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Quantum Mechanics and Superconductivity in a Magnetic Field*

A. H. MacDonald,[A] *Hiroshi Akera*[B] *and M. R. Norman*[C]

[A] Department of Physics, Indiana University,
Bloomington, IN 47405, U.S.A.
[B] Faculty of Engineering, Hokkaido University,
Sapporo 060, Japan.
[C] Materials Science Division, Argonne National Laboratory,
Argonne, IL 60439, U.S.A.

Abstract

The influence of a magnetic field on superconductivity is usually described either phenomenologically, using Ginzburg–Landau theory, or semiclassically, using Gor'kov theory. In this article we discuss the influence of magnetic fields on the mean-field theory of the superconducting instability from a completely quantum-mechanical point of view. The suppression of superconductivity by an external magnetic field is seen in this more physically accurate picture to be due to the impossibility, in quantum mechanics, of precisely specifying both the centre-of-mass state of a pair and the individual electron kinetic energies. We also discuss the possibility of novel aspects of superconductivity at extremely strong magnetic fields, where recent work has shown that the transition temperature may be enhanced rather than suppressed by a magnetic field and where a quantum treatment is essential.

1. Introduction

The relationship between superconductivity and magnetic fields is both of practical importance in the design of superconducting devices and of fundamental importance to the superconductivity phenomenon. In the absence of an external magnetic field, superconductivity is associated with the pairing of time-reversed electron states. As we discuss in detail below, magnetic fields break time-reversal-invariance symmetry and frustrate this pairing. For sufficiently weak external magnetic fields, superconductors prefer to completely expel any external magnetic flux (the Meissner effect) in order to avoid this frustration. At stronger magnetic fields type-II superconductors, which are used in the construction of superconducting magnets, can form a mixed state in which superconductivity coexists with magnetic flux. Superconductivity in the mixed state is usually described in terms of Ginzburg–Landau theory (see e.g. de Gennes 1966; Tinkham 1975) which predicts a decrease in the temperature to which superconductivity can survive (T_c) proportional to the external magnetic field strength. For

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sufficiently weak external fields and temperatures close to T_c Ginzburg–Landau theory has been derived microscopically by Gor'kov (1958). This theory has a wider range of validity than Ginzburg–Landau theory and predicts (see e.g. Gor'kov 1959; Helfand and Werthamer 1966; Werthamer *et al.* 1966) that T_c decreases monotonically with increasing magnetic field and is eventually driven to zero. (Non-monotonic behaviour is possible in special circumstances when the electron g-factor is different from zero.) However, Gor'kov's theory treats the magnetic field in a semiclassical approximation which is not valid when the temperature is sufficiently low and the disorder is sufficiently weak that the Landau quantisation of motion in planes perpendicular to the field direction becomes important. In the past few years, following seminal work by Rasolt, Tešanović and collaborators, it has been realised (see e.g. Rasolt 1987; Tešanović and Rasolt 1989; Tešanović *et al.* 1989, 1991; Norman 1990; Rieck *et al.* 1990; Rasolt and Tešanović 1992) that, at least within the standard mean-field theory known to be accurate at weak magnetic fields, superconductivity can survive to arbitrarily strong magnetic fields once Landau quantisation is accounted for. In this article we discuss the superconducting instability in a magnetic field from a completely quantum-mechanical point of view. We explain how the results of Ginzburg–Landau theory and Gor'kov theory can be understood in terms of the microscopic quantum mechanics of charged particles in a magnetic field, and why Gor'kov theory can fail at sufficiently strong fields.

