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Cylindrical and Spherical Solitons at the Critical Density of Negative Ions in a Generalised Multicomponent Plasma

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Abstract

Propagation of nonlinear ion–acoustic waves in generalised multicomponent plasmas bounded by cylindrical and spherical geometries is investigated. At the critical density of negative ions where the nonlinearity of the Korteweg–deVries (K–dV) equation vanishes, the ion–acoustic solitary wave is described by a modified K–dV (mK–dV) equation. It is also emphasised that near the critical density neither the K–dV nor mK–dV equation is sufficient to describe fully the ion–acoustic waves and thus there is a need to derive a further mK–dV (fmK–dV) equation in the vicinity of this critical density. Furthermore, the amplitude variations of the K–dV and mK–dV solitons depending on the limitations of geometrical effects are also discussed, emphasising that the results could be of interest for diagnosing the soliton properties of laboratory plasmas.

1. Introduction

The concept of soliton propagation and its interactions has become of increasing interest in plasma dynamics, from both the theoretical and experimental points of view. The characteristics of long wavelength ion–acoustic waves have been explored thoroughly and it is well known that the reductive perturbation method is one of the most established approaches to study the K–dV solitons in relation to laboratory plasmas. Beginning with Washimi and Taniuti (1966), many authors such as Su and Gardner (1969), Jeffrey and Kakutani (1972), Ikezi (1973), Tran (1974), Das and Tagare (1975), Lonngren (1983), Raychaudhuri *et al.* (1985) pioneered the study of solitons in simple as well as multicomponent plasmas. Das (1975) and Das and Tagare (1975) have shown the existence of negative ion concentrations at which the nonlinear coefficient of the K–dV equation vanishes. Later, the existence of solitons at the critical density of negative ions in the form of compressive and rarefactive solitons was observed theoretically (Watanabe 1984; Hase *et al.* 1985; Verheest 1988) and experimentally (Nakamura and Tsukabayashi 1985; Nakamura *et al.* 1985; Nakamura 1987).

The present investigation is a sequel to earlier work (Singh and Das 1989; Das *et al.* 1989; Das and Singh 1990, 1991) dealing with the interaction of negative ions on solitary wave propagations. We consider a generalised multicomponent bounded plasma that includes multiple electron temperatures with ions of both kinds and ion beams. In a geometrically bounded plasma

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the results from cylindrical and spherical geometries are compared with the planar K–dV solitons, raising possible implications for further verification in laboratory plasmas.

2. Mathematical Derivation of the K–dV and mK–dV Equations

To derive the K–dV equation from the basic equations governing the generalised multicomponent plasma we have the basic one-dimensional normalised equations (Das 1978)

$$\frac{\partial \bar{n}_\alpha}{\partial \bar{t}} + \nabla . (\bar{n}_\alpha \bar{v}_\alpha) = 0, \tag{1}$$

$$\frac{\partial \bar{v}_\alpha}{\partial \bar{t}} + \bar{v}_\alpha . \nabla \bar{v}_\alpha + q_\alpha \mu_\alpha \nabla \bar{\phi} = 0, \tag{2}$$

supplemented by the Poisson equation

$$\nabla^2 \bar{\phi} = \bar{n}_{\rm el} + \bar{n}_{\rm eh} - \sum_\alpha q_\alpha \bar{n}_\alpha , \tag{3}$$

where $\alpha = i, j, b$ stands for positive ions, negative ions and ion beams respectively, and where $q_i = q_b = -q_j = 1$. The normalised plasma parameters are defined as

$$\bar{n}_\alpha = n_\alpha / n_0, \quad \bar{n}_{\rm el,h} = n_{\rm el,h}/n_0, \quad \mu_\alpha = m_i/m_\alpha ,$$

$$\bar{v}_\alpha = v_\alpha (KT_{\rm ef}/m_\alpha)^{-1/2}, \quad T_{\rm ef} = T_{\rm el} T_{\rm eh}/(\mu T_{\rm eh} + \nu T_{\rm el}),$$

$$\bar{r} = r(KT_{\rm ef}/4\pi e^2 n_0)^{1/2}, \quad \bar{t} = t(4\pi e^2 n_0/m_\alpha)^{1/2},$$

