\tau\int d^2r'\int dt'\,B_f(\mathbf{u},\tau)$$

$$\times\, e^{[-i(\mathbf{K}\cdot\mathbf{u}-\Omega\tau)+i(\mathbf{K}'-\mathbf{K})\cdot\mathbf{r}'+i(\Omega-\Omega')t']}\,. \tag{2·22}$$

Integrating with respect to $\mathbf{r}'$ and t' yields $(2\pi)^2\delta(\mathbf{K}'-\mathbf{K})$ and $2\pi\,\delta(\Omega-\Omega')$ respectively and therefore (2·22) reduces to

$$|\,\mathbf{k}_0\cdot\mathbf{K}||\,\mathbf{k}_0\cdot\mathbf{K}'|\,\langle\tilde{f}(\mathbf{K},\Omega)\,\tilde{f}^*(\mathbf{K}',\Omega')\rangle=\frac{1}{(2\pi)^3}\int d^2u\int d\tau\,B_f(\mathbf{u},\tau)\,e^{-i(\mathbf{K}\cdot\mathbf{u}-\Omega\tau)}$$

$$\times\,\delta(\mathbf{K}'-\mathbf{K})\,\delta(\Omega-\Omega')\,. \tag{2·23}$$

On the other hand from (2·16) we have

$$|\,\mathbf{k}_0\cdot\mathbf{K}\,|^2 S(\mathbf{K},\Omega)=\frac{1}{(2\pi)^3}\int d^2u\int d\tau\,B_f(\mathbf{u},\tau)\,e^{-i(\mathbf{K}\cdot\mathbf{u}-\Omega\tau)}\,,$$

and thus

$$\langle\tilde{f}(\mathbf{K},\Omega)\,\tilde{f}^*(\mathbf{K}',\Omega')\rangle=S(\mathbf{K},\Omega)\,\delta(\mathbf{K}'-\mathbf{K})\,\delta(\Omega-\Omega')\,. \tag{2·24a}$$

Similarly, from (2·19) we have

$$\langle\tilde{f}(\mathbf{K},\Omega)\,\tilde{f}(\mathbf{K}',\Omega')\rangle=S(\mathbf{K},\Omega)\,\delta(\mathbf{K}'+\mathbf{K})\,\delta(\Omega+\Omega')\,. \tag{2·24b}$$

If we define $S_{\mathbf{K}}$ such that

$$\langle f_{\mathbf{K}}\,f^*_{\mathbf{K}'}\rangle=\tfrac{1}{2}S_{\mathbf{K}}\,\delta(\mathbf{K}'-\mathbf{K})\,, \tag{2·25}$$

then from (2·24) we have

$$S(\mathbf{K},\Omega) = \langle \tilde{f}(\mathbf{K},\Omega)\, \tilde{f}^*(\mathbf{K},\Omega) \rangle$$

and substituting for $\tilde{f}(\mathbf{K},\Omega)$ using (2·19) yields

$$S(\mathbf{K},\Omega) = \tfrac{1}{2}\left(S_{\mathbf{K}}\,\delta(\Omega - \Omega_{\mathbf{K}}) + S_{-\mathbf{K}}\,\delta(\Omega + \Omega_{-\mathbf{K}})\right). \qquad (2\cdot 26)$$

This enables us to write our radar cross section to first order (2·18) as

$$\sigma^{(1)}(\omega) = 2^3\pi^3\,|c|^2\,|\mathbf{k}_0 \cdot \Delta\mathbf{k}|^2\,(S_{\Delta\mathbf{K}}\,\delta(\Delta\omega - \Omega_{\Delta\mathbf{K}}) + S_{-\Delta\mathbf{K}}\,\delta(\Delta\omega + \Omega_{-\Delta\mathbf{K}})). \qquad (2\cdot 27)$$

Small amplitude sea surface gravity waves are characterised by the well known dispersion equation (derived in Kinsman 1965, Chapter 3)

$$\Omega^2 = \Omega_{\mathbf{K}}^2 = gK\tanh(KH) \qquad (2\cdot 28)$$

which for deep water ($KH \gtrsim 3$) reduces to

$$\Omega_{\mathbf{K}}^2 = gK. \qquad (2\cdot 29)$$

In the presence of a sea surface current of velocity $\mathbf{v}$ we have (see Appendix B, equation B·19)

$$\Omega_{\mathbf{K}} = \mathbf{K} \cdot \mathbf{v} + \sqrt{gK}. \qquad (2\cdot 30)$$

If we now consider the case of backscatter where we have the receiver and source co-located, then for grazing incidence the wavevector of the backscattered radiation is $\mathbf{k} = -\mathbf{k}_0$, i.e.

$$\Delta\mathbf{k} = \mathbf{k} - \mathbf{k}_0 = -2\mathbf{k}_0. \qquad (2\cdot 31)$$

From (2·30) it then follows that

$$\Omega_{-2\mathbf{k}_0} = -2\mathbf{k}_0 \cdot \mathbf{v} + \sqrt{2gk_0}, \quad \Omega_{2\mathbf{k}_0} = 2\mathbf{k}_0 \cdot \mathbf{v} + \sqrt{2gk_0}.$$

The backscatter radar cross section to first order is thus given by

$$\sigma^{(1)}(\omega) = 2^5\pi^3\,|c|^2\,k_0^4\,[S_{-2\mathbf{k}_0}\,\delta(\omega - \omega_0 + 2\mathbf{k}_0 \cdot \mathbf{v} - \sqrt{2gk_0})$$
$$+ S_{2\mathbf{k}_0}\,\delta(\omega - \omega_0 + 2\mathbf{k}_0 \cdot \mathbf{v} + \sqrt{2gk_0})]. \qquad (2\cdot 32)$$

The first-order backscatter cross section consists of two impulse functions at frequencies

$$\omega = \omega_0 - 2\mathbf{k}_0 \cdot \mathbf{v} \pm \sqrt{2gk_0}. \qquad (2\cdot 33)$$

The energy in the first-order lines, the so-called Bragg lines, is proportional to the ocean wave-height directional spectrum at the Bragg wavevectors $\pm 2\mathbf{k}_0$.

In the absence of surface current equation (2·33) reduces to the earlier expressions (Barrick 1972*a*, *b*) for the first-order backscatter cross section provided we choose our arbitrary constant c such that $|c|^2 = 2/\pi^2$. The effect of a surface current of velocity $\mathbf{v}$ is to shift both first-order lines by an equal amount $-2\mathbf{k}_0 \cdot \mathbf{v}$.

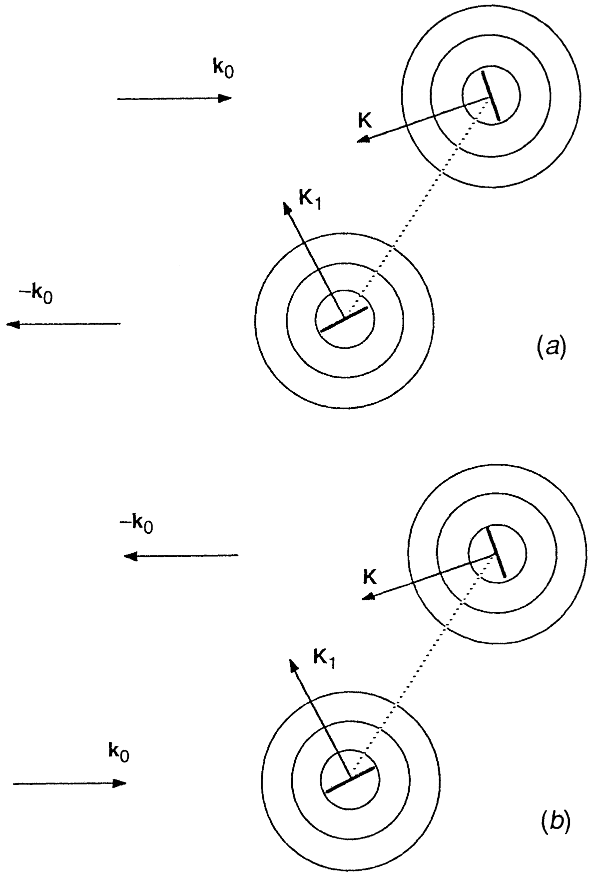

Fig. 2. Schematic representation of double scatter process. The interaction displayed in (*a*) is indiscernable from that in (*b*).

3. Second-order Cross Section: Electromagnetic Contribution

The radar cross section expanded to second order contains two terms; an electromagnetic term due to double scatter of the incident plane wave by two sea surface waves in the first-order wave-height spectrum, and a hydrodynamic term due to single scatter of our incident plane wave by a second-order sea surface wave. In this section we derive the electromagnetic contribution to the second-order backscatter cross section.

The incident plane wave of wavevector $\mathbf{k}_0$ is scattered at $\mathbf{r}$ into a spherical wave of wavenumber k. This spherical wave undergoes a second scattering at $\mathbf{r}_1$, into a second spherical wave of wavenumber k_1 (refer Fig. 1). The model employed here is identical to that used to solve the single scattering problem:

$$dE_S^{(1)}(r_1,t) = c\,E_0\, e^{i(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)}\, f(\mathbf{r},t)\, \frac{e^{ik|\mathbf{r}_1-\mathbf{r}|}}{|\mathbf{r}_1-\mathbf{r}|}\, d^2r \qquad (3\cdot 1)$$

is the amplitude of the first spherical wave at $\mathbf{r}_1$ due to scattering from area d^2r. This is subsequently scattered to produce our second spherical wave, which has the amplitude

$$dE_S^{(2)}(t) = c\,E_0\, e^{i(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)}\, f(\mathbf{r},t)\, \frac{e^{ik|\mathbf{r}_1-\mathbf{r}|}}{|\mathbf{r}_1-\mathbf{r}|}\, d^2r\, c_1\, f(\mathbf{r}_1,t_1)\, \frac{e^{ik_1|\mathbf{R}-\mathbf{r}_1|}}{|\mathbf{R}-\mathbf{r}_1|}\, d^2r_1 \qquad (3\cdot 2)$$

at the receiver. Once again we assume the receiver distance is large compared with the dimensions of the scattering area, i.e. $R \gg r_1$ and hence

$$dE_S^{(2)}(t) = c\,E_0\, \frac{e^{ik_1R}}{R} e^{i(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)}\, f(\mathbf{r},t)\, \frac{e^{ik|\mathbf{r}_1-\mathbf{r}|}}{|\mathbf{r}_1-\mathbf{r}|}\, c_1\, f(\mathbf{r}_1,t_1)\; e^{-i\mathbf{k}_1\cdot\mathbf{r}_1}\, d^2r\, d^2r_1\,. \qquad (3\cdot 3)$$

Replacing $f(\mathbf{r},t)$ and $f(\mathbf{r}_1,t_1)$ by expressions of the form defined in equation $(2\cdot 6)$, assuming $t_1 = t$, and integrating over the entire scattering region, we find

$$E_S^{(2)}(t) = c\,c_1\,E_0 \frac{e^{ik_1R}}{R} \int d^2K \int d\Omega \int d^2K_1 \int d\Omega_1\; e^{-i(\omega_0+\Omega+\Omega_1)t}$$

$$\times \tilde{f}(\mathbf{K},\Omega)\, \tilde{f}(\mathbf{K}_1,\Omega_1)\, I(\mathbf{K},\mathbf{K}_1)\,, \qquad (3\cdot 4\mathrm{a})$$

where

$$I(\mathbf{K},\mathbf{K}_1) = \int_A d^2r \int_A d^2r_1\; e^{[i(\mathbf{k}_0+\mathbf{K})\cdot\mathbf{r}-i(\mathbf{k}_1-\mathbf{K}_1)\cdot\mathbf{r}_1]}\, (\mathbf{k}_0\cdot\mathbf{K})(\mathbf{k}\cdot\mathbf{K}_1)\, \frac{e^{ik|\mathbf{r}_1-\mathbf{r}|}}{|\mathbf{r}_1-\mathbf{r}|}\,. \qquad (3\cdot 4\mathrm{b})$$

If we restrict our attention to backscatter, i.e. $\mathbf{k}_1 = -\mathbf{k}_0$, then schematically we have the interaction displayed in Fig. 2*a*, but quite obviously, this is indiscernable from that displayed in Fig. 2*b*. If we apply our model to the second situation (Fig. 2*b*), however, we obtain:

$$E_S^{(2)}(t) = c\,c_1\,E_0 \frac{e^{ik_1R}}{R} \int d^2K \int d\Omega \int d^2K_1 \int d\Omega_1\; e^{-i(\omega_0+\Omega+\Omega_1)t}$$

$$\times \tilde{f}(\mathbf{K}_1,\Omega_1)\, \tilde{f}(\mathbf{K},\Omega)\, I(\mathbf{K}_1,\mathbf{K})\,, \qquad (3\cdot 5\mathrm{a})$$

where

$$I(\mathbf{K}_1,\mathbf{K}) = \int_A d^2r_1 \int_A d^2r\; e^{[i(\mathbf{k}_0+\mathbf{K}_1)\cdot\mathbf{r}_1-i(\mathbf{k}_1-\mathbf{K})\cdot\mathbf{r}]}\, (\mathbf{k}_0\cdot\mathbf{K}_1)(\mathbf{k}\cdot\mathbf{K})\, \frac{e^{ik|\mathbf{r}-\mathbf{r}_1|}}{|\mathbf{r}-\mathbf{r}_1|}\,. \qquad (3\cdot 5\mathrm{b})$$