2. T_c at Zero Magnetic Field

It is useful to begin by discussing the familiar implicit equation for T_c in the absence of a magnetic field (see e.g. Schrieffer 1964, 1969):

$$1 = \frac{V}{\Omega} \sum_{\boldsymbol{k},\boldsymbol{k}'}{}' \left[\frac{1 - f(\varepsilon_{\boldsymbol{k}}) - f(\varepsilon_{\boldsymbol{k}'})}{\varepsilon_{\boldsymbol{k}} + \varepsilon_{\boldsymbol{k}'}} \right] \delta_{\boldsymbol{k}+\boldsymbol{k}',\boldsymbol{P}} \,. \tag{1}$$

(Energies are measured from the chemical potential μ, and Ω is the volume of three-dimensional systems or the area of two-dimensional systems. The Fermi energy, $\varepsilon_F = mV_F^2/2$, is the zero-temperature limit of μ.) This equation is for the usual BCS model with attractive interactions of constant strength V. All the discussion in this article will be in terms of this simple model (our discussion is readily adapted to the case of strong-coupling superconductors). The prime on the sum over wavevectors denotes the usual separable high-energy cutoff that requires both electron energies to be within E^+ of the Fermi level. The numerator of the factor in square brackets in (1) expresses through the Fermi occupation numbers the requirement that the pairing comes either from electrons outside the Fermi sea, as in the Cooper problem, or from holes inside the Fermi sea. Note that this factor vanishes at finite temperature, and even at $T = 0$ for $\boldsymbol{P} \neq 0$, when $\varepsilon_{\boldsymbol{k}} + \varepsilon_{\boldsymbol{k}'}$ is near zero. In a superconductor a bound state occurs for the relative motion of electrons in a Cooper pair and the temperature at which the bound state first occurs, T_c, depends on the centre-of-mass (COM) momentum of the pair, $\boldsymbol{P}$, as we discuss in the following paragraph.

Defining an effective pairing density of states by

$$\nu^p(\varepsilon : \boldsymbol{P}, T) \equiv \frac{1}{\Omega} \sum_{\boldsymbol{k}}{}' \, [1 - f(\varepsilon_{\boldsymbol{k}}) - f(\varepsilon_{\boldsymbol{P}-\boldsymbol{k}})] \delta(\varepsilon - \varepsilon_{\boldsymbol{k}} - \varepsilon_{\boldsymbol{P}-\boldsymbol{k}}) \,, \tag{2}$$

the T_c equation can be rewritten in the form

$$1 = V \int_{-\infty}^{\infty} \nu^p(\varepsilon : \boldsymbol{P}, T)/\varepsilon \,. \tag{3}$$

At $\boldsymbol{P} = 0$ and $T = 0$, $\nu^p(\varepsilon) = \theta(2E^+ - |\varepsilon|)(\varepsilon/|\varepsilon|)\nu(2\varepsilon)/2$ where $\nu(\varepsilon)$ is the single-electron density of states per spin. At finite $\boldsymbol{P}$ and T, $\nu^p(\varepsilon : \boldsymbol{P}, T)$ is reduced toward zero for $|\varepsilon| \lesssim E^- \equiv \sup(k_B T, V_F P)$ because of the combination of Fermi factors appearing in (2), but otherwise is nearly constant. [Here V_F is the Fermi velocity. Low-energy pairs tend to be composed of states on opposite sides of the Fermi energy for $\boldsymbol{P} \neq 0$ (see Fig. 1). The pairing densities of states for two dimensions (2D) and three dimensions (3D) are reduced as a consequence when $|(\varepsilon - 2\varepsilon_F)/V_F P| < 1$.] For $T = 0$ the reduction in ν^p is illustrated in Fig. 2. For $|(\varepsilon - 2\varepsilon_F)/V_F P| < 1$,

$$\nu^p(\varepsilon : \boldsymbol{P}) = \frac{\nu(2\varepsilon)}{2} \left[\frac{\varepsilon - 2\varepsilon_F}{V_F P} \right] \tag{4}$$

for 3D, and

$$\nu^p(\varepsilon : \boldsymbol{P}) = \frac{\nu(2\varepsilon)}{2} \left(1 - \frac{2}{\pi} \cos^{-1}[(\varepsilon - 2\varepsilon_F)/V_F P] \right) \tag{5}$$