where n_α is the density of the α-type charged particle moving with velocity v_α and $n_{\rm el}$, $n_{\rm eh}$ are the densities of the low and high temperature electrons normalised to the background total electron density n_0. Further, r is the radial distance for the cylindrical and spherical geometries (r becomes x along the x direction for planar geometry), t is the time and ϕ is the electrostatic potential. Furthermore, we assume the following boundary conditions at $|r| \to \infty$ (omitting bars hereafter):

$$\text{(i)} \quad n_\alpha \to n_\alpha^{(0)}, \quad n_{\rm el} \to \mu, \quad n_{\rm eh} \to \nu,$$

$$\text{(ii)} \quad v_\alpha \to 0, \quad \phi \to 0, \tag{4}$$

and (iii) the overall charge neutrality condition is maintained in the plasma and given by

$$\sum_\alpha q_\alpha n_\alpha^{(0)} = \mu + \nu, \tag{5}$$

where μ and ν are the initial densities of the low and high electron temperatures.

Moreover, the plasma is assumed to be isothermal and the isothermality is obtained through the Boltzmann relations (Das *et al.* 1986) as

$$n_{\rm el} = \mu \exp\left(\frac{\phi}{\mu + \nu\beta}\right), \qquad n_{\rm eh} = \nu \exp\left(\frac{\beta\phi}{\mu + \nu\beta}\right), \tag{6}$$

where $\beta = T_{el}/T_{eh}$.

Now, we first transform the (r, t) coordinates to the new variables in the form

$$\overline{X} = r - \lambda t, \qquad \overline{T} = \lambda t, \tag{7}$$

and then stretching coordinates ξ and τ are expressed as

$$\xi = \epsilon^{1/2}\overline{X}, \qquad \tau = \epsilon^{3/2}\overline{T}, \tag{8}$$

where λ is the phase velocity of the wave propagation along the r direction and will be determined in a self-consistent manner. Furthermore, all the plasma parameters $\Psi \equiv (n_\alpha, v_\alpha, \phi)$ are expanded as a power series in ϵ about the equilibrium state as

$$\Psi = \sum_{s=0}^{\infty} \epsilon^s \Psi^{(s)}, \tag{9}$$

with the conditions

$$v_\alpha^{(0)} = \phi^{(0)} = 0. \tag{10}$$

Now employing the expansions (9) and (10) together with (6)–(8) in the basic equations (1)–(3), and then equating the lowest order terms in ϵ, we get the first order perturbed plasma parameters for the planar, cylindrical and spherical geometries:

$$n_\alpha^{(1)} = q_\alpha \mu_\alpha n_\alpha^{(0)} \phi^{(1)}/\lambda^2, \tag{11}$$

$$v_\alpha^{(1)} = q_\alpha \mu_\alpha \phi^{(1)}/\lambda, \tag{12}$$

$$\sum_\alpha q_\alpha n_\alpha^{(1)} = \phi^{(1)}, \tag{13}$$

from which the phase velocity λ is expressed as

$$\lambda^2 = \sum_\alpha \mu_\alpha n_\alpha^{(0)}. \tag{14}$$

The next higher order terms in ϵ give a system of equations involving the second order perturbed quantities in the following form:

$$\lambda \frac{\partial n_\alpha^{(1)}}{\partial \tau} - \lambda \frac{\partial n_\alpha^{(2)}}{\partial \xi} + n_\alpha^{(0)} \frac{\partial v_\alpha^{(2)}}{\partial \xi} + \frac{\partial}{\partial \xi}(n_\alpha^{(1)} v_\alpha^{(1)})$$
$$- \frac{2L\lambda\xi}{\tau}\frac{\partial n_\alpha^{(1)}}{\partial \xi} + \frac{2Ln_\alpha^{(0)}}{\tau}\frac{\partial}{\partial \xi}(\xi v_\alpha^{(1)}) = 0, \tag{15}$$

$$\lambda \frac{\partial v_\alpha^{(1)}}{\partial \tau} - \lambda \frac{\partial v_\alpha^{(2)}}{\partial \xi} + v_\alpha^{(1)} \frac{\partial v_\alpha^{(1)}}{\partial \xi} + q_\alpha \mu_\alpha \frac{\partial \phi^{(2)}}{\partial \xi} = 0, \tag{16}$$

$$\sum_\alpha q_\alpha n_\alpha^{(2)} = \phi^{(2)} + \frac{\mu + \nu\beta^2}{2(\mu + \nu\beta)^2}(\phi^{(1)})^2 - \frac{\partial^2 \phi^{(1)}}{\partial \xi^2}, \tag{17}$$

where L represents the geometry of the plasma: for the planar $L = 0$, the cylindrical $L = \frac{1}{2}$, while for spherical geometry $L = 1$.

