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Boson Bound States near a Kerr–Newman Naked Singularity

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Abstract

We discuss boson bound states near a Kerr–Newman (KN) naked singularity by means of spectroscopic eigenvalue analysis. The results show that in the background of a KN naked singularity, the self-conjugate extension operator of a boson Hamiltonian has and only has discrete eigenvalues. The discrete eigenvalues exist in the interval $(-\mu, \mu)$. Thus, we find that the boson bound states may appear only for $\mu \neq 0$. Here μ is the boson mass.

1. Introduction

Investigations of particle bound states about a black hole have been a significant area of work in black hole physics. Much research has been done in this field; some have discussed the problem of boson bound states by solving the Klein–Gordon equation (de Felic 1979; Adler and Pearson 1978; Zang and Shu 1982); others have discussed fermion bound states using the same method (Brill and Cohen 1966; Soffel 1977; Xu and Xie 1980). We have also studied a similar problem (Li and Zhang 1986). However, all these works do not touch upon the massive boson bound states in the field of a naked singularity. In this paper we discuss this case by means of spectroscopic eigenvalue analysis.

2. Hamiltonian Form

In the background of a KN black hole, the Klein–Gordon equation of a single boson is (Liu 1987)

$$\left(\partial_r \Delta \partial_r - \frac{1}{\Delta}[(r^2+a^2)\partial_t + a\partial_\phi + \mathrm{i}\, eQr]^2 - \mu^2 r^2 + \frac{1}{\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta) + \frac{1}{\sin\theta}\,(\partial_\phi + a\sin\theta\,\partial_t)^2 - \mu^2 a^2\cos^2\theta\right)\phi = 0\,. \qquad (1)$$

Separating variables in (1), we deduce the radial equation

$$\frac{\mathrm{d}^2\Psi(r)}{\mathrm{d}u^2} = \{\Delta(\mu^2 r^2 + K) - [\omega(r^2+a^2) - am - eQr]^2\}\Psi(r)\,, \qquad (2)$$

where Q and m are the charge and mass of the black hole, a is the angular momentum per mass, while $\Delta = r^2 - 2mr + a^2 + Q^2$ and $\mathrm{d}u = \mathrm{d}r/\Delta$. Equation

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(2) can be rewritten as a matrix equation in the canonical Hamiltonian form

$$\mathbf{HF} = E\mathbf{F}\,, \tag{3}$$

with

$$\mathbf{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \qquad \mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

where the matrix elements H_{ij} are given by

$$\begin{aligned} H_{11} = &- \frac{A^4 \mathrm{d}^2}{2\mu\, \mathrm{d}u^2} + A^2(am + eQr) \\ &- \frac{A^4}{2\mu}[(am + eQr)^2 - \Delta(\mu^2 r^2 + K)] + \frac{\mu}{2}, \\ H_{12} = &- \frac{A^4 \mathrm{d}^2}{2\mu\, \mathrm{d}u^2} - A^2(am + eQr) \\ &- \frac{A^4}{2\mu}[(am + eQr)^2 - \Delta(\mu^2 r^2 + K)] - \frac{\mu}{2}, \\ H_{21} = &\frac{A^4 \mathrm{d}^2}{2\mu\, \mathrm{d}u^2} + A^2(am + eQr) \\ &+ \frac{A^4}{2\mu}[(am + eQr)^2 - \Delta(\mu^2 r^2 + K)] + \frac{\mu}{2}, \\ H_{22} = &\frac{A^4 \mathrm{d}^2}{2\mu\, \mathrm{d}u^2} - A^2(am + eQr) \\ &+ \frac{A^4}{2\mu}[(am + eQr)^2 - \Delta(\mu^2 r^2 + K)] - \frac{\mu}{2}. \end{aligned}$$

Here $A^2 = 1/(a^2 + r^2)$. For more details, we refer to the Appendix.
From (3) we get

$$\left(- \frac{2A^4}{\mu + E}\, \partial_u^2 + p_1(r) \right) f_1 = \frac{2E^2}{\mu + E}\, f_1\,, \tag{4}$$

and

$$\begin{aligned} p_1(r) = &- \frac{2\mu}{\mu + E} \left(\frac{A^4}{\mu}[(am + eQr)^2 \right. \\ &\left. - \Delta(\mu^2 r^2 + K)] - \frac{2A^2 E}{\mu}(am + eQr) \right). \end{aligned} \tag{5}$$