(Note that here we have $-\mathbf{k}$ for the intermediate wavevector.) We take account of this symmetry by combining (3·4) and (3·5) to produce

$$E_S^{(2)}(t) = \tfrac{1}{2} c\, c_1\, E_0 \frac{e^{ik_1 R}}{R} \int d^2K \int d\Omega \int d^2K_1 \int d\Omega_1\, e^{-i(\omega_0+\Omega+\Omega_1)t}$$

$$\times \tilde{f}(\mathbf{K},\Omega)\, \tilde{f}(\mathbf{K}_1,\Omega_1)\, I_S(\mathbf{K},\mathbf{K}_1), \tag{3·6}$$

where

$$I_S(\mathbf{K},\mathbf{K}_1) = \int_A d^2r \int_A d^2r_1\, e^{[i(\mathbf{k}_0+\mathbf{K}_1)\cdot\mathbf{r}_1 + i(\mathbf{k}_0+\mathbf{K})\cdot\mathbf{r}]} \frac{e^{ik|\mathbf{r}_1-\mathbf{r}|}}{|\mathbf{r}_1-\mathbf{r}|}$$

$$\times [(\mathbf{k}_0 \cdot \mathbf{K})(\mathbf{k} \cdot \mathbf{K}_1) - (\mathbf{k}_0 \cdot \mathbf{K}_1)(\mathbf{k} \cdot \mathbf{K})]. \tag{3·7}$$

The autocorrelation function for the second-order scattered field is

$$\langle E_S^{(2)}(t) E_S^{(2)*}(t')\rangle = \frac{|c|^2 |c_1|^2 |E_0|^2}{4R^2} \int d^2K \int d^2K_1 \int d^2K' \int d^2K_1' \int d\Omega \int d\Omega_1 \int d\Omega' \int d\Omega_1'$$

$$\times e^{[-i(\omega_0+\Omega+\Omega_1)t + i(\omega_0+\Omega'+\Omega_1')t']}\, I_S(\mathbf{K},\mathbf{K}_1)\, I_S^*(\mathbf{K}',\mathbf{K}_1')$$

$$\times \langle \tilde{f}(\mathbf{K},\Omega)\, \tilde{f}(\mathbf{K}_1,\Omega_1)\, \tilde{f}^*(\mathbf{K}',\Omega')\, \tilde{f}^*(\mathbf{K}_1',\Omega_1')\rangle. \tag{3·8}$$

Kinsman (1965) indicated that a Gaussian distribution closely represents actual wave-height distributions. Although there are obvious reasons why the wave-height distribution cannot be truly Gaussian, for mathematical simplicity we will nevertheless assume it is. In that case (Isserlis 1918; Thomas 1969) we get

$$\begin{aligned}\langle \tilde{f}(\mathbf{K},\Omega)\, \tilde{f}(\mathbf{K}_1,\Omega_1)\, \tilde{f}^*(\mathbf{K}',\Omega')\, \tilde{f}^*(\mathbf{K}_1',\Omega_1')\rangle &= \langle \tilde{f}(\mathbf{K},\Omega)\, \tilde{f}^*(\mathbf{K}',\Omega')\rangle\langle \tilde{f}(\mathbf{K}_1,\Omega_1)\, \tilde{f}^*(\mathbf{K}_1',\Omega_1')\rangle \\ &+ \langle \tilde{f}(\mathbf{K},\Omega)\, \tilde{f}^*(\mathbf{K}_1',\Omega_1')\rangle\langle \tilde{f}(\mathbf{K}_1,\Omega_1)\, \tilde{f}^*(\mathbf{K}',\Omega')\rangle \\ &+ \langle \tilde{f}(\mathbf{K},\Omega)\, \tilde{f}(\mathbf{K}_1,\Omega_1)\rangle\langle \tilde{f}^*(\mathbf{K}',\Omega')\, \tilde{f}^*(\mathbf{K}_1',\Omega_1')\rangle,\end{aligned} \tag{3·9}$$

and from (2·24) we have

$$\begin{aligned}\langle \tilde{f}(\mathbf{K},\Omega)\, \tilde{f}(\mathbf{K}_1,\Omega_1) \tilde{f}^*(\mathbf{K}',\Omega')\, \tilde{f}^*(\mathbf{K}_1',\Omega_1')\rangle = \\ S(\mathbf{K},\Omega)\, \delta(\mathbf{K}'-\mathbf{K})\, \delta(\Omega'-\Omega)\, S(\mathbf{K}_1,\Omega_1)\, \delta(\mathbf{K}_1'-\mathbf{K}_1)\, \delta(\Omega_1'-\Omega_1) \\ + S(\mathbf{K},\Omega)\, \delta(\mathbf{K}_1'-\mathbf{K})\, \delta(\Omega_1'-\Omega)\, S(\mathbf{K}_1,\Omega_1)\, \delta(\mathbf{K}'-\mathbf{K}_1)\, \delta(\Omega'-\Omega_1) \\ + S(\mathbf{K},\Omega)\, \delta(\mathbf{K}_1+\mathbf{K})\, \delta(\Omega_1+\Omega)\, S(\mathbf{K}',\Omega')\, \delta(\mathbf{K}_1'+\mathbf{K}')\, \delta(\Omega_1'+\Omega').\end{aligned} \tag{3·10}$$

The delta functions enable us to do the $\mathbf{K}'$ and $\mathbf{K}_1'$ integrations and the Ω' and Ω_1' integrations.

The first two terms of (3 · 10) make equal contributions, while the contribution from the third term will be neglected momentarily. Thus we get

$$\langle E_S^{(2)}(t)\, E_S^{(2)*}(t')\rangle = \frac{|c|^2 |c_1|^2 |E_0|^2}{4R^2} \int d^2K \int d^2K_1 \int d\Omega \int d\Omega_1\, e^{[-i(\omega_0+\Omega+\Omega_1)(t-t')]}$$
$$\times\, 2\, S(\mathbf{K},\Omega)\, S(\mathbf{K}_1,\Omega_1)\, |I_S(\mathbf{K},\mathbf{K}_1)|^2 \,. \qquad (3\cdot 11)$$

Consider our expression for $I_S(\mathbf{K},\mathbf{K}_1)$; if we define new spatial coordinates $\mathbf{u}$ and $\mathbf{w}$, where

$$\mathbf{u} = \mathbf{r} - \mathbf{r}_1 \quad \text{and} \quad \mathbf{w} = \tfrac{1}{2}(\mathbf{r}+\mathbf{r}_1), \qquad (3\cdot 12)$$

then

$$I_S(\mathbf{K},\mathbf{K}_1) = \int d^2u\, \frac{e^{iku+\frac{1}{2}i(\mathbf{a}+\mathbf{b})\cdot\mathbf{u}}}{u} \int d^2w\, e^{i(\mathbf{a}-\mathbf{b})\cdot\mathbf{w}}[(\mathbf{k}_0\cdot\mathbf{K})(\mathbf{k}\cdot\mathbf{K}_1) - (\mathbf{k}_0\cdot\mathbf{K}_1)(\mathbf{k}\cdot\mathbf{K})], \qquad (3\cdot 13)$$

where

$$\mathbf{a} = \mathbf{k}_0 + \mathbf{K} \quad \text{and} \quad \mathbf{b} = \mathbf{k}_1 - \mathbf{K}_1 = -(\mathbf{k}_0 + \mathbf{K}_1). \qquad (3\cdot 14)$$

Thus we get

$$|I_S(\mathbf{K},\mathbf{K}_1)|^2 = \int d^2u\, \frac{e^{iku+\frac{1}{2}i(\mathbf{a}+\mathbf{b})\cdot\mathbf{u}}}{u} \int d^2u'\, \frac{e^{-iku'-\frac{1}{2}i(\mathbf{a}+\mathbf{b})\cdot\mathbf{u}'}}{u'} \int d^2w\, e^{i(\mathbf{a}-\mathbf{b})\cdot\mathbf{w}}$$
$$\times \int d^2w'\, e^{-i(\mathbf{a}-\mathbf{b})\cdot\mathbf{w}'}[(\mathbf{k}_0\cdot\mathbf{K})(\mathbf{k}\cdot\mathbf{K}_1) - (\mathbf{k}_0\cdot\mathbf{K}_1)(\mathbf{k}\cdot\mathbf{K})]^2. \qquad (3\cdot 15)$$

Consider the integration over $\mathbf{w}$ and $\mathbf{w}'$, and note the identity

$$\left|\int_A d^2w\, e^{i(\mathbf{a}-\mathbf{b})\cdot\mathbf{w}}\right|^2 = \int_A d^2w \int_A d^2w'\, e^{i(\mathbf{a}-\mathbf{b})\cdot(\mathbf{w}-\mathbf{w}')}. \qquad (3\cdot 16)$$

Changing coordinates to

$$\mathbf{x} = \tfrac{1}{2}(\mathbf{w}+\mathbf{w}') \quad \text{and} \quad \mathbf{z} = \mathbf{w} - \mathbf{w}' \qquad (3\cdot 17)$$

and integrating over $\mathbf{x}$, we find

$$\left|\int_A d^2w\, e^{i(\mathbf{a}-\mathbf{b})\cdot\mathbf{w}}\right|^2 = A \int_A d^2z\, e^{i(\mathbf{a}-\mathbf{b})\cdot\mathbf{z}}, \qquad (3\cdot 18)$$

and if we extend the range of the integration over the new variable $\mathbf{z}$ to infinity, then

$$\left|\int_A e^{i(\mathbf{a}-\mathbf{b})\cdot\mathbf{w}}\, d^2w\right|^2 = A\,(2\pi)^2\,\delta(\mathbf{a}-\mathbf{b})$$
$$= A\,(2\pi)^2\,\delta(\mathbf{K}+\mathbf{K}_1+2\mathbf{k}_0). \qquad (3\cdot 19)$$

We now see that neglecting the third term in (3·10) is justified, for this term requires $\mathbf{K}+\mathbf{K}_1=0$, while (3·19) requires $\mathbf{K}+\mathbf{K}_1=-2\mathbf{k}_0$. Quite obviously both these conditions cannot be satisfied simultaneously unless $\mathbf{k}_0$ is a null vector, which, of course, it is not.

To complete our calculation of $|I_S(\mathbf{K},\mathbf{K}_1)|^2$ we need to consider the integral

$$\int_A d^2u\,\frac{e^{iku+i\mathbf{a}\cdot\mathbf{u}}}{u},$$

and note that (3·19) requires $\mathbf{a}=\mathbf{b}$. If we allow the integral limits to tend to infinity and change to polar coordinates, we have

$$\begin{aligned}\int_A d^2u\,\frac{e^{iku+i\mathbf{a}\cdot\mathbf{u}}}{u} &\approx \int_{-\infty}^{\infty} du\, e^{iku}\int_{-\pi}^{\pi} e^{iau\cos\theta}\,d\theta \\ &= 2\pi\int_0^{\infty} du\, e^{iku}\,J_0(au) \\ &= \frac{2\pi i}{\sqrt{(k^2-a^2)}}. \end{aligned} \tag{3·20}$$

Combining (3·20), (3·19) and (3·15) yields

$$|I_S(\mathbf{K},\mathbf{K}_1)|^2 = 16\,(2\pi)^4 A\,\delta(\mathbf{K}+\mathbf{K}_1+2\mathbf{k}_0)\,|\Gamma(\mathbf{K},\mathbf{K}_1)|^2, \tag{3·21a}$$

where

$$|\Gamma(\mathbf{K},\mathbf{K}_1)|^2 = \left(\frac{(\mathbf{k}_0\cdot\mathbf{K})(\mathbf{k}\cdot\mathbf{K}_1)-(\mathbf{k}_0\cdot\mathbf{K}_1)(\mathbf{k}\cdot\mathbf{K})}{4\,|\sqrt{(k^2-a^2)}|}\right)^2. \tag{3·21b}$$

According to lowest order scattering theory (Rice 1951; Tatarski 1961; Hasselmann 1966), an incident electromagnetic wave with horizontal wavevector $\mathbf{k}_I$ is scattered by a surface gravity wave of wavevector $\mathbf{k}_G$ into two waves with horizontal wavevectors given by the Bragg (resonant interaction) condition $\mathbf{k}_S=\mathbf{k}_I\pm\mathbf{k}_G$. Our model assumes that an incident electromagnetic wave with horizontal wavevector $\mathbf{k}_I$ is scattered by a surface gravity wave of wavevector $\mathbf{k}_G$ into a spherical wave of wavenumber $\mathbf{k}_S$, where $\mathbf{k}_S=|\mathbf{k}_I+\mathbf{k}_G|$. Returning our attention to Figs. 2*a* and 2*b* we see that the wavenumber of the scattered radiation after successive scattering on the sea surface from wavevectors $\mathbf{K}$ and $\mathbf{K}_1$ is $\mathbf{k}_S=|\mathbf{k}_I+\mathbf{K}+\mathbf{K}_1|$. The incident wavevector $\mathbf{k}_I=\mathbf{k}_0$, and from (3·19) $\mathbf{K}+\mathbf{K}_1=-2\mathbf{k}_0$, and thus $\mathbf{k}_S=|-\mathbf{k}_0|$. Substituting $\mathbf{k}=\mathbf{k}_0+\mathbf{K}$ and $-\mathbf{k}=\mathbf{k}_0+\mathbf{K}_1$ in (3·21) yields

$$|\Gamma(\mathbf{K},\mathbf{K}_1)|^2 = \left(\frac{(\mathbf{k}_0\cdot\mathbf{K})(\mathbf{k}_0\cdot\mathbf{K}_1)-k_0^2(\mathbf{K}\cdot\mathbf{K}_1)}{2\,|\sqrt{(k^2-a^2)}|}\right)^2. \tag{3·22}$$