Following the usual procedure (Das *et al.* 1989*a*, 1989*b*) and using the first order results, the relations (15)–(17) yield the desired K–dV equation in the general form:

$$\frac{\partial \phi^{(1)}}{\partial \tau} + L\frac{\phi^{(1)}}{\tau} + A_1 \phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \xi} + \frac{1}{2}\frac{\partial^3 \phi^{(1)}}{\partial \xi^3} = 0, \tag{18}$$

where

$$A_1 = \frac{1}{2\lambda^4}\left(3\sum_\alpha q_\alpha \mu_\alpha^2 n_\alpha^{(0)} - \lambda^4(\mu + \nu\beta^2)/(\mu + \nu\beta)^2\right). \tag{19}$$

In order to obtain the steady state solution of the K–dV equation (18) we introduce a new parameter $\tilde{\phi} = A_1 \phi^{(1)}$ which transforms the K–dV equation to the simplified form

$$\frac{\partial \tilde{\phi}}{\partial \tau} + L\frac{\tilde{\phi}}{\tau} + \tilde{\phi}\frac{\partial \tilde{\phi}}{\partial \xi} + \frac{1}{2}\frac{\partial^3 \tilde{\phi}}{\partial \xi^3} = 0. \tag{20}$$

The usual procedure (Jeffrey and Kakutani 1972) gives the soliton solution of (20) in the form

$$\tilde{\phi} = \tilde{\phi}_0(\tau_0/\tau)^L \operatorname{sech}^2 [\sim], \tag{21}$$

where $\tilde{\phi}_0$ is the amplitude of the corresponding planar soliton and the expression within the square brackets is connected with dispersiveness and the geometry varies from model to model. The solution of (2) for the planar case ($L = 0$) is

$$\tilde{\phi} = \tilde{\phi}_0 \operatorname{sech}^2[(\xi - U\tau)/\delta], \tag{22}$$

where $\delta\,[= (2/U)^{1/2}]$ is the width of the planar soliton and U is the velocity of the frame of the transformed coordinate. The corresponding cylindrical ($L = \frac{1}{2}$)and spherical ($L = 1$) soliton solutions (Maxon and Viecelli 1974*a*, 1974*b*) are

$$\tilde{\phi} = \tilde{\phi}_0(\tau_0/\tau)^{1/2} \operatorname{sech}^2\left[\left(\frac{\tilde{\phi}_0}{6}(\tau_0/\tau)^{1/2}\right)^{1/2} \times \left(\xi - \xi_0 + \frac{2\tilde{\phi}_0}{3}|\tau_0|^{1/2}(|\tau|^{1/2} - |\tau_0|^{1/2})\right)\right], \tag{23}$$

$$\tilde{\phi} = \tilde{\phi}_0(\tau_0/\tau) \operatorname{sech}^2\left[\left(\frac{\tilde{\phi}_0}{6}(\tau_0/\tau)\right)^{1/2} \times \left(\xi - \xi_0 - \tfrac{1}{3}\tau_0\tilde{\phi}_0 \log(\tau/\tau_0)\right)\right]. \tag{24}$$

From the solutions (22)–(24) it is obvious that the soliton amplitude plays a vital role in exhibiting ion-acoustic solitary waves in the plasma and it depends functionally on the variation of the nonlinear coefficient A_1 which, in fact, depends on the plasma parameters. The observations first noted by Das (1975) and later extensively observed by others (Nakamura and Tsukabayashi 1985; Das *et al.* 1986; Hase *et al.* 1985; Verheest 1988) concluded that the nonlinear coefficient A_1 can be positive, zero or negative and this variation of A_1 arises generally due to the multiple electron temperatures and negative ionic species. The critical density of negative ion concentrations, at which the nonlinear term in the K–dV equation disappears, shows that the amplitude variation has the singularity and forms a precursor therein. However, from earlier knowledge it has been concluded that, whatever the plasma constituents, the positive particles always introduce barriers from which the ion–acoustic waves are reflected. The possibility of large amplitude waves does not arise and, as a result, the reductive perturbation technique is capable of yielding finite amplitude K–dV solitons from the basic equations. In the present plasma model, apart from the case $A_1 = 0$, there are always two types of K–dV solitons, either the compressive soliton when $A_1 > 0$ or the rarefactive one when $A_1 < 0$. However, with the addition of heavier negative ions in the plasma, the range of the compressive soliton becomes wider. The solution of the K–dV equation shows that in cylindrical geometry the amplitude increases faster than $\tau^{-1/2}$ and the width decreases faster than $\tau^{1/4}$, as compared with the planar soliton, whereas in spherical geometry the amplitude changes faster than τ^{-1} while the width decreases with the order of $\tau^{1/2}$. Thus, in both geometries, the amplitude increases and the width decreases as the soliton moves inwards with an increasing propagation speed, and thereby the solitons in a bounded plasma travel faster than the planar soliton. However, the presence of multiple electron temperatures or negative ions changes the characteristic properties of the soliton. The negatively charged particles introduce a critical density dividing the range of the concentration variation into two regions where the compressive and rarefactive solitons, similar to the case of planar solitons, are also observed in the bounded plasma.