for 2D. For weak coupling ($E^- \ll E^+$) the T_c equation reduces to $E^- \sim E^+ \exp(-1/\lambda)$ which will have no solution once $V_F P$ exceeds $\sim k_B T_c(\boldsymbol{P} = 0)$. [$\lambda \equiv V\nu(0)$. It is assumed that $\nu(\varepsilon)$ is nearly constant over the energy range E^+.] Later we will relate this result for the dependence of T_c on the COM momentum of the Cooper pair directly to the dependence of T_c on an external magnetic field.

3. Pair States in a Magnetic Field

Note that the T_c equation (3) depends both on the COM state of the pair and, through the Pauli exclusion principle requirements expressed by Fermi factors, on the states of the individual electrons making up the pair. The states of a pair of electrons may be described either in terms of COM and relative motion states, or in terms of the individual electron states. In the absence of a magnetic field this connection is trivial. To describe superconductivity in a magnetic field quantum-mechanically, we must start by discussing the relationship between these two descriptions in a magnetic field. The Hamiltonian (note that motion parallel to the field direction is unaffected by a magnetic field so we restrict our attention

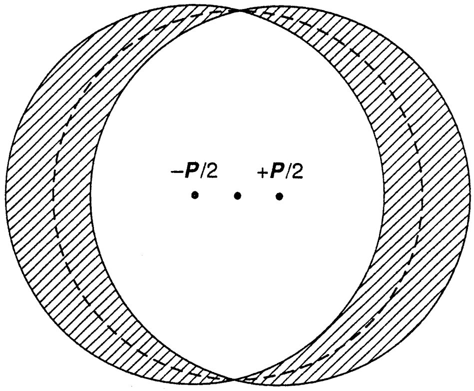

Fig. 1. Pauli blocking of low-energy pair states at finite COM momentum, $\boldsymbol{P}$, of the pair. In terms of the relative momentum of the pair the Fermi surfaces for the two electrons composing the pair are displaced by $\boldsymbol{P}$. The pair must be composed of unoccupied electron states or hole states. At zero temperature the allowed values of relative momentum are either inside both Fermi surfaces or outside both Fermi surfaces. The shaded regions where the energies are close to the Fermi energy are forbidden.

to planes perpendicular to the field direction) for two non-interacting electrons, h, is

$$h = \frac{1}{2m}\Big(-i\hbar\nabla_1 + \frac{e}{c}\boldsymbol{A}(\boldsymbol{r}_1)\Big)^2 + \frac{1}{2m}\Big(-i\hbar\nabla_2 + \frac{e}{c}\boldsymbol{A}(\boldsymbol{r}_2)\Big)^2 \tag{6}$$

or

$$h = \frac{1}{2M}\Big(-i\hbar\nabla_{\boldsymbol{R}} + \frac{2e}{c}\boldsymbol{A}(\boldsymbol{R})\Big)^2 + \frac{1}{2\mu}\Big(-i\hbar\nabla_r + \frac{e}{2c}\boldsymbol{A}(\boldsymbol{r})\Big)^2 . \tag{7}$$

Here we have assumed a gauge where the vector potential is linear in the coordinates, $M = 2m$, $\mu = m/2$, $\boldsymbol{R} = (\boldsymbol{r}_1 + \boldsymbol{r}_2)/2$ and $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$. Notice that the charge appearing in the COM term is $2e$ while the charge appearing in the relative motion term is $e/2$, so that both the relative and COM kinetic energies (KEs) are quantised in the same units as for individual electrons, $\hbar\omega_c = eB/mc$. [The individual electron eigenvalues measured from the chemical potential are $\varepsilon_N = \hbar\omega_c(N + 1/2) - \mu \equiv \hbar\omega_c(N - N_{\mathrm{F}})$, where N_{F} is the Landau level index at the Fermi level.] In the Landau gauge the eigenfunctions for individual electrons are well known:

$$\psi_{N,X}(\boldsymbol{r}_i) = \exp(-iXy_i/\ell^2)\phi_N((x_i - X))/\sqrt{L_y}\,, \tag{8}$$

where L_y is the length of the system in the y direction, $\ell \equiv (\hbar c/eB)^{\frac{1}{2}}$ is the magnetic length, and $\phi_N(x)$ is a one-dimensional harmonic oscillator eigenstate for