From (3·14), we have $\mathbf{a} = \mathbf{k}_0 + \mathbf{K} = -(\mathbf{k}_0 + \mathbf{K}_1)$. Therefore $a^2 - k_0^2 = -\mathbf{K} \cdot \mathbf{K}_1$ and

$$|\Gamma(\mathbf{K},\mathbf{K}_1)|^2 = \left(\frac{(\mathbf{k}_0 \cdot \mathbf{K})(\mathbf{k}_0 \cdot \mathbf{K}_1) - k_0^2(\mathbf{K} \cdot \mathbf{K}_1)}{2\,|\sqrt{\mathbf{K} \cdot \mathbf{K}_1 + (k^2 - k_0^2)}|} \right)^2 \tag{3·23}$$

The difference between this term and that derived by Robson (1984), is due to Robson's ad hoc assumption of an average value for $\mathbf{k} \cdot \mathbf{K}_1$ of $(\mathbf{k}_0 + 2\mathbf{K}) \cdot \mathbf{K}_1$, and an average value for $\mathbf{k} \cdot \mathbf{K}$ of $(\mathbf{k}_0 + 2\mathbf{K}_1) \cdot \mathbf{K}$.

Combining (3·21) and (3·11) gives

$$\langle E_S^{(2)}(t)\, E_S^{(2)*}(t') \rangle = 8(2\pi)^4 A\, \frac{|c|^2 |c_1|^2 |E_0|^2}{R^2}$$
$$\times \int d^2K \int d^2K_1 \int d\Omega \int d\Omega_1\, S(\mathbf{K},\Omega)\, S(\mathbf{K}_1,\Omega_1)$$
$$\times\, \delta(\mathbf{K} + \mathbf{K}_1 + 2\mathbf{k}_0)\, e^{[-i(\omega_0+\Omega+\Omega_1)(t-t')]}\, |\Gamma(\mathbf{K},\mathbf{K}_1)|^2 \tag{3·24}$$

The second-order spectral power density $\Theta^{(2)}(\omega)$ is given by

$$\Theta^{(2)}(\omega) = \frac{1}{2\pi} \int e^{i\omega\tau} \langle E_S^{(2)}(t)\, E_S^{(2)*}(t') \rangle d\tau$$
$$= 8(2\pi)^4 A\, \frac{|c|^2 |c_1|^2 |E_0|^2}{R^2} \int d^2K \int d^2K_1 \int d\Omega \int d\Omega_1\, S(\mathbf{K},\Omega)\, S(\mathbf{K}_1,\Omega_1)$$
$$\times\, \delta(\Delta\omega - \Omega - \Omega_1)\, |\Gamma(\mathbf{K},\mathbf{K}_1)|^2\, \delta(\mathbf{K} + \mathbf{K}_1 + 2\mathbf{k}_0), \tag{3·25}$$

where $\Delta\omega = \omega - \omega_0$ is the frequency shift. The backscatter sea surface radar cross section to second order for grazing incidence is then found by substitution of (3·25) into (2·1)

$$\sigma^{(2)}(\omega) = 16(2\pi)^5\, |c|^2 |c_1|^2 \int d^2K \int d^2K_1\, |\Gamma(\mathbf{K},\mathbf{K}_1)|^2\; \delta(\mathbf{K} + \mathbf{K}_1 + 2\mathbf{k}_0)$$
$$\times\, \tfrac{1}{4}\,(S_{\mathbf{K}}\, S_{\mathbf{K}_1}\, \delta(\Delta\omega - \Omega_{\mathbf{K}} - \Omega_{\mathbf{K}_1}) + S_{\mathbf{K}}\, S_{-\mathbf{K}_1}\, \delta(\Delta\omega - \Omega_{\mathbf{K}} + \Omega_{-\mathbf{K}_1})$$
$$+ S_{-\mathbf{K}}\, S_{\mathbf{K}_1}\, \delta(\Delta\omega + \Omega_{-\mathbf{K}} - \Omega_{\mathbf{K}_1}) + S_{-\mathbf{K}}\, S_{-\mathbf{K}_1}\, \delta(\Delta\omega + \Omega_{-\mathbf{K}} + \Omega_{-\mathbf{K}_1}))\,. \tag{3·26}$$

The integration over $\mathbf{K}_1$ may be performed immediately because of the delta function. With $\mathbf{K}_1 = -2\mathbf{k}_0 - \mathbf{K}$ we have

$$\sigma^{(2)}(\omega) = 4(2\pi)^5\, |c|^2 |c_1|^2 \int d^2K \sum_{m=\pm1} \sum_{m_1=\pm1} S_{m\mathbf{K}} S_{m_1\mathbf{K}_1}\, |\Gamma(\mathbf{K},\mathbf{K}_1)|^2$$
$$\times\, \delta(\Delta\omega + 2\mathbf{k}_0 \cdot \mathbf{v} - m\sqrt{gK} - m_1\sqrt{gK_1})\,. \tag{3·27}$$

Notice how the entire frequency spectrum is shifted uniformly due to a surface current, just as in first-order scattering.

Equation (3·27) is of the same form as Barrick's second-order cross section. We have set $|c|^2 = 2/\pi^2$, to be consistent with Barrick (1972*a*) to first order, and if we are to be consistent to second order we must choose $|c_1|^2 = 1/(2\pi)^2$, which is what we would expect given that our intermediate radio wave is spherical.

Returning to expression (3·23), if the surface is assumed to be perfectly conducting, $k = k_0$ and our expression for $\Gamma(\mathbf{K},\mathbf{K}_1)$ reduces to

$$|\Gamma(\mathbf{K},\mathbf{K}_1)|^2 = \left(\frac{(\mathbf{k}_0 \cdot \mathbf{K})(\mathbf{k}_0 \cdot \mathbf{K}_1) - k_0^2(\mathbf{K} \cdot \mathbf{K}_1)}{2|\sqrt{\mathbf{K} \cdot \mathbf{K}_1}|}\right)^2 . \tag{3·28}$$

This expression is clearly singular when $\mathbf{K} \cdot \mathbf{K}_1 = 0$, i.e. when the two ocean waves from which scattering takes place are propagating perpendicular to each other. This singularity is removed by taking account of the finite conductivity σ_c of the ocean surface.

For good conductors such as sea water and only slightly rough scattering surfaces, the effects of finite conductivity are most apparent for vertically polarised grazing propagation. Under these conditions, deviations from perfect conductor theory are explained by the small component of wave propagation directed into the surface. Elementary considerations (Jordan 1950; Ramo and Whinnery 1953) on propagation over such a surface show that to a good approximation

$$k^2 - k_0^2 \approx (k_0 \Delta)^2 , \tag{3·29}$$

where

$$\Delta = (1+i)\,\frac{\omega\epsilon_0}{2\sigma_c} \tag{3·30}$$

represents the so-called normalised surface impedance. Thus we can write our electromagnetic coupling coefficient $\Gamma(\mathbf{K},\mathbf{K}_1)$ as

$$|\Gamma(\mathbf{K},\mathbf{K}_1)|^2 = \left(\frac{(\mathbf{k}_0 \cdot \mathbf{K})(\mathbf{k}_0 \cdot \mathbf{K}_1) - k_0^2(\mathbf{K} \cdot \mathbf{K}_1)}{2|\sqrt{\mathbf{K} \cdot \mathbf{K}_1 + (k_0\Delta)^2}|}\right)^2 . \tag{3·31}$$

This completes the discussion on the electromagnetic, double scattering effect.

4. Second-order Cross Section: Hydrodynamic Contribution

We now seek the contribution to the radar cross section due to single scatter of the incident radiation by second-order sea surface waves, i.e. the second-order terms of $\tilde{f}(\mathbf{K},\Omega)$ in equations (2·5) and (2·6). Much of the analysis is standard and we incorporate it in Appendix A. By (2·6) we have

$$\langle f(\mathbf{r},t)\, f^*(\mathbf{r}',t')\rangle = \int d^2K \int d\omega \int d^2K' \int d\omega'\, (\mathbf{k}_0 \cdot \mathbf{K})\, (\mathbf{k}_0 \cdot \mathbf{K}')$$
$$\times \langle \tilde{f}(\mathbf{K},\omega)\, \tilde{f}^*(\mathbf{K}',\omega')\rangle\, e^{i(\mathbf{K}\cdot\mathbf{r} - \mathbf{K}'\cdot\mathbf{r}' - \omega t + \omega' t')} , \tag{4·1}$$

where

$$\langle \tilde{f}(\mathbf{K},\omega)\,\tilde{f}^*(\mathbf{K}',\omega')\rangle = \langle \tilde{f}^{(1)}(\mathbf{K},\omega)\,\tilde{f}^{(1)*}(\mathbf{K}',\omega')\rangle + \langle \tilde{f}^{(2)}(\mathbf{K},\omega)\,\tilde{f}^{(2)*}(\mathbf{K}',\omega')\rangle + \cdots \qquad (4\cdot 2)$$

and we have assumed that the terms of different order are uncorrelated. From (A·30) we can determine the average of the product of second-order terms:

$$\langle \tilde{f}^{(2)}(\mathbf{K},\omega)\,\tilde{f}^{(2)*}(\mathbf{K}',\omega')\rangle = \int d^2K_1 \int d\omega_1 \int d^2K'_1 \int d\omega'_1\; \Upsilon(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1,\mathbf{K}_1,\omega_1) \times \langle \cdots \rangle\; \Upsilon^*(\mathbf{K}'-\mathbf{K}_1{}',\omega'-\omega_1{}',\mathbf{K}_1{}',\omega_1{}'), \qquad (4\cdot 3)$$

where

$$\langle \cdots \rangle = \langle \tilde{f}^{(1)}(\mathbf{K}_1,\omega_1)\,\tilde{f}^{(1)}(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1)\,\tilde{f}^{(1)*}(\mathbf{K}_1{}',\omega_1{}')\,\tilde{f}^{(1)*}(\mathbf{K}'-\mathbf{K}_1{}',\omega'-\omega_1{}')\rangle$$

and

$$\Upsilon(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1,\mathbf{K}_1,\omega_1) = \left(\frac{2g\omega^2}{\omega^2-gK}-g\right)\frac{\mathbf{K}_1\cdot(\mathbf{K}-\mathbf{K}_1)-K_1\,|\mathbf{K}-\mathbf{K}_1|}{2(\omega-\omega_1)\omega_1}+\tfrac{1}{2}(K_1+|\mathbf{K}-\mathbf{K}_1|).$$

If we assume the random amplitude? $\tilde{f}^{(1)}(\mathbf{K},\omega)$ are Gaussian distributed, then by (2·24)

$$\begin{aligned}\langle \cdots \rangle =& \langle \tilde{f}^{(1)}(\mathbf{K}_1,\omega_1)\,\tilde{f}^{(1)}(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1)\rangle\langle \tilde{f}^{(1)*}(\mathbf{K}_1{}',\omega_1{}')\,\tilde{f}^{(1)*}(\mathbf{K}'-\mathbf{K}_1{}',\omega'-\omega_1{}')\rangle \\ &+\langle \tilde{f}^{(1)}(\mathbf{K}_1,\omega_1)\,\tilde{f}^{(1)*}(\mathbf{K}_1{}',\omega_1{}')\rangle\langle \tilde{f}^{(1)}(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1)\,\tilde{f}^{(1)*}(\mathbf{K}'-\mathbf{K}_1{}',\omega'-\omega_1{}')\rangle \\ &+\langle \tilde{f}^{(1)}(\mathbf{K}_1,\omega_1)\,\tilde{f}^{(1)*}(\mathbf{K}'-\mathbf{K}_1{}',\omega'-\omega_1{}')\rangle\langle \tilde{f}^{(1)}(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1)\,\tilde{f}^{(1)*}(\mathbf{K}_1{}',\omega_1{}')\rangle \\ =& S(\mathbf{K}_1,\omega_1)\,S(\mathbf{K}_1{}',\omega_1{}')\,\delta(\mathbf{K})\,\delta(\omega)\,\delta(\mathbf{K}')\,\delta(\omega')+S(\mathbf{K}_1,\omega_1)\,S(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1) \\ &\times[\,\delta(\mathbf{K}_1-\mathbf{K}_1{}')\,\delta(\omega_1-\omega_1{}')\,\delta(\mathbf{K}-\mathbf{K}_1-\mathbf{K}'+\mathbf{K}_1{}')\,\delta(\omega-\omega_1-\omega'+\omega_1{}') \\ &+\;\delta(\mathbf{K}_1-\mathbf{K}'+\mathbf{K}_1{}')\,\delta(\omega_1-\omega'+\omega_1{}')\,\delta(\mathbf{K}-\mathbf{K}_1-\mathbf{K}_1{}')\,\delta(\omega-\omega_1-\omega_1{}')\,].\end{aligned} \qquad (4\cdot 4)$$