The expression for $A_1 = 0$ shows that the critical density occurs at the point where the A_1 variation crosses the axis of concentration of negative ions. Now, the variation of A_1 plotted against negative ion concentration exhibits a unique critical density, whereas $A_1 = 0$ shows that there will be two roots in μ indicating the existence of two critical densities of electrons. When the temperature ratio is $\beta \approx 1$, the later critical density disappears. Thus, when the plasma contains multiple electron temperatures along with multiple ionic species of both kinds, there is the possibility of having multiple critical densities due to which the exhibition of multi-layer solitary waves is possible. The analysis then becomes more complicated and one has to be careful tackling such problems in laboratory plasmas.

A numerical estimation of the soliton characteristics in planar geometry has been done experimentally with the ionic species (Ar^+, F^-) and (Ar^+, SF_6^-) by Nakamura and Tsukabayashi (1985) and Nakamura (1987). Such observable features of solitons have also been theoretically discussed in plasmas involving multiple-electron temperatures (Singh and Das 1989). However, due to the appearance of the singularity it seems that the K–dV equation cannot describe the ion–acoustic waves for all the plasma parameters. This happens because of negative ions in the plasmas. Furthermore, experimental evidence shows that an appreciable fraction of negative ions can be found in other plasmas, such as in the Thermonuclear Fusion and Q-machine device in which the ion–acoustic waves are not explored at all. From earlier investigations we conclude that the characteristic role of ion–acoustic waves could be exhibited in laboratory alkaline plasmas. However, experimental verification in the case of cylindirical and spherical geometries could be obtained with the ionic species (Ar^+, F^-), (Ar^+, SF_6^-) and with suitable beams, so that our theoretical results may be of interest in laboratory plasmas.

The K–dV equation, so far, analyses compressive and rarefactive solitons in plasma and predicts the formation of a precursor at the critical density of negative ions. However, the characteristics of the waves at this critical density have not yet been discussed, which may be a new area for studying solitary waves in generalised multicomponent plasmas. In order to examine the behaviour of ion–acoustic solitary waves at this critical density, we consider the higher order nonlinearities together with the modified stretched coordinates ξ and τ defined by

$$\xi = \epsilon(r - \lambda t), \qquad \tau = \epsilon^3 \lambda t. \tag{25}$$

Using the expansion of parameters given by (9) and (10) along with the newly defined coordinates (25) in the basic equations and then the equating of the lowest order terms in ϵ yields the same expression for the phase velocity λ. The changes occurred when the next higher order in ϵ is taken into consideration, giving the following relations:

$$n_\alpha^{(2)} = \frac{3n_\alpha^{(0)}\mu_\alpha^2}{2\lambda^4}(\phi^{(1)})^2 + \frac{q_\alpha n_\alpha^{(0)}\mu_\alpha}{\lambda^2}\phi^{(2)}, \tag{26}$$

$$v_\alpha^{(2)} = \frac{\mu_\alpha^2}{2\lambda^3}(\phi^{(1)})^2 + \frac{q_\alpha \mu_\alpha}{\lambda}\phi^{(2)}, \tag{27}$$

$$\sum_\alpha q_\alpha n_\alpha^{(2)} = \phi^{(2)} + \frac{\mu + \nu\beta^2}{2(\mu+\nu\beta)^2}(\phi^{(1)})^2. \tag{28}$$