We make the transformation

$$\frac{\mathrm{d}r'}{\mathrm{d}r} = \frac{a^2 + r^2}{\Delta} \tag{6}$$

when $r \to 0$, $r' \to 0$ and $r \to \infty$, $r' \to \infty$, and let

$$p_0 = \frac{2(r^2 + a^2)^2}{\mu + E}. \tag{7}$$

Using equations (5)–(7), we can rewrite (4) as

$$L(f_1) \equiv [-\partial_{r'}(p_0\, \partial_{r'}) + p_1] f_1 = \frac{2E^2}{\mu + E} f_1 . \tag{8}$$

Now we introduce a theorem of self-conjugate differential operators (Naemark 1954).

Theorem: Let $L(y)$ be a self-conjugate differential operator in the interval (a, b), then

$$L(y) = \sum_{i=0}^{n} (-1)^{n-i} [p_i\, y^{(n-i)}]^{(n-i)} , \tag{9}$$

if (1) $L(y)$ is canonical at the end point a, (2) $\lim_{x \to b} P_n(x) = B$, and (3) for any x approaching close to b, $P_i(x) \geq 0$.

Thus, the spectrum of the self-conjugate extension of the operator $L(x)$ is definitely discrete in $(-\infty, B)$. According to this theorem, we can get spectroscopic eigenvalues of the operator (8)

$$\lambda = \frac{2E^2}{\mu + E},$$

which has to be discrete in the interval $[-\infty,\ 2\mu^2/(\mu + E)]$, or

$$-\infty < \frac{2E^2}{\mu + E} < \frac{2\mu^2}{\mu + E}. \tag{10}$$

Thus, we have

$$-\mu < E < \mu \tag{11}$$

where E can only take some discrete value.

3. Conclusions

The condition (11) shows that in the field of a naked singularity we can find massive boson bound states, but we cannot find massless boson bound states. This conclusion is different to de Felic's (1979) result. The energy of the bound states E has to satisfy the condition (11). It is found that the method of spectroscopic eigenvalue analysis is more effective and simpler than solving the equation. Specially, this method avoids some errors that arise from approximate methods.

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Appendix

Let $\xi = \eta$ and $i\eta = i\partial_t\xi$, and then equation (2) becomes

$$\begin{aligned}\frac{\mathrm{d}^2\xi}{\mathrm{d}u^2} &= A^{-4}\partial_t\eta + 2A^{-2}(am+eQr)\mathrm{i}\eta \\ &\quad - [(am+eQr)^2 - \Delta(\mu^2r^2+K)]\xi\,. \end{aligned} \tag{A1}$$

Taking the following transformation

$$\xi = \frac{f_1+f_2}{\sqrt{2}}\,, \qquad \eta = \frac{\mu}{\mathrm{i}}\,\frac{f_1-f_2}{\sqrt{2}}\,,$$

then from (A1) we obtain

$$\begin{aligned}\mathrm{i}\frac{\partial f_1}{\partial t} &= \bigg(-\frac{A^4\mathrm{d}^2}{2\mu\,\mathrm{d}u^2} + A^2(am+eQr) \\ &\quad - \frac{A^4}{2\mu}[(am+eQr)^2 - \Delta(\mu^2r^2+K)] + \frac{\mu}{2}\bigg)f_1 \\ &\quad + \bigg(-\frac{A^4\mathrm{d}^2}{2\mu\,\mathrm{d}u^2} - A^2(am+eQr) \\ &\quad - \frac{A^4}{2\mu}[(am+eQr)^2 - \Delta(\mu^2r^2+K)] - \frac{\mu}{2}\bigg)f_2\,,\end{aligned}$$

$$\begin{aligned}\mathrm{i}\frac{\partial f_2}{\partial t} &= \bigg(\frac{A^4\mathrm{d}^2}{2\mu\,\mathrm{d}u^2} - A^2(am+eQr) \\ &\quad + \frac{A^4}{2\mu}[(am+eQr)^2 - \Delta(\mu^2r^2+K)] + \frac{\mu}{2}\bigg)f_1 \\ &\quad + \bigg(\frac{A^4\mathrm{d}^2}{2\mu\,\mathrm{d}u^2} + A^2(am+eQr) \\ &\quad - \frac{A^4}{2\mu}[(am+eQr)^2 - \Delta(\mu^2r^2+K)] - \frac{\mu}{2}\bigg)f_2\,.\end{aligned}$$

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