If we momentarily ignore the first term on the right-hand side of (4·4), then the second-order contribution to (4·1) is

$$\begin{aligned}\langle f(\mathbf{r},t)\,f^*(\mathbf{r}',t')\rangle =& \int d^2K \int d\omega \int d^2K_1 \int d\omega_1 \int d^2K_1{}' \int d\omega_1{}'\,(\mathbf{k}_0\cdot\mathbf{K})^2 \\ &\times \Upsilon(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1,\mathbf{K}_1,\omega_1)\,\Upsilon^*(\mathbf{K}-\mathbf{K}_1{}',\omega-\omega_1{}',\mathbf{K}_1{}',\omega_1{}') \\ &\times e^{[i(\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]}\,S(\mathbf{K}_1,\omega_1)\,S(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1) \\ &\times[\,\delta(\mathbf{K}_1-\mathbf{K}_1{}')\,\delta(\omega_1-\omega_1{}')+\delta(\mathbf{K}-\mathbf{K}_1-\mathbf{K}_1{}')\,\delta(\omega-\omega_1-\omega_1{}')\,],\end{aligned} \qquad (4\cdot 5)$$

where we have integrated over $\mathbf{K}'$ and ω' by enforcing the delta functions $\delta(\mathbf{K}-\mathbf{K}')$ and $\delta(\omega-\omega')$ implicit in (4·4). We can further simplify (4·5) by integrating over $\mathbf{K}'$ and ω' and enforcing the remaining delta functions:

$$\langle f(\mathbf{r},t)\, f^*(\mathbf{r}',t')\rangle = \int d^2K \int d\omega \int d^2K_1 \int d\omega_1\, (\mathbf{k}_0 \cdot \mathbf{K})^2\, e^{[i(\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]}$$

$$\times S(\mathbf{K}_1,\omega_1)\, S(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1)\, 2Y^2(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1,\mathbf{K}_1,\omega_1), \quad (4\cdot 6)$$

where

$$Y(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1,\mathbf{K}_1,\omega_1)\,(Y^*(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1,\mathbf{K}_1,\omega_1)+Y^*(\mathbf{K}_1,\omega_1,\mathbf{K}-\mathbf{K}_1,\omega-\omega_1))$$

$$= 2Y^2(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1,\mathbf{K}_1,\omega_1).$$

From the definition of the spectral power density (2·2) we have, to second order,

$$\Theta^{(2)}(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\Omega\tau}\, B_f^{(2)}(\tau)\, d\tau,$$

and thus, combining this with (2·12) and (4·6):

$$\Theta^{(2)}(\Omega) = \frac{|c|^2\,|E_0|^2}{2\pi R^2}\int_A d^2r \int_A d^2r' \int d^2K \int d\omega \int d^2K_1 \int d\omega_1\, (\mathbf{k}_0 \cdot \mathbf{K})^2$$

$$\times \int e^{[i(\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')-\omega\tau]}\, e^{[-i(\Delta\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\Delta\omega\tau)]}\, d\tau$$

$$\times 2Y^2(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1,\mathbf{K}_1,\omega_1)\, S(\mathbf{K}_1,\omega_1)\, S(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1), \quad (4\cdot 7)$$

where

$$\Delta\mathbf{k} = \mathbf{k}_S - \mathbf{k}_0 \quad \text{and} \quad \Delta\omega = \Omega - \omega_0 .$$

Introducing the variables

$$\mathbf{a} = \mathbf{r}-\mathbf{r}' \quad \text{and} \quad \mathbf{b} = \tfrac{1}{2}(\mathbf{r}+\mathbf{r}')$$

and integrating over $\mathbf{b}$ results in the appearance of a factor of A, the scattering area. By extending the integral limits on $\mathbf{a}$ to infinity we can perform the integration which introduces a factor $(2\pi)^2\,\delta(\mathbf{K}-\Delta\mathbf{k})$, while integrating with respect to τ will bring down a second delta function $2\pi\,\delta(\omega-\Delta\omega)$. Carrying out these three integrations enables us to rewrite (4·7):

$$\Theta^{(2)}(\Omega) = \frac{2\,|c|^2\,|E_0|^2\,(2\pi)^2 A}{R^2}$$

$$\times \int d^2K \int d\omega \int d^2K_1 \int d\omega_1\, (\mathbf{k}_0 \cdot \mathbf{K})^2\, \delta(\mathbf{K}-\Delta\mathbf{k})\, \delta(\omega-\Delta\omega)$$

$$\times Y^2(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1,\mathbf{K}_1,\omega_1)\, S(\mathbf{K}_1,\omega_1)\, S(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1), \quad (4\cdot 8)$$

and further integrating over $\mathbf{K}$ and ω by enforcing the delta functions in (4·8) yields

$$\Theta^{(2)}(\Omega) = \frac{2\,|c|^2\,|E_0|^2\,(2\pi)^2 A}{R^2} \int d^2K_1 \int d\omega_1\,(\mathbf{k}_0 \cdot \Delta\mathbf{k})^2\,\Upsilon^2(\mathbf{K}_2,\omega_2,\mathbf{K}_1,\omega_1)$$

$$\times S(\mathbf{K}_1,\omega_1)\,S(\mathbf{K}_2,\omega_2), \qquad (4\cdot 9)$$

with

$$\mathbf{K}_2 = \Delta\mathbf{k} - \mathbf{K}_1 \quad \text{and} \quad \omega_2 = \Delta\omega - \omega_1 .$$

Expanding $S(\mathbf{K}_1,\omega_1)$ and $S(\mathbf{K}_2,\omega_2)$ using (2·26), we have

$$S(\mathbf{K}_1,\omega_1)\,S(\mathbf{K}_2,\omega_2) = \tfrac{1}{4}\Big(S_{\mathbf{K}_1}\,\delta(\omega_1 - \omega_{\mathbf{K}_1})\,[S_{\mathbf{K}_2}\,\delta(\omega_2 - \omega_{\mathbf{K}_2}) + S_{-\mathbf{K}_2}\,\delta(\omega_2 + \omega_{-\mathbf{K}_2})]$$

$$+ S_{-\mathbf{K}_1}\,\delta(\omega_1 + \omega_{-\mathbf{K}_1})[\,S_{\mathbf{K}_2}\,\delta(\omega_2 - \omega_{\mathbf{K}_2}) + S_{-\mathbf{K}_2}\,\delta(\omega_2 + \omega_{-\mathbf{K}_2})]\Big) \qquad (4\cdot 10)$$

and enforcing the delta functions on ω_1, allows us to integrate with respect to ω_1 in (4·9):

$$\Theta^{(2)}(\Omega) = \frac{2\,|c|^2\,|E_0|^2\,(2\pi)^2 A}{R^2} \int d^2K_1\,(\mathbf{k}_0 \cdot \Delta\mathbf{k})^2 \sum_{m_1=\pm 1}\,\sum_{m_2=\pm 1} \tfrac{1}{4}\,S_{m_1\mathbf{K}_1}\,S_{m_2\mathbf{K}_2}$$

$$\times \Upsilon^2(\mathbf{K}_2, m_2\omega_{m_2\mathbf{K}_2}, \mathbf{K}_1, m_1\omega_{m_1\mathbf{K}_1})\,\delta(\Delta\omega - m_1\omega_{m_1\mathbf{K}_1} - m_2\omega_{m_2\mathbf{K}_2}). \qquad (4\cdot 11)$$

Making use of the first-order dispersion equation for deep water (2·30) and restricting our interest to backscatter for grazing incidence, i.e.

$$\mathbf{k}_S = -\mathbf{k}_0 \quad \text{thus} \quad \Delta\mathbf{k} = -2\mathbf{k}_0 = \mathbf{K}_1 + \mathbf{K}_2 ,$$

we obtain

$$\Theta^{(2)}(\Omega) = \frac{|c|^2\,|E_0|^2\,(2\pi)^2 A}{2R^2} \int d^2K_1\,(2k_0^2)^2 \sum_{m_1=\pm 1}\,\sum_{m_2=\pm 1} S_{m_1\mathbf{K}_1}\,S_{m_2\mathbf{K}_2}$$

$$\times \Upsilon^2(\mathbf{K}_2, m_2\omega_{m_2\mathbf{K}_2}, \mathbf{K}_1, m_1\omega_{m_1\mathbf{K}_1})\,\delta(\Omega - \omega_0 - m_1\sqrt{gK_1} - m_2\sqrt{gK_2}). \qquad (4\cdot 12)$$

The contribution to the radar cross section is therefore

$$\sigma^{(2)}(\Omega) = \frac{4\pi R^2\,\Theta^{(2)}(\Omega)}{A\,|E_0|^2}$$

$$= (2\pi)^3\,|c|^2\,(2k_0^2)^2 \int d^2K_1 \sum_{m_1=\pm 1}\,\sum_{m_2=\pm 1} S_{m_1\mathbf{K}_1}\,S_{m_2\mathbf{K}_2}$$

$$\times \Upsilon^2(\mathbf{K}_2, m_2\omega_{m_2\mathbf{K}_2}, \mathbf{K}_1, m_1\omega_{m_1\mathbf{K}_1})\,\delta(\Omega - \omega_0 - m_1\sqrt{gK_1} - m_2\sqrt{gK_2}). \qquad (4\cdot 13)$$

The full expression for $\Upsilon^2(\mathbf{K}_2, m_2\omega_{m_2\mathbf{K}_2}, \mathbf{K}_1, m_1\omega_{m_1\mathbf{K}_1})$ is

$$\Upsilon^2(\cdot\cdot\cdot) = \left(\frac{1}{2}\left[g\frac{(\Delta\omega^2+2gk_0)}{(\Delta\omega^2-2gk_0)}\frac{(\mathbf{K}_1\cdot\mathbf{K}_2-K_1K_2)}{m_1m_2\sqrt{gK_1}\sqrt{gK_2}}+K_1+K_2\right]\right)^2, \qquad (4\cdot 14)$$

where

$$\Delta\omega = m_1\sqrt{gK_1}+m_2\sqrt{gK_2}.$$

This agrees with the expression derived by Barrick (1972*a*) and Johnstone (1975). The presence of a current $\mathbf{v}$ not only alters the dispersion equation, it also alters the hydrodynamic equations and boundary conditions which must be satisfied by the water surface. The contribution to the radar cross section due to second-order hydrodynamic considerations in the presence of a current $\mathbf{v}$ is presented in Appendix B. There it is shown that one effect of a current $\mathbf{v}$ on the hydrodynamic contribution to the second-order cross section is to shift the entire frequency spectrum by an amount $\Delta\omega = -2\mathbf{k}_0\cdot\mathbf{v}$, just as for the second-order electromagnetic contribution and the first-order cross section. However, the presence of a current also alters the hydrodynamic coupling term $\Upsilon^2(\mathbf{K}_2, m_2\omega_{m_2\mathbf{K}_2}, \mathbf{K}_1, m_1\omega_{m_1\mathbf{K}_1})$. Thus we find

$$\Theta^{(2)}(\Omega) = \frac{|c|^2|E_0|^2(2\pi)^2A}{2R^2}\int d^2K_1\,(2k_0^2)^2\sum_{m_1=\pm1}\sum_{m_2=\pm1}S_{m_1\mathbf{K}_1}S_{m_2\mathbf{K}_2} \qquad (4\cdot 15)$$

$$\times\Upsilon^2(\mathbf{K}_2, m_2\omega_{m_2\mathbf{K}_2}, \mathbf{K}_1, m_1\omega_{m_1\mathbf{K}_1})\delta(\Omega-\omega_0+2\mathbf{k}_0\cdot\mathbf{v}-m_1\sqrt{gK_1}-m_2\sqrt{gK_2})$$

where $\Upsilon^2(\mathbf{K}_2, m_2\omega_{m_2\mathbf{K}_2}, \mathbf{K}_1, m_1\omega_{m_1\mathbf{K}_1})$ is given by

$$\Upsilon^2(\cdot\cdot\cdot) = \left(\frac{1}{2}\left[K_1+K_2+\frac{(\mathbf{K}_1\cdot\mathbf{K}_2-K_1K_2)}{m_1m_2\sqrt{K_1}\sqrt{K_2}}\frac{(1-2v^2k_0/g)\Delta\omega^2+\omega_B^2}{\Delta\omega^2-\omega_B^2}\right]\right)^2. \qquad (4\cdot 16)$$

Combining (2·32), (3·27) and (4·15), we can write the backscatter radar cross section of the sea surface to grazing incidence electromagnetic radiation of wavenumber $\mathbf{k}_0$ as

$$\sigma(\omega) = 2^6\pi k_0^4\sum_{m=\pm1}S_{2m\mathbf{k}_0}\,\delta(\omega-\omega_0+2\mathbf{k}_0\cdot\mathbf{v}+m\sqrt{2gk_0})$$

$$+\,2^6\pi k_0^4\int d^2K_1\sum_{m_1=\pm1}\sum_{m_2=\pm1}S_{m_1\mathbf{K}_1}S_{m_2\mathbf{K}_2}$$

$$\times\,\delta(\omega-\omega_0+2\mathbf{k}_0\cdot\mathbf{v}-m_1\sqrt{gK_1}-m_2\sqrt{gK_2})$$

$$\times\left(|\Gamma(\mathbf{K}_1,\mathbf{K}_2)|^2+\Upsilon^2(\mathbf{K}_2, m_2\omega_{m_2\mathbf{K}_2}, \mathbf{K}_1, m_1\omega_{m_1\mathbf{K}_1})\right), \qquad (4\cdot 17)$$

where

$$|\Gamma(\mathbf{K}_1,\mathbf{K}_2)|^2 = \left(\frac{(\mathbf{k}_0\cdot\mathbf{K}_1)(\mathbf{k}_0\cdot\mathbf{K}_2)-k_0^2(\mathbf{K}_1\cdot\mathbf{K}_2)}{2|\sqrt{\mathbf{K}_1\cdot\mathbf{K}_2+(k_0\Delta)^2}|}\right)^2$$

and

$$\Delta = (1 + i)\,\frac{\omega\epsilon_0}{2\sigma_c}.$$

Equation (4·17) is the complete expression for σ to second order in electromagnetic and hydrodynamic effects.