The elimination of $n_\alpha^{(2)}$ from (26) and (28) gives

$$\left(\frac{1}{\lambda^2}\sum_\alpha \mu_\alpha n_\alpha^{(0)} - 1\right)\phi^{(2)} + \tfrac{1}{2}\left(\frac{3}{\lambda^4}\sum_\alpha q_\alpha \mu_\alpha^2 n_\alpha^{(0)} - (\mu+\nu\beta^2)/(\mu+\nu\beta)^2\right)(\phi^{(1)})^2 = 0, \tag{29}$$

which shows that the Poisson equation remains valid even when the nonlinear coefficient A_1 vanishes. Based on these results, the next higher order terms in ϵ give

$$\lambda\frac{\partial n_\alpha^{(1)}}{\partial \tau} - \lambda\frac{\partial n_\alpha^{(3)}}{\partial \xi} + n_\alpha^{(0)}\frac{\partial v_\alpha^{(3)}}{\partial \xi} + \frac{\partial}{\partial \xi}(n_\alpha^{(1)} v_\alpha^{(2)}) + \frac{\partial}{\partial \xi}(n_\alpha^{(2)} v_\alpha^{(1)}) - \frac{2L\lambda\xi}{\tau}\frac{\partial n_\alpha^{(1)}}{\partial \xi} + \frac{2Ln_\alpha^{(0)}}{\tau}\frac{\partial}{\partial \xi}(\xi v_\alpha^{(1)}) = 0, \tag{30}$$

$$\lambda\frac{\partial v_\alpha^{(1)}}{\partial \tau} - \lambda\frac{\partial v_\alpha^{(3)}}{\partial \xi} + \frac{\partial}{\partial \xi}(v_\alpha^{(1)} v_\alpha^{(2)}) + q_\alpha\mu_\alpha\frac{\partial \phi^{(3)}}{\partial \xi} = 0, \tag{31}$$

$$\sum_{\alpha} q_\alpha n_\alpha^{(3)} = \phi^{(3)} + \frac{\mu+\nu\beta^2}{(\mu+\nu\beta)^2}(\phi^{(1)}\phi^{(2)}) + \frac{\mu+\nu\beta^3}{6(\mu+\nu\beta)^3}(\phi^{(1)})^3 - \frac{\partial^2\phi^{(1)}}{\partial\xi^2}, \tag{32}$$

which yield, using the first and second order results in ϵ, the mK–dV equation as

$$\frac{\partial\phi^{(1)}}{\partial\tau} + L\frac{\phi^{(1)}}{\tau} + A_2(\phi^{(1)})^2\frac{\partial\phi^{(1)}}{\partial\xi} + \frac{1}{2}\frac{\partial^3\phi^{(1)}}{\partial\xi^3} = 0, \tag{33}$$

where

$$A_2 = \frac{1}{4\lambda^6}\left(15\sum_{\alpha}\mu_\alpha^3 n_\alpha^{(0)} - \lambda^6(\mu+\nu\beta^3)/(\mu+\nu\beta)^3\right). \tag{34}$$

The solution of the mK–dV equation (33), based on the usual procedure, is of the soliton 'sech' type given by

$$\phi^{(1)} = \pm\phi_0 \operatorname{sech}\left(\frac{\phi_0^2}{6}\right)^{1/2}\left(\xi + \frac{\phi_0^2}{6}\tau\right) \tag{35}$$

for the planar soliton ($L = 0$). The corresponding solutions for the bounded plasmas are similarly obtainable. In the case of the ion–beam plasma with negative ions, the nonlinear coefficient A_2 is positive at the critical density for which $A_1 = 0$ and for all the plasma models irrespective of the constituents of the ionic species. The compressive and rarefactive solitons occur simultaneously arising due to the $\pm$ sign of the soliton amplitude. Similar salient features have been observed experimentally (Nakamura and Tsukabayashi 1985) confirming the theoretical results. But in the case of multiple electron temperatures, the variation of the nonlinear coefficient in the mK–dV equation with electron density exhibits two critical densities, each differing from the critical density derived for the K–dV equation, similar to the case of negative ions. The observations indicate that the critical density occurs when the plasma involves multiple electron temperatures as well as negative ions in isolation, and correspondingly exhibits the compressive and rarefactive solitons in the plasmas. However, the case of a plasma involving only multiple positive ions and electrons exhibits only the compressive solitons.

In the planar plasma with negative ions the nonlinearity of an ion–acoustic wave can be controlled, while in the bounded (cylindrical and spherical) system the energy of a wave depends on the position of the wave due to the geometrical effect. However, a geometrical concentration of the wave energy is expected to depend sensitively on the nonlinearity and consequently on the amplitude.

From the present analysis it is obvious that neither the K–dV equation nor the mK–dV equation is sufficient to describe fully the essential features of ion–acoustic solitary waves. For this reason we examine the further mK–dV (fmK–dV) equation involving higher order nonlinearities near the critical density.

In this case, the nonlinear coefficient A_1 of the K–dV equation is not zero but is of O(ϵ), and the Poisson equation of O(ϵ^2) yields the relation:

$$\phi^{(2)} + \frac{\mu+\nu\beta^2}{2(\mu+\nu\beta)^2}(\phi^{(1)})^2 - \sum_\alpha q_\alpha n_\alpha^{(2)} = \left(\frac{3}{\lambda^4}\sum_\alpha q_\alpha \mu_\alpha^2 n_\alpha^{(0)} - \frac{\mu+\nu\beta^2}{(\mu+\nu\beta)^2}\right)\frac{(\phi^{(1)})^2}{2}, \tag{36}$$

where the coefficient of $(\phi^{(1)})^2/2$ is not zero but of O(ϵ). Then the charge density in (36) is of O(ϵ^3) and must be included in the Poisson equation of O(ϵ^3). Thus an evolution equation of the mK–dV equation (33) near the critical density of negative ions (Das and Singh 1990) is obtained as

$$\frac{\partial\phi^{(1)}}{\partial\tau} + L\frac{\phi^{(1)}}{\tau} + (A_1\phi^{(1)} + A_2(\phi^{(1)})^2)\frac{\partial\phi^{(1)}}{\partial\xi} + \tfrac{1}{2}\frac{\partial^3\phi^{(1)}}{\partial\xi^3} = 0, \tag{37}$$

which is the fmK–dV equation consisting of the nonlinear terms of the K–dV (18) and mK–dV (33) equations. Thus, the fmK–dV (37) can be studied for the particular cases of the planar (or cylindrical or spherical) K–dV or mK–dV equations and serves especially as the transitive link between the various K–dV equations.

3. Amplitude Variation of K–dV and mK–dV Soliton Solutions

We investigate the change of the wave amplitude due to the geometrical effect by means of the K–dV and mK–dV equations. To examine the amplitude variation we first consider the case of cylindrical geometry by taking simple K–dV and mK–dV equations corresponding to (18) and (33). For this we assume the simple cylindrical K–dV equation

$$\frac{\partial\phi}{\partial t} + \frac{\phi}{2t} + \phi\frac{\partial\phi}{\partial x} + \frac{\partial^3\phi}{\partial x^3} = 0, \tag{38}$$

and a planar K–dV solution

$$\phi(x, t_0) = \phi_0 \operatorname{sech}^2\left(\frac{\phi_0}{12}\right)^{1/2}(x - x_0) \tag{39}$$

is used as an initial wave at $t = t_0$ (where ϕ_0 denotes the initial wave amplitude and x_0 is the initial position).

Employing (39) in equation (38) the orders of magnitude (Watanabe and Yajima 1984) of the second, third and fourth terms of (38) are

$$\frac{\phi}{2t} \propto \frac{\phi_0}{T}, \qquad \phi\phi_x \approx \phi_{xxx} \propto \phi_0^{5/2},$$

where T is the time interval considered. When the geometrical effect is much stronger than the nonlinear and dispersive effects, we obtain

$$1/\sqrt{\phi_0} \gg \phi_0 T, \tag{40}$$

showing that the width of the initial wave is much larger than the propagation distance during time T. This implies that the initial wave amplitude ϕ_0 is small or the time interval is short and thereby we can neglect the third and fourth terms in (38). Then we have

$$\frac{\partial \phi}{\partial t} + \frac{\phi}{2t} = 0, \tag{41}$$

which immediately gives

$$\phi(t) = \phi_0 (t_0/t)^{1/2}. \tag{42}$$

Thus, the solution of (38) under the initial solution (39) reads as

$$\phi(x, t) = | t_0/t |^{1/2} \phi_0 \operatorname{sech}^2 \left(\frac{\phi_0}{12}\right)^{1/2} (x - x_0), \tag{43}$$

showing that the position and width do not change, but the amplitude increases as $t \to 0$. Again, if the geometrical effect is much weaker than the nonlinear and dispersive effects, the propagation distance is greater than the initial soliton width and then the solution of (38) is modified to (Kako and Yajima 1982; Hase *et al.* 1985)

$$\phi(x, t) = \left(\frac{t_0}{t}\right)^{2/3} \phi_0 \operatorname{sech}^2 \left\{\left(\frac{t_0}{t}\right)^{2/3} \frac{\phi_0}{12}\right\}^{1/2} \left\{x - x_0 - \tfrac{1}{3}\int_{t_0}^{t} \phi_0 \left(\frac{t_0}{t}\right)^{2/3} dt\right\}. \tag{44}$$