5. Summary and Discussion

A simple and physically appealing model for the scatter of electromagnetic radiation from the sea surface has been adopted. We assume the incident plane wave (radar beam) is diffusely scattered from the ocean surface into a spherical wave. Our model is such that the electric field amplitude of the scattered wave is proportional to

(1) the amplitude of the incident wave,

(2) the amplitude of the scattering water wave,

(3) the area from which scattering takes place.

This approach circumvents much of the lengthy mathematical detail associated with the derivations of Barrick (1972*a, b*) and Johnstone (1975) who employed the boundary perturbation approach initially developed by Rayleigh and generalised by Rice (1951). They expressed the total electromagnetic field above the sea surface as a sum of the incident field, fields reflected in the absence of surface roughness, and those fields scattered because of surface roughness. The scattered fields and the sea surface were then expanded in terms of the same double Fourier series, and the unknown coefficients in the expansion of the scattered fields were determined by imposing the boundary conditions at the surface. The expressions developed by Barrick (1972*a, b*) and Johnstone (1975) correspond to coherent addition and incoherent addition of the scattered fields respectively.

These two approaches give slightly different theoretical cross sections to second order. Our model, which assumes that the electric fields associated with different order scattering are uncorrelated, uses incoherent addition. Coherent addition is based on the contention that, although individual first-order ocean waves trains may be uncorrelated, second-order wave trains are related to the first-order waves that interact to generate them. Which, if either, form of addition is correct has yet to be resolved.

The expression for the backscatter cross section of the sea surface to grazing incidence electromagnetic radiation predicted by our model agrees with the alternative derivations to first order. However, when our analysis was extended to second order we arrived at an expression which is slightly different from the expressions of Barrick (1972*a, b*) and Johnstone (1975). This is attributed to the approximate nature of the model.

The effect of a surface current on the radar cross section predicted by our model has been examined. A small change in the form of the second-order hydrodynamic contribution gives rise to a small change in the Doppler spectrum. For practical purposes the only discernable change to the radar cross section is a shift of the entire frequency spectrum by an amount $\Delta\omega = -2\mathbf{k}_0 \cdot \mathbf{v}$. This agrees with the widely accepted, though previously unproven, effect of a sea surface current.

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Appendix A: Equations of Motion and Boundary Condition

We consider here Fig. 3. The fluid equation of motion is

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho}\nabla p + \mathbf{g}, \tag{A·1}$$

where we have included gravity but neglected other forces associated with fluids such as friction, surface tension and coriolis, as these have little effect on ocean wavelengths of interest (Kinsman 1965, p. 23). Writing the total time derivative of **u** explicitly (A·1) becomes

$$\frac{\partial \mathbf{u}}{\partial t} + \tfrac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) + (\nabla\times\mathbf{u})\times\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g}, \tag{A·2}$$

where we have used the identity

$$\mathbf{u}\cdot\nabla\mathbf{u} = \tfrac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) + (\nabla\times\mathbf{u})\times\mathbf{u}.$$

By assuming the fluid is irrotational, i.e. vorticity $\nabla\times\mathbf{u}$ is zero, we may express the velocity **u** as the gradient of a scalar ϕ, which in fluid dynamics is known as the velocity potential:

$$\mathbf{u} = -\nabla\phi. \tag{A·3}$$

This allows us to simplify (A·2) in the following way:

$$-\nabla\frac{\partial\phi}{\partial t} + \tfrac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) = -\frac{1}{\rho}\nabla p + \mathbf{g}. \tag{A·4}$$

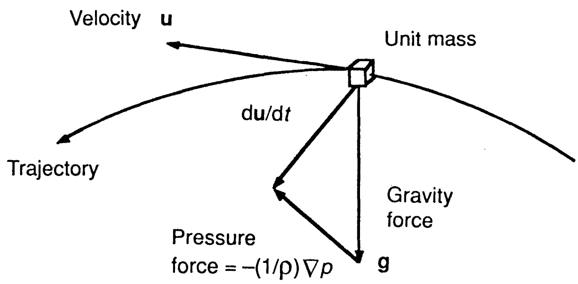

Fig. 3. Forces on a unit mass of fluid.

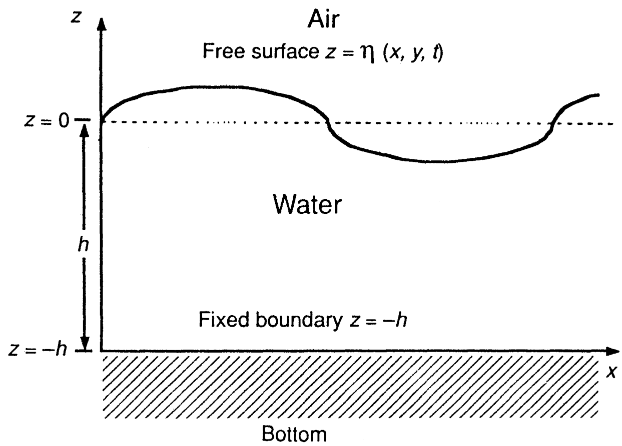

Fig. 4. Schematic representation of fluid boundaries.

Expressing **g** in terms of the gravitational potential $\psi = gz$, i.e. $\mathbf{g} = -\nabla\psi$, enables us to integrate (A·4), yielding

$$-\frac{\partial\phi}{\partial t} + \tfrac{1}{2}u^2 = -\frac{p}{\rho} - \psi, \tag{A·5}$$

where we have assumed ρ is a constant. Since we are interested only in wave-generated or second-order ocean waves, we assign the overlying atmospheric pressure to be zero. At the surface (Fig. 4), equation (A·5) becomes

$$\left[\frac{\partial\phi}{\partial t} - \tfrac{1}{2}u^2 - gz\right]_{z=\eta(x,y,t)} = 0, \tag{A·6}$$

where we have taken g in the $-z$ direction. This condition (A·6) is the dynamic free surface boundary condition (DFSBC). The second free surface boundary condition, the kinematic free surface boundary condition (KFSBC), requires that surface particles remain at the surface, i.e.

$$\left.\frac{d}{dt}[z - \eta(x,y,t)]\right|_{z=\eta} = 0. \tag{A·7}$$

In addition, the fluid is assumed incompressible, i.e.

$$\nabla \cdot \mathbf{u} = 0 . \tag{A·8}$$

If we couple this latter requirement with our irrotational fluid assumption (A·3), then ϕ satisfies the Laplace equation:

$$\nabla^2 \phi = 0 . \tag{A·9}$$

The velocity potential ϕ is also subject to a boundary condition at the ocean bottom $z = -h$. Here we insist there be no component of velocity perpendicular to the ocean bottom, i.e.

$$u_z = -\frac{\partial \phi}{\partial z} = 0 \bigg|_{z=-h} . \tag{A·10}$$

In this work we shall assume deep water, i.e. we let $-h \to -\infty$ in (A·10). We now require the solution of equations (A·6), (A·7), (A·9) and (A·10) involving $\phi(x,y,z,t)$ and $\eta(x,y,t)$. To achieve this we initially write these equations in component form. The DFSBC in component form is

$$\frac{\partial \phi}{\partial t} - \tfrac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] - gz = 0 , \tag{A·11}$$

while expanding the KFSBC yields

$$\frac{dz}{dt} - \frac{\partial \eta}{\partial t} - \frac{\partial \eta}{\partial x}\frac{dx}{dt} - \frac{\partial \eta}{\partial y}\frac{dy}{dt} = 0 . \tag{A·12}$$

Note, however, that

$$\frac{dz}{dt} = u_z = -\frac{\partial \phi}{\partial z} ,$$

and this enables us to write the KFSBC as

$$-\frac{\partial \phi}{\partial z} - \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial \eta}{\partial y}\frac{\partial \phi}{\partial y}\bigg|_{z=\eta} = 0 . \tag{A·13}$$

Laplace's equation (A·9) in expanded form is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 . \tag{A·14}$$

Perturbation Solution

We solve the above system of equations by expanding $\phi(x,y,z,t)$ in a Taylor series about the point $z = 0$, i.e.

$$\phi(x,y,z,t)\,|_{z=\eta} = \phi(x,y,0,t) + \eta \frac{\partial}{\partial z}\phi(x,y,0,t) + \cdots . \tag{A·15}$$

We then substitute this expression in equations (A·11) and (A·12) and retain terms up to second order in ϕ and η. The DFSBC becomes*

$$\phi_t + \eta\phi_{tz} - \tfrac{1}{2}[\phi_x^2 + \phi_y^2 + \phi_z^2] - g\eta = 0\,, \tag{A·16}$$

while the KFSBC is

$$\phi_z + \eta\phi_{zz} = \eta_x\phi_x + \eta_y\phi_y - \eta_t\,. \tag{A·17}$$

Solving (A·16) to first order yields

$$\eta^{(1)} = \frac{1}{g}\phi_t\,. \tag{A·18}$$

Substituting (A·18) back into (A·16) gives the following expression, accurate to second order,

$$\phi_t + \frac{1}{g}\phi_t\phi_{tz} - \tfrac{1}{2}[\phi_x^2 + \phi_y^2 + \phi_z^2] - g\eta = 0\,,$$

from which we obtain η to second order:

$$\eta^{(2)} = \frac{1}{g}\phi_t + \frac{1}{g^2}\phi_t\phi_{tz} - \frac{1}{2g}[\phi_x^2 + \phi_y^2 + \phi_z^2]\,. \tag{A·19}$$

In order to evaluate the right-hand side of (A·19) we also require ϕ to both first and second order. Combining equations (A·17) and (A·19), there follows

$$\begin{aligned}\phi_z + \frac{1}{g}\phi_t\phi_{zz} = \frac{1}{g}\phi_{xt}\phi_x + \frac{1}{g}\phi_{yt}\phi_y - \frac{1}{g}\phi_{tt} - \frac{1}{g^2}\phi_{tt}\phi_{tz} - \frac{1}{g^2}\,\phi_t\phi_{ttz}\\ + \frac{1}{2g}(2\phi_x\phi_{tx} + 2\phi_y\phi_{ty} + 2\phi_z\phi_{tz})\,,\end{aligned} \tag{A·20}$$

and therefore, to first order,

$$\phi_z + \frac{1}{g}\phi_{tt} = 0\,. \tag{A·21}$$

Substituting in (A·20) for ϕ_{tt} and ϕ_{ttz} using (A·21) gives the following equation for ϕ, valid to second order:

$$g\phi_z + \phi_{tt} = 2[\phi_{xt}\phi_{\mathbf{x}} + \phi_{yt}\phi_y + \phi_{zt}\phi_z]\,. \tag{A·22}$$

Our perturbation solution therefore requires that we solve (A·22) to obtain $\phi(x,y,0,t)$ to second order. We then substitute this in (A·19) to yield $\eta(x,y,t)$ to second order.

* We now change to the more compact notation of indicating partial differentiation by a subscript, i.e. $\partial\phi/\partial t = \phi_t$.