In this case we obtain a growing soliton as the amplitude and velocity increase, with the width decreasing as $t \to 0$, and hence it corresponds to a large initial amplitude ϕ_0 (or for a long time interval T).

Now, similar to the mK–dV (33), we consider the mK–dV equation

$$\frac{\partial \phi}{\partial t} + \frac{\phi}{2t} + \phi^2 \frac{\partial \phi}{\partial x} + \frac{\partial^3 \phi}{\partial x^3} = 0, \tag{45}$$

which under the initial wave at $t = t_0$ given by

$$\phi(x, t_0) = \pm \phi_0 \operatorname{sech} \left(\frac{\phi_0^2}{6}\right)^{1/2} (x - x_0) \tag{46}$$

yields, when the geometrical effect is much stronger than the nonlinear and dispersive effects, the solution (Hase *et al.* 1985)

$$\phi(x, t) = \pm | t_0/t |^{1/2} \phi_0 \operatorname{sech} \left(\frac{\phi_0^2}{6}\right)^{1/2} (x - x_0). \tag{47}$$

Again this has the same time dependence of the amplitude as in (43). When the propagation distance is larger than the initial soliton width, the solution of (45) is obtained as

$$\phi(x,t) = \pm |\, t_0/t \,|\, \phi_0 \operatorname{sech}\left\{\left(\frac{t_0}{t}\right)^2 \frac{\phi_0^2}{6}\right\}^{1/2} \left\{x - x_0 - \frac{1}{6}\int_{t_0}^{t} \left(\frac{t_0}{t}\right)^2 \phi_0^2 \, dt\right\}, \tag{48}$$

showing that the time dependence of the amplitude is different from that given by solution (44). Thus, we conclude that the solutions of the cylindrical K–dV or mK–dV equation have different time ranges. When the propagation distance is smaller than the initial soliton width, the wave grows as $(\tau_0/\tau)^{1/2}$, but after the distance becomes much larger than the soliton width, the wave grows as $(\tau_0/\tau)^{2/3}$ for the K–dV equation and as τ_0/τ for the mK–dV equation. In view of the solutions of the K–dV and mK–dV equations, we can further conclude that the solution of the cylindrical fmK–dV (37) has three time ranges $(\tau_0/\tau)^{1/2}$, $(\tau_0/\tau)^{2/3}$ and τ_0/τ.

The amplitude variations of the spherical K–dV and mK–dV equations can be derived more conveniently by writing $L = 1$ and considering the similar simple spherical K–dV and mK–dV equations as in the cylindrical case. As in the cylindrical K–dV equation, the spherical K–dV equation (20), where $L = 1$, also has two time ranges. The wave grows as τ_0/τ when the amplitude is small, but when the amplitude is large the wave increases in proportion to $(\tau_0/\tau)^{4/3}$. Furthermore, the spherical mK–dV equation also has two time ranges: the wave grows as τ_0/τ when the amplitude is small, but when the amplitude becomes large, the wave increases as $(\tau_0/\tau)^2$. Thus the solution of the spherical fmK–dV (37) has the three time ranges τ_0/τ, $(\tau_0/\tau)^{4/3}$ and $(\tau_0/\tau)^2$.

4. Conclusions

We have derived the generalised K–dV, mK–dV and fmK–dV equations for three different geometrical plasmas ($L = 0, \frac{1}{2}, 1$). As in the planar case, cylindrical and spherical solitons have similar characteristics in the form of compressive and rarefactive solitons, depending on the sign of the nonlinear coefficient A_1. As the negative ion concentration increases, A_1 decreases and consequently the amplitude increases remarkably. With a higher concentration of negative ions, the soliton amplitude tends to be very large and the charge separation providing the dispersive effect will not be sufficient to prevent steepening of the ion–acoustic wave and also breaking up of the soliton into multiple solitons. However, the wave gets reflected from barriers introduced by positive ions and ion beams before the wave can attain very large amplitude. At the critical concentration of negative ions the ion–acoustic wave is described by the mK–dV equation which exhibits both compressive and rarefactive solitons simultaneously. Furthermore, near this critical density the transformation of the three (planar and bounded) geometrical K–dV and mK–dV equations is given by the fmK–dV equation (37). However, near the critical density neither the K–dV nor the mK–dV equation in isolation can describe fully the ion–acoustic waves in the generalised multicomponent plasma.