If we define the Fourier transforms

$$\Phi(\mathbf{K},\omega,z) = \frac{1}{(2\pi)^3}\int\int\int dx\,dy\,dt\; e^{-i(\mathbf{K}\cdot\mathbf{r}-\omega t)}\,\phi(x,y,z,t) \tag{A$\cdot$23}$$

and

$$\tilde{f}(\mathbf{K},\omega) = \frac{1}{(2\pi)^3}\int\int\int dx\,dy\,dt\; e^{-i(\mathbf{K}\cdot\mathbf{r}-\omega t)}\,\eta(x,y,t)\,, \tag{A$\cdot$24}$$

then taking the Fourier transform of (A·14) yields

$$-K^2\,\Phi(\mathbf{K},\omega,z) + \frac{d^2}{dz^2}\Phi(\mathbf{K},\omega,z) = 0\,,$$

which has the solution

$$\Phi(\mathbf{K},\omega,z) = \Psi(\mathbf{K},\omega)\,e^{Kz} \tag{A$\cdot$25}$$

consistent with the boundary condition (A·10) as $h \to -\infty$. Taking the Fourier transform of (A·22) and recalling that the transform of a product leads to the convolution of the transforms, we find

$$(gK-\omega^2)\Psi^{(2)}(\mathbf{K},\omega) = 2\int d^2K_1 \int d\omega_1\{\mathbf{K}_1\cdot(\mathbf{K}-\mathbf{K}_1) - K_1\,|\,\mathbf{K}-\mathbf{K}_1|\}i(\omega-\omega_1)$$

$$\times\,\Psi^{(1)}(\mathbf{K}_1,\omega_1)\;\Psi^{(1)}(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1)\,. \tag{A$\cdot$26}$$

Had we taken the Fourier transform of (A·22) with the products on the right-hand side reversed, i.e.

$$g\phi_z + \phi_{tt} = 2[\phi_x\phi_{xt} + \phi_y\phi_{yt} + \phi_z\phi_{zt}]\,,$$

then we would have arrived at a different, but entirely equivalent, expression to (A·26), i.e.

$$(gK-\omega^2)\Psi^{(2)}(\mathbf{K},\omega) = 2\int d^2K_1 \int d\omega_1\{\mathbf{K}_1\cdot(\mathbf{K}-\mathbf{K}_1) - K_1\,|\,\mathbf{K}-\mathbf{K}_1|\}i\omega_1$$

$$\times\,\Psi^{(1)}(\mathbf{K}_1,\omega_1)\;\Psi^{(1)}(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1)\,. \tag{A$\cdot$27}$$

Taking the Fourier transform of (A·18) allows us to express $\Psi^{(2)}(\mathbf{K},\omega)$ in terms of $\tilde{f}^{(1)}(\mathbf{K},\omega)$,

$$\tilde{f}^{(1)}(\mathbf{K},\omega) = -\frac{i\omega}{g}\,\Psi^{(1)}(\mathbf{K},\omega)\,.$$

It then follows from (A·26) and (A·27) that

$$\Psi^{(2)}(\mathbf{K},\omega) = -\,\frac{2ig^2}{(gK-\omega^2)}\int d^2K_1\int d\omega_1\,\frac{\omega}{2\omega_1(\omega-\omega_1)}\,\{\mathbf{K}_1\cdot(\mathbf{K}-\mathbf{K}_1) - K_1\,|\,\mathbf{K}-\mathbf{K}_1|\}$$

$$\times\,\tilde{f}^{(1)}(\mathbf{K}_1,\omega_1)\,\tilde{f}^{(1)}(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1)\,. \tag{A$\cdot$28}$$

Having $\Psi^{(2)}(\mathbf{K},\omega)$ in terms of $\tilde{f}^{(1)}(\mathbf{K},\omega)$, we now seek $\tilde{f}^{(2)}(\mathbf{K},\omega)$ by taking the Fourier transform of (A·19):

$$\tilde{f}^{(2)}(\mathbf{K},\omega) = -\frac{i\omega}{g}\Psi^{(2)}(\mathbf{K},\omega) + \frac{1}{g^2}\int d^2K_1 \int d\omega_1(-i\omega_1)\,|\mathbf{K}-\mathbf{K}_1|\,(-i(\omega-\omega_1))$$
$$\times \Psi^{(1)}(\mathbf{K}_1,\omega_1)\,\Psi^{(1)}(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1) - \frac{1}{2g}\int d^2K_1 \int d\omega_1$$
$$\times\{K_1\,|\mathbf{K}-\mathbf{K}_1| - \mathbf{K}_1\cdot(\mathbf{K}-\mathbf{K}_1)\}\Psi^{(1)}(\mathbf{K}_1,\omega_1)\,\Psi^{(1)}(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1). \qquad \text{(A·29)}$$

Once again our expression on the right could be altered by interchanging product terms on the right of equation (A·19). In the above case the relevant term is $\phi_t\phi_{tz}/g^2$. Interchanging would yield

$$\frac{1}{g^2}\int d^2K_1 \int d\omega_1(-i\omega_1)\,\mathbf{K}_1\,(-i(\omega-\omega_1))\,\Psi^{(1)}(\mathbf{K}_1,\omega_1)\,\Psi^{(1)}(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1)$$

instead of the second term in (A·29). We take account of symmetry to obtain the desired expression for $\tilde{f}^{(2)}(\mathbf{K},\omega)$:

$$\tilde{f}^{(2)}(\mathbf{K},\omega) = \left(\frac{2g\omega^2}{\omega^2 - g\mathbf{K}} - g\right)\int d^2K_1 \int d\omega_1 \left(\frac{\mathbf{K}_1\cdot(\mathbf{K}-\mathbf{K}_1) - K_1\,|\mathbf{K}-\mathbf{K}_1|}{2(\omega-\omega_1)\omega_1}\right)$$
$$\times \tilde{f}^{(1)}(\mathbf{K}_1,\omega_1)\,\tilde{f}^{(1)}(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1) + \tfrac{1}{2}\int d^2K_1 \int d\omega_1$$
$$\times[\,K_1 + |\mathbf{K}-\mathbf{K}_1|\,]\tilde{f}^{(1)}(\mathbf{K}_1,\omega_1)\,\tilde{f}^{(1)}(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1). \qquad \text{(A·30)}$$

The perturbation solution is now complete; we have obtained an expression for $\tilde{f}^{(2)}(\mathbf{K},\omega)$ involving a product of first-order transforms $\tilde{f}^{(1)}(\mathbf{K},\omega)$.

Appendix B: Hydrodynamic Contribution with Current

In this section we solve the hydrodynamic equations with boundary conditions which must be satisfied by the water surface in the presence of a surface current. As in Appendix A we solve the equations to second order in surface elevation and velocity potential. The boundary conditions which must be satisfied by the free surface are the kinematic free surface boundary condition,

$$\eta_t + \varphi_z - \varphi_x\eta_x - \varphi_y\eta_y = 0|_{z=\eta(x,y,t)}\,, \qquad \text{(B·1)}$$

and the dynamic free surface boundary condition,

$$g\eta - \varphi_t + \tfrac{1}{2}[\,\varphi_x^2 + \varphi_y^2 + \varphi_z^2\,] + \frac{p}{\rho} = 0\Big|_{z=\eta(x,y,t)}\,. \qquad \text{(B·2)}$$

The bottom boundary condition is that there be no component of velocity perpendicular to the bottom boundary:

$$\varphi_z = 0|_{z=-h} \to -\infty\,. \qquad \text{(B·3)}$$

To include the effects of a current $(v_1, v_2, 0)$ we assume that the velocity potential $\varphi(x,y,z,t)$ can be expressed as

$$\varphi(x,y,z,t) = -v_1 x - v_2 y + \phi(x,y,z,t). \tag{B·4}$$

The assumptions of an irrotational and incompressible fluid lead to the requirement that $\varphi(x,y,z,t)$ be a harmonic function. Clearly (B·4) requires that $\phi(x,y,z,t)$ also be harmonic, i.e.

$$\nabla^2 \phi = 0. \tag{B·5}$$

With the velocity potential defined by (B·4), the DFSBC and the KFSBC become

$$\eta_t + \phi_z + v_1 \eta_x + v_2 \eta_y - \phi_x \eta_x - \phi_y \eta_y = 0|_{z=\eta(x,y,t)} \tag{B·6}$$

and

$$g\eta - \phi_t + \tfrac{1}{2}[\, v_1^2 - 2v_1 \phi_x + \phi_x^2 + v_2^2 - 2v_2 \phi_y + \phi_y^2 + \phi_z^2 \,] + \frac{p}{\rho} = 0\Big|_{z=\eta(x,y,t)} \tag{B·7}$$

respectively, and the bottom boundary condition (B·3) becomes

$$\phi_z = 0\,|_{z=-h\to-\infty}\,. \tag{B·8}$$

We expand $\phi(x,y,z,t)$ in a Taylor series about the point $z=0$,

$$\phi(x,y,z,t)\,|_{z=\eta(x,y,t)} = \phi(x,y,0,t) + \eta \frac{\partial}{\partial z}\phi(x,y,0,t) + \cdots, \tag{B·9}$$

and substitute this expression in (B·6) and (B·7). Retaining terms up to second order in ϕ and η we find

$$\eta_t + \phi_z + \eta\phi_{zz} + v_1 \eta_x + v_2 \eta_y - \phi_x \eta_x - \phi_y \eta_y = 0\,|_{z=\eta(x,y,t)} \tag{B·10}$$

and

$$g\eta - \phi_t - \eta\phi_{tz} + \tfrac{1}{2}[\, \phi_x^2 + \phi_y^2 + \phi_z^2 \,] + \tfrac{1}{2}[\, v_1^2 - 2v_1\phi_x - 2v_1\eta\phi_{xz} \,]$$
$$+ \tfrac{1}{2}[\, v_2^2 - 2v_2\phi_y - 2v_2\eta\phi_{yz} \,] = 0\Big|_{z=\eta(x,y,t)}\,. \tag{B·11}$$

Solving (B·11) to first order, we find

$$\phi_t = g\eta + \tfrac{1}{2}[(v_1^2 + v_2^2) - 2v_1\phi_x - 2v_2\phi_y], \tag{B·12}$$

and therefore

$$\eta^{(1)} = \frac{1}{g}\phi_t - \frac{v^2}{2g} + \frac{1}{g}(v_1\phi_x + v_2\phi_y). \tag{B·13}$$

Substituting this expression for $\eta^{(1)}$ back into (B·11), we get the second-order expression

$$\eta^{(2)} = \frac{1}{g}\Bigg(\phi_t + \frac{1}{g}\phi_t\phi_{tz} - \frac{v^2}{2g}\phi_{tz} + \frac{\phi_{tz}}{g}(v_1\phi_x + v_2\phi_y)$$
$$- \frac{v^2}{2} + v_1\phi_x + v_2\phi_y - \tfrac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2)$$
$$+ \frac{1}{g}(v_1\phi_{xz} + v_2\phi_{yz})[\phi_t - \frac{v^2}{2} + (v_1\phi_x + v_2\phi_y)] \Bigg). \qquad (B\cdot 14)$$

To obtain η to second order we require ϕ also to second order. We obtain the latter by substituting (B·13) and (B·14) into (B·10) [note that as we are interested in wave effects we can ignore the constant terms in (B·13) and (B·14)]:

$$\phi_z + \frac{\phi_{zz}}{g}\{ \phi_t - \frac{v^2}{2} + v_1\phi_x + v_2\phi_y \} =$$
$$-\frac{v_1}{g}\Bigg(\phi_{xt} + \frac{\phi_{xtz}}{g}(\phi_t - \frac{v^2}{2} + v_1\phi_x + v_2\phi_y) + v_1\phi_{xx} + v_2\phi_{xy}$$
$$+ \frac{\phi_{tz}}{g}(\phi_{xt} + v_1\phi_{xx} + v_2\phi_{xy}) - (\phi_x\phi_{xx} + \phi_y\phi_{xy} + \phi_z\phi_{xz})$$
$$+ \frac{1}{g}(v_1\phi_{xxz} + v_2\phi_{xyz})[\phi_t - \frac{v^2}{2} + v_1\phi_x + v_2\phi_y]$$
$$+ \frac{1}{g}(v_1\phi_{xz} + v_2\phi_{yz})[\phi_{xt} + v_1\phi_{xx} + v_2\phi_{xy}]\Bigg)$$
$$-\frac{v_2}{g}\Bigg(\phi_{yt} + \frac{\phi_{ytz}}{g}(\phi_t - \frac{v^2}{2} + v_1\phi_x + v_2\phi_y) + v_1\phi_{yx} + v_2\phi_{yy}$$
$$+ \frac{\phi_{tz}}{g}(\phi_{yt} + v_1\phi_{yx} + v_2\phi_{yy}) - (\phi_x\phi_{yx} + \phi_y\phi_{yy} + \phi_z\phi_{yz})$$
$$+ \frac{1}{g}(v_1\phi_{yxz} + v_2\phi_{yyz})[\phi_t - \frac{v^2}{2} + v_1\phi_x + v_2\phi_y]$$
$$+ \frac{1}{g}(v_1\phi_{xz} + v_2\phi_{yz})[\phi_{yt} + v_1\phi_{yx} + v_2\phi_{yy}]\Bigg)$$
$$-\frac{1}{g}\Bigg(\phi_{tt} + \frac{\phi_{ttz}}{g}(\phi_t - \frac{v^2}{2} + v_1\phi_x + v_2\phi_y) + v_1\phi_{tx} + v_2\phi_{ty}$$
$$+ \frac{\phi_{tz}}{g}(\phi_{tt} + v_1\phi_{tx} + v_2\phi_{ty}) - (\phi_x\phi_{tx} + \phi_y\phi_{ty} + \phi_z\phi_{tz})$$
$$+ \frac{1}{g}(v_1\phi_{txz} + v_2\phi_{tyz})[\phi_t - \frac{v^2}{2} + v_1\phi_x + v_2\phi_y]$$
$$+ \frac{1}{g}(v_1\phi_{xz} + v_2\phi_{yz})[\phi_{tt} + v_1\phi_{tx} + v_2\phi_{ty}]\Bigg)$$
$$+\frac{\phi_x}{g}(\phi_{xt} + v_1\phi_{xx} + v_2\phi_{xy}) + \frac{\phi_y}{g}(\phi_{yt} + v_1\phi_{yx} + v_2\phi_{yy}). \qquad (B\cdot 15)$$

To first order (B·15) becomes

$$g\phi_z - \frac{v^2}{2}\phi_{zz} + \phi_{tt} = -v_1\phi_{xt} + \frac{v_1 v^2}{2g}\phi_{xtz} - v_1^2\phi_{xx} - v_1 v_2\phi_{xy} + \frac{v_1 v^2}{2g}(v_1\phi_{xxz} + v_2\phi_{xyz})$$

$$- v_2\phi_{yt} + \frac{v_2 v^2}{2g}\phi_{ytz} - v_2^2\phi_{yy} - v_2 v_1\phi_{yx} + \frac{v_2 v^2}{2g}(v_1\phi_{yxz} + v_2\phi_{yyz})$$

$$+ \frac{v^2}{2g}\phi_{ttz} - v_1\phi_{tx} - v_2\phi_{ty} + \frac{v^2}{2g}(v_1\phi_{txz} + v_2\phi_{tyz}). \qquad (B\cdot 16)$$

We now substitute for ϕ_{tt} and ϕ_{ttz} in (B·15) using (B·16), and neglect terms of order 2 in ϕ which are also of order 2 in (v/g). This elimination procedure reduces (B·15) to

$$g\phi_z + \phi_{tt} + 2(v_1\phi_{xt} + v_2\phi_{yt}) + v_1^2\phi_{xx} + 2v_1 v_2\phi_{xy} + v_2^2\phi_{yy} =$$

$$2\{(\phi_x\phi_{tx} + \phi_y\phi_{ty} + \phi_z\phi_{tz}) + v_1(\phi_x\phi_{xx} + \phi_y\phi_{xy} + \phi_z\phi_{xz})$$

$$+ v_2(\phi_x\phi_{yx} + \phi_y\phi_{yy} + \phi_z\phi_{yz})\}. \qquad (B\cdot 17)$$

If we define $\Phi(\mathbf{K}, \omega, z)$ and $\tilde{f}(\mathbf{K}, \omega)$ as in (A·24) and (A·25) then we again require that $\Phi(\mathbf{K}, \omega, z)$ be of the form:

$$\Phi(\mathbf{K}, \omega, z) = \Psi(\mathbf{K}, \omega)\, e^{Kz}.$$

Taking the Fourier transform of (B·17) gives

$$(gK + (-i\omega)^2 2[v_1(iK_x)(-i\omega) + v_2(iK_y)(-i\omega)]$$

$$+ v_1^2(iK_x)^2 + 2v_1 v_2(iK_x)(iK_y) + v_2^2(iK_y)^2)\Psi^{(2)}(\mathbf{K}, \omega) =$$

$$2\int d^2K_1 \int d\omega_1\{(-i(\omega - \omega_1))\,[(i(K_x - K_{1x})(iK_{1x})$$

$$+ (i(K_y - K_{1y}))(iK_{1y}) + K_1|\mathbf{K} - \mathbf{K}_1|\,]$$

$$+ v_1(i(K_x - K_{1x}))\,[(i(K_x - K_{1x}))(iK_{1x}) + (i(K_y - K_{1y}))(iK_{1y}) + K_1|\mathbf{K} - \mathbf{K}_1|\,]$$

$$+ v_2(i(K_y - K_{1y}))\,[(i(K_x - K_{1x}))(iK_{1x}) + (i(K_y - K_{1y}))(iK_{1y}) + K_1|\mathbf{K} - \mathbf{K}_1|\,]\}$$

$$\times \Psi^{(1)}(\mathbf{K}_1, \omega_1)\,\Psi^{(1)}(\mathbf{K} - \mathbf{K}_1, \omega - \omega_1). \qquad (B\cdot 18)$$

(Note that subscripts no longer designate partial differentiation but rather indicate cartesian components.)

We can condense equation (B·18) to

$$[gK + (\omega - \mathbf{v}\cdot\mathbf{K})^2]\Psi^{(2)}(\mathbf{K}, \omega) =$$

$$2i\int d^2K_1 \int d\omega_1\,(((\omega - \omega_1) - \mathbf{v}\cdot(\mathbf{K} - \mathbf{K}_1))(\mathbf{K}_1\cdot(\mathbf{K} - \mathbf{K}_1) - K_1|\mathbf{K} - \mathbf{K}_1|))$$

$$\times \Psi^{(1)}(\mathbf{K}_1, \omega_1)\,\Psi^{(1)}(\mathbf{K} - \mathbf{K}_1, \omega - \omega_1). \qquad (B\cdot 19)$$

Obviously we could reverse the order of the product terms on the right-hand side of (B·17), which would lead to a different, but equivalent expression to (B·18). To account for this we replace (B·19) by the symmetrised quantity

$$[\,gK+(\omega-\mathbf{v}\cdot\mathbf{K})^2]\Psi^{(2)}(\mathbf{K},\omega)=i\int d^2K_1\int d\omega_1\,((\omega-\mathbf{v}\cdot\mathbf{K})(\mathbf{K}_1\cdot(\mathbf{K}-\mathbf{K}_1)-K_1|\mathbf{K}-\mathbf{K}_1|))$$

$$\times\Psi^{(1)}(\mathbf{K}_1,\omega_1)\,\Psi^{(1)}(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1)\,. \qquad \text{(B·20)}$$

From the Fourier transform of (B·13) we have

$$\tilde f^{(1)}(\mathbf{K},\omega)=-\frac{i}{g}(\omega-\mathbf{v}\cdot\mathbf{K})\,\Psi^{(1)}(\mathbf{K},\omega)\,. \qquad \text{(B·21)}$$

Thus our expression for $\Psi^{(2)}(\mathbf{K},\omega)$ in terms of $\tilde f^{(1)}(\mathbf{K},\omega)$ is

$$\Psi^{(2)}(\mathbf{K},\omega)=-\frac{ig(\omega-\mathbf{v}\cdot\mathbf{K})}{(gK-(\omega-\mathbf{v}\cdot\mathbf{K})^2)}\int d^2K_1\int d\omega_1\,(\mathbf{K}_1\cdot(\mathbf{K}-\mathbf{K}_1)-K_1|\mathbf{K}-\mathbf{K}_1|)$$

$$\times\frac{\tilde f^{(1)}(\mathbf{K}_1,\omega_1)}{(\omega_1-\mathbf{v}\cdot\mathbf{K}_1)}\,\frac{\tilde f^{(1)}(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1)}{((\omega-\omega_1)-\mathbf{v}\cdot(\mathbf{K}-\mathbf{K}_1))}\,. \qquad \text{(B·22)}$$

Having determined $\Psi^{(2)}(\mathbf{K},\omega)$ we can now obtain $\tilde f^{(2)}(\mathbf{K},\omega)$ by taking the Fourier transform of (B·14):

$$\tilde f^{(2)}(\mathbf{K},\omega)=\frac{1}{g}\Bigg(-i\omega-\frac{v^2}{2g}(-i\omega)K+v_1(iK_x)+v_2(iK_y)$$

$$-\frac{v^2K}{2g}[v_1(iK_x)+v_2(iK_y)]\Bigg)\Psi^{(2)}(\mathbf{K},\omega)$$

$$+\frac{1}{g^2}\int d^2K_1\int d\omega_1\Bigg(-\frac{g}{2}(K_1|\mathbf{K}-\mathbf{K}_1|\,-\mathbf{K}_1\cdot(\mathbf{K}-\mathbf{K}_1))-i\omega_1K_1(-i(\omega-\omega_1))$$

$$+iv_1(K_x-K_{1x})+iv_2(K_y-K_{1y}))-iK_1(\omega-\omega_1)(iv_1K_{1x}+iv_2K_{1y})$$

$$+K_1[\,v_1^2(iK_{1x})i(K_x-K_{1x})$$

$$+v_1v_2(iK_{1x})i(K_y-K_{1y})+v_2v_1(K_{1y})i(K_x-K_{1x})$$

$$+iv_2^2(K_{1y})i(K_y-K_{1y})]\Bigg)\Psi^{(1)}(\mathbf{K}_1,\omega_1)\,\Psi^{(1)}(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1)\,. \qquad \text{(B·23)}$$

Rewriting the expression in the large parentheses gives

$$\Big(\cdots\Big)=\frac{1}{g^2}\Bigg(-\omega_1K_1((\omega-\omega_1)+\mathbf{v}\cdot(\mathbf{K}-\mathbf{K}_1))-K_1[(\mathbf{v}\cdot\mathbf{K}_1)(\mathbf{v}\cdot(\mathbf{K}-\mathbf{K}_1))]$$

$$+\frac{g}{2}(K_1|\mathbf{K}-\mathbf{K}_1|\,-\mathbf{K}_1\cdot(\mathbf{K}-\mathbf{K}_1))+(\omega-\omega_1)K_1(\mathbf{v}\cdot\mathbf{K}_1)\Bigg)\,. \qquad \text{(B·24)}$$

Once again we need to symmetrise this quantity by considering the product terms in (B·14) in the reverse order. The symmetrised form of (B·24) is

$$\left(\cdots\right)=\frac{1}{2g^2}(K_1+|\mathbf{K}-\mathbf{K}_1|)(\mathbf{v}\cdot(\mathbf{K}-\mathbf{K}_1)\omega_1$$
$$+\mathbf{v}\cdot\mathbf{K}_1(\omega-\omega_1)-\omega_1(\omega-\omega_1)-\mathbf{v}\cdot\mathbf{K}_1-\mathbf{v}\cdot(\mathbf{K}-\mathbf{K}_1))$$
$$+\frac{1}{2g}(K_1|\mathbf{K}-\mathbf{K}_1|-\mathbf{K}_1\cdot(\mathbf{K}-\mathbf{K}_1)). \tag{B·25}$$

Substituting (B·22) and (B·25) in (B·23) we get

$$\tilde{f}^{(2)}(\mathbf{K},\omega)=$$
$$\frac{(\omega-\mathbf{v}\cdot\mathbf{K})}{((\omega-\mathbf{v}\cdot\mathbf{K})^2-gK)}\left((g-\frac{v^2K}{2})(\omega-\mathbf{v}\cdot\mathbf{K})\right)\int d^2K_1\int d\omega_1$$
$$\times\frac{\mathbf{K}_1\cdot(\mathbf{K}-\mathbf{K}_1)-K_1|\mathbf{K}-\mathbf{K}_1|}{(\omega_1-\mathbf{v}\cdot\mathbf{K}_1)(\omega-\omega_1+\mathbf{v}\cdot(\mathbf{K}-\mathbf{K}_1))}\tilde{f}^{(1)}(\mathbf{K}_1,\omega_1)\,\tilde{f}^{(1)}(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1)$$
$$+\tfrac{1}{2}\int d^2K_1\int d\omega_1\{(K_1+|\mathbf{K}-\mathbf{K}_1|)[(\omega-\omega_1)(\omega_1-\mathbf{v}\cdot\mathbf{K}_1)+(\mathbf{v}\cdot(\mathbf{K}-\mathbf{K}_1))(\mathbf{v}\cdot\mathbf{K}_1-\omega_1)]$$
$$+g(K_1|\mathbf{K}-\mathbf{K}_1|-\mathbf{K}_1\cdot(\mathbf{K}-\mathbf{K}_1))\}\frac{\tilde{f}^{(1)}(\mathbf{K}_1,\omega_1)}{(\omega_1-\mathbf{v}\cdot\mathbf{K}_1)}\frac{\tilde{f}^{(1)}(\mathbf{K}-\mathbf{K}_1,\omega-\omega_1)}{(\omega-\omega_1-\mathbf{v}\cdot(\mathbf{K}-\mathbf{K}_1))}. \tag{B·26}$$