In considering two limiting cases, the amplitude variations of the (cylindrical and spherical) K–dV and mK–dV equations have been discussed. For the cylindrical [spherical] K–dV soliton, the wave grows as $(\tau_0/\tau)^{1/2}$ $[\tau_0/\tau]$ when the amplitude is small, but when the amplitude is large the wave increases as $(\tau_0/\tau)^{2/3}$ $[(\tau_0/\tau)^{4/3}]$. The cylindrical [spherical] mK–dV soliton solution also has two time ranges: the wave grows as $(\tau_0/\tau)^{1/2}$ $[\tau_0/\tau]$ for the small amplitude, but for a large amplitude the growth of the wave changes as τ_0/τ $[(\tau_0/\tau)^2]$. Thus, we conclude that the K–dV and mK–dV solutions show the existence of two time ranges depending on the two limiting cases of whether the amplitude is small or large. Furthermore, the presence of negative ions and ion beams along with multiple electron temperatures leads to a slower exhibition of the critical density of negative ions and, consequently, the later formation of rarefactive solitons in the plasma, as in the case of planar geometry. Moreover, we believe that the present results will be definitely applicable in laboratory plasmas, but one has to be careful about the choice of plasma parameters.

References

Das, G. C. (1975). *IEEE Plasma Sci.* **PS-3**, 68.
Das, G. C. (1978). *Ind. J. Phys. Math.* **28**, 2706.
Das, G. C., Paul, S. N., and Karmakar, B. (1986). *Phys. Fluids* **29**, 2192.
Das, G. C., and Singh, Kh.I. (1990). *Aust. J. Phys.* **43**, 319.
Das, G. C., and Singh, Kh.I. (1991). *Contrib. Plasma Phys.* **31**, 1.
Das, G. C., Singh, Kh.I., and Karmakar, B. (1989*a*). *Plasma Phys.* **31**, 69.
Das, G. C., Singh, Kh.I., and Karmakar, B. (1989*b*). *Can. J. Phys.* **67**, 609.
Das, G. C., and Tagare, S. G. (1975). *Plasma Phys.* **17**, 1025.
Hase, H., Watanabe, S., and Tanaca, H. (1985). *J. Phys. Soc. Jpn* **54**, 4115.
Ikezi, H. (1973). *Phys. Fluids* **16**, 1668.
Jeffrey, A., and Kakutani, T. (1972). *SIAM-Review* **14**, 582.
Kako, F., and Yajima, N. (1982). *J. Phys. Soc. Jpn* **51**, 311.
Lonngren, K. E. (1983). *Plasma Phys.* **25**, 943.
Maxon, S., and Viecelli, J. (1974*a*). *Phys. Rev. Lett.* **32**, 4.
Maxon, S., and Viecelli, J. (1974*b*). *Phys. Fluids* **17**, 1614.
Nakamura, Y. (1987). *J. Plasma Phys.* **38**, 461.
Nakamura, Y., Ferreira, J. L., and Ludwig, G. O. (1985). *J. Plasma Phys.* **33**, 237.
Nakamura, Y., and Tsukabayashi, I. (1985). *J. Plasma Phys.* **34**, 401.
Raychaudhuri, S., Gabl, E. F., Das, K. P., and Sengupta, S. N. (1985). *Plasma Phys.* **27**, 299.
Singh, Kh.I., and Das, G. C. (1989). *IEEE Plasma Sci.* **17**, 863.
Su, C. H., and Gardner, C. S. (1969). *J. Math. Phys. (NY)* **10**, 536.
Tran, M. Q. (1974). *Plasma Phys.* **16**, 1167.
Verheest, F. (1988). *J. Plasma Phys.* **39**, 71.
Washimi, H., and Taniuti, T. (1966). *Phys. Rev. Lett.* **17**, 996.
Watanabe, S. (1984). *J. Phys. Soc. Jpn* **53**, 950.
Watanabe, S., and Yajima, N. (1984). *J. Phys. Soc. Jpn* **53**, 3325.

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