Having determined the expression for $\tilde{f}^{(2)}(\mathbf{K},\omega)$ we are now in a position to determine the radar cross section due to single scatter, to second order. The quantity we need to evaluate is the autocorrelation function for the scatter field as defined in (2·9). By analogy with (4·1) and (4·2) we can write the product of second-order terms as

$$\langle\tilde{f}^{(2)}(\mathbf{K},\Omega)\,\tilde{f}^{(2)*}(\mathbf{K}',\Omega')\rangle=$$
$$\int d^2K_1\int d\omega_1\int d^2K_1'\int d\omega_1'\,Y(\mathbf{K}-\mathbf{K}_1,\Omega-\Omega_1,\mathbf{K}_1,\Omega_1)$$
$$\times Y^*(\mathbf{K}'-\mathbf{K}_1',\Omega'-\Omega_1',\mathbf{K}_1',\Omega_1')\,\langle\tilde{f}^{(1)}(\mathbf{K}_1,\Omega_1)\,\tilde{f}^{(1)}(\mathbf{K}-\mathbf{K}_1,\Omega-\Omega_1)$$
$$\times\tilde{f}^{(1)*}(\mathbf{K}_1',\Omega_1')\,\tilde{f}^{(1)*}(\mathbf{K}'-\mathbf{K}_1',\Omega'-\Omega_1')\rangle, \tag{B·27}$$

where $Y(\mathbf{K}-\mathbf{K}_1,\Omega-\Omega_1,\mathbf{K}_1,\Omega_1)$ is given by

$$Y(\cdots)=\left(\frac{(\Omega-\mathbf{v}\cdot\mathbf{K})^2(2g-v^2K)}{2((\Omega-\mathbf{v}\cdot\mathbf{K})^2-gK)}-\frac{g}{2}\right)$$
$$\times\frac{\mathbf{K}_1\cdot(\mathbf{K}-\mathbf{K}_1)-K_1|\mathbf{K}-\mathbf{K}_1|}{(\Omega_1-\mathbf{v}\cdot\mathbf{K}_1)(\Omega-\Omega_1-\mathbf{v}\cdot(\mathbf{K}-\mathbf{K}_1))}+\tfrac{1}{2}K|\mathbf{K}-\mathbf{K}_1|\;. \tag{B·28}$$

If we assume the random amplitudes $\tilde{f}^{(1)}(\mathbf{K},\Omega)$ are Gaussian distributed, then by (3·9) and (3·10):

$$\langle \tilde f^{(1)}(\mathbf{K}_1,\Omega_1)\tilde f^{(1)}(\mathbf{K}-\mathbf{K}_1,\Omega-\Omega_1)\,\tilde f^{(1)*}(\mathbf{K}_1{}',\Omega'_1)\,\tilde f^{(1)*}(\mathbf{K}'-\mathbf{K}_1{}',\Omega'-\Omega_1{}')\rangle =$$
$$S(\mathbf{K}_1,\Omega_1)\,S(\mathbf{K}_1{}',\Omega_1{}')\,\delta(\mathbf{K})\,\delta(\Omega)\,\delta(\mathbf{K}')\,\delta(\Omega')$$
$$+S(\mathbf{K}_1,\Omega_1)\,S(\mathbf{K}-\mathbf{K}_1,\Omega-\Omega_1)\,\delta(\mathbf{K}_1-\mathbf{K}_1{}')\,\delta(\Omega_1-\Omega_1{}')$$
$$\times\delta(\mathbf{K}-\mathbf{K}_1-\mathbf{K}'+\mathbf{K}_1{}')\,\delta(\Omega-\Omega_1-\Omega'+\Omega_1{}')$$
$$+S(\mathbf{K}_1,\Omega_1)\,S(\mathbf{K}-\mathbf{K}_1,\Omega-\Omega_1)\,\delta(\mathbf{K}_1-\mathbf{K}'+\mathbf{K}_1{}')\,\delta(\Omega_1-\Omega'+\Omega_1{}')$$
$$\times\delta(\mathbf{K}-\mathbf{K}_1-\mathbf{K}_1{}')\,\delta(\Omega-\Omega_1-\Omega_1{}')\,. \qquad (B\cdot 29)$$

We momentarily ignore the first term as it requires

$$\mathbf{K}=\mathbf{K}'=0\,.$$

If we perform the integrations over $\mathbf{K}'$ and Ω' in (B·27) by enforcing the respective delta functions we obtain

$$\langle f(\mathbf{r},t)\,f^*(\mathbf{r}',t')\rangle = \int d^2K\int d\Omega\int d^2K_1\int d\Omega_1\int d^2K_1{}'\int d\Omega_1{}'\,(\mathbf{k}_0\cdot\mathbf{K})^2$$
$$\times Y(\mathbf{K}-\mathbf{K}_1,\Omega-\Omega_1,\mathbf{K}_1,\Omega_1)\,Y^*(\mathbf{K}-\mathbf{K}_1{}',\Omega-\Omega_1{}',\mathbf{K}_1{}',\Omega_1{}')$$
$$\times e^{[i\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')-i\Omega(t-t')]}\,S(\mathbf{K}_1,\Omega_1)\,S(\mathbf{K}-\mathbf{K}_1,\Omega-\Omega_1)$$
$$\times\delta(\mathbf{K}_1-\mathbf{K}_1{}')\,\delta(\Omega_1-\Omega_1{}')\,\delta(\mathbf{K}-\mathbf{K}_1-\mathbf{K}_1{}')\,\delta(\Omega-\Omega_1-\Omega_1{}')\,. \qquad (B\cdot 30)$$

By enforcing the remaining delta functions we can integrate over $\mathbf{K}_1{}'$ and $\Omega_1{}'$ in (B·30):

$$\langle f(\mathbf{r},t)\,f^*(\mathbf{r}',t')\rangle = \int d^2K\int d\Omega\int d^2K_1\int d\Omega_1\,(\mathbf{k}_0\cdot\mathbf{K})^2 e^{[i\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')-i\Omega(t-t')]}$$
$$\times S(\mathbf{K}_1,\Omega_1)\,S(\mathbf{K}-\mathbf{K}_1,\Omega-\Omega_1)\,[Y^2(\mathbf{K}-\mathbf{K}_1,\Omega-\Omega_1,\mathbf{K}_1,\Omega_1)$$
$$+\,Y(\mathbf{K}-\mathbf{K}_1,\Omega-\Omega_1,\mathbf{K}_1,\Omega_1)Y(\mathbf{K}_1,\Omega_1,\mathbf{K}-\mathbf{K}_1,\Omega-\Omega_1)]\,. \qquad (B\cdot 31)$$

From (B·28) $Y(\mathbf{K}-\mathbf{K}_1,\Omega-\Omega_1,\mathbf{K}_1,\Omega_1)=Y(\mathbf{K}_1,\Omega_1,\mathbf{K}-\mathbf{K}_1,\Omega-\Omega_1)$ and thus we can determine the spectral power density due to the second-order terms in $\tilde f(\mathbf{K},\Omega)$ by combining (B·31), (2·9) and (2·2):

$$\Theta^{(2)}(\omega)=\frac{1}{2\pi}\int d\tau\, e^{i\omega\tau}\,\frac{|c|^2|E_0|^2}{R^2}\int_A d^2r\int_A d^2r'2\int d^2K\int d\Omega\int d^2K_1\int d\Omega_1\,(\mathbf{k}_0\cdot\mathbf{K})^2$$
$$\times 2Y^2(\mathbf{K}-\mathbf{K}_1,\Omega-\Omega_1,\mathbf{K}_1,\Omega_1)\,S(\mathbf{K}_1,\Omega_1)\,S(\mathbf{K}-\mathbf{K}_1,\Omega-\Omega_1)$$
$$\times e^{[i\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')-i\Omega(t-t')]}\,e^{[-i\Delta\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-i\Omega\tau]}\,. \qquad (B\cdot 32)$$

We carry out the spatial integration by changing to a new set of coordinates:

$$\mathbf{u} = \mathbf{r} - \mathbf{r}' \quad \text{and} \quad \mathbf{w} = \tfrac{1}{2}(\mathbf{r} + \mathbf{r}')$$

Integrating over $\mathbf{w}$ introduces a factor of A (scattering area), while extending the limits on the $\mathbf{u}$ integral to infinity will introduce a factor $(2\pi)^2\,\delta(\mathbf{K} - \Delta\mathbf{k})$. Integrating over τ introduces a second delta function $(2\pi)\,\delta(\Omega - \omega + \omega_0)$ yielding

$$\Theta^{(2)}(\omega) = \frac{2\,|c|^2 |E_0|^2 A(2\pi)^2}{R^2} \int d^2K \int d\Omega \int d^2K_1 \int d\Omega_1$$

$$\times\, (\mathbf{k}_0 \cdot \mathbf{K})^2\, S(\mathbf{K}_1, \Omega_1)\, S(\mathbf{K} - \mathbf{K}_1, \Omega - \Omega_1)$$

$$\times\, Y^2(\mathbf{K} - \mathbf{K}_1, \Omega - \Omega_1, \mathbf{K}_1, \Omega_1)\, \delta(\mathbf{K} - \Delta\mathbf{k})\, \delta(\Omega - \Delta\omega), \qquad \text{(B·33)}$$

where

$$\Delta\mathbf{k} = \mathbf{k}_S - \mathbf{k}_0 \quad \text{and} \quad \Delta\omega = \omega - \omega_0 .$$

Enforcing the remaining delta functions enables us to integrate over $\mathbf{K}$ and Ω,

$$\Theta^{(2)}(\omega) = \frac{2\,|c|^2 |E_0|^2 A(2\pi)^2}{R^2} \int d^2K_1 \int d\Omega_1$$

$$\times\, (\mathbf{k}_0 \cdot \Delta\mathbf{k})^2\, Y^2(\mathbf{K}_2, \Omega_2, \mathbf{K}_1, \Omega_1)\, S(\mathbf{K}_1, \Omega_1)\, S(\mathbf{K}_2, \Omega_2), \qquad \text{(B·34)}$$

where

$$\mathbf{K}_2 = \Delta\mathbf{k} - \mathbf{K}_1 \quad \text{and} \quad \Omega_2 = \Delta\omega - \Omega_1 .$$

Substituting for Ω_2, integrating over Ω_1, and making use of (4·10) to evaluate the product term $S(\mathbf{K}_1, \Omega_1)\, S(\mathbf{K}_2, \Omega_2)$ allows us to rewrite (B·34) as

$$\Theta^{(2)}(\omega) = \frac{2\,|c|^2 |E_0|^2 A(2\pi)^2}{R^2} \int d^2K_1 \sum_{m_1=\pm 1} \sum_{m_2=\pm 1} (\mathbf{k}_0 \cdot \Delta\mathbf{k})^2\, \tfrac{1}{4} S_{m_1\mathbf{K}_1} S_{m_2\mathbf{K}_2}$$

$$\times\, Y^2(\mathbf{K}_2, m_2\Omega_{m_2\mathbf{K}_2}, \mathbf{K}_1, m_1\Omega_{m_1\mathbf{K}_1})\,\delta(\Delta\omega - m_1\Omega_{m_1\mathbf{K}_1} - m_2\Omega_{m_2\mathbf{K}_2}). \qquad \text{(B·35)}$$

The first-order dispersion equation in water moving with velocity $\mathbf{v}$ is

$$\Omega_{\mathbf{K}} = \sqrt{gK} + \mathbf{v} \cdot \mathbf{K}. \qquad \text{(B·36)}$$

Making this substitution, and considering backscatter of our incident radiation, i.e.

$$\mathbf{k}_S = -\mathbf{k}_0 \quad \text{hence} \quad \Delta\mathbf{k} = -2\mathbf{k}_0 = \mathbf{K}_1 + \mathbf{K}_2 ,$$

we find

$$\Theta^{(2)}(\omega)=\frac{|c|^2|E_0|^2 A(2\pi)^2}{2R^2}\int d^2K_1\,(2k_0^2)^2$$

$$\times\sum_{m_1=\pm1}\sum_{m_2=\pm1} S_{m_1\mathbf{K}_1}S_{m_2\mathbf{K}_2}\,\Upsilon^2(\mathbf{K}_2,m_2\Omega_{m_2\mathbf{K}_2},\mathbf{K}_1,m_1\Omega_{m_1\mathbf{K}_1})$$

$$\times\delta(\Delta\omega+2\mathbf{k}_0\cdot\mathbf{v}-m_1\sqrt{gK_1}-m_2\sqrt{gK_2})\,. \qquad \text{(B}\cdot\text{37)}$$

The contribution to the radar cross section due to second-order hydrodynamic considerations is thus

$$\sigma^{(2)}(\omega)=(2\pi)^3\,|c|^2\,(2k_0^2)^2\int d^2K_1$$

$$\times\sum_{m_1=\pm1}\sum_{m_2=\pm1} S_{m_1\mathbf{K}_1}S_{m_2\mathbf{K}_2}\,\Upsilon^2(\mathbf{K}_2,m_2\Omega_{m_2\mathbf{K}_2},\mathbf{K}_1,m_1\Omega_{m_1\mathbf{K}_1})$$

$$\times\delta(\Delta\omega+2\mathbf{k}_0\cdot\mathbf{v}-m_1\sqrt{gK_1}-m_2\sqrt{gK_2})\,, \qquad \text{(B}\cdot\text{38)}$$

where $\Upsilon^2(\mathbf{K}_2,m_2\Omega_{m_2\mathbf{K}_2},\mathbf{K}_1,m_1\Omega_{m_1\mathbf{K}_1})$ is given by

$$\Upsilon^2(\cdot\cdot\cdot)=\left(\tfrac{1}{2}(K_1+K_2+\frac{(\mathbf{K}_1\cdot\mathbf{K}_2-K_1K_2)}{m_1m_2\sqrt{K_1}\sqrt{K_2}}\;\frac{(1-2v^2k_0/g)\zeta^2+\omega_B^2}{\zeta^2-\omega_B^2}\right)^2, \qquad \text{(B}\cdot\text{39)}$$

where $\zeta=m_1\sqrt{gK_1}+m_2\sqrt{gK_2}$ and ω_B is the Bragg frequency ($\sqrt{2gk_0}$).

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