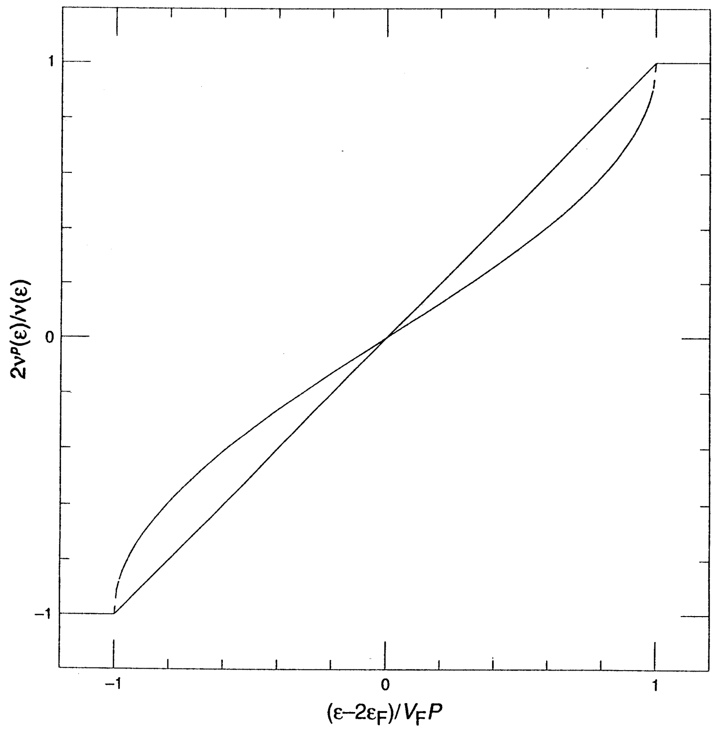

Fig. 2. Pairing density of states at finite COM momentum for three dimensions (solid line) and two dimensions (broken curve).

mass m^* and frequency ω_c. The expressions for the COM and relative eigenstates, ψ^R and ψ^r, are identical except that the characteristic lengths are scaled to account for the changes of charge and mass. (The effective magnetic lengths are $\ell^R = \ell/\sqrt{2}$ and $\ell^r = \sqrt{2}\ell$ for the COM and relative eigenstates respectively.)

In the lowest Landau level $\phi_{N=0}(x) \sim \exp(-x^2/4\ell^2)$ so that

$$\psi_{0,X+Y/2}(\boldsymbol{r}_1)\,\psi_{0,X-Y/2}(\boldsymbol{r}_2) = \psi^R_{0,X}(\boldsymbol{R})\,\psi^r_{0,Y}(\boldsymbol{r})\,. \tag{9}$$

The relationship is easily generalised to higher Landau levels by writing the Hamiltonian in terms of ladder operators,

$$h = \hbar\omega_c\,(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) = \hbar\omega_c\,(a_R^\dagger a_R + a_r^\dagger a_r + 1)\,, \tag{10}$$

and noting that $a_R = (a_1 + a_2)/\sqrt{2}$ and $a_r = (a_1 - a_2)/\sqrt{2}$. Here $a_j = (\ell/\sqrt{2}\hbar)(\pi_{xj} - i\pi_{yj})$ and $\boldsymbol{\pi}_j = -i\hbar\nabla_j + (e/c)\boldsymbol{A}_j$. It follows that

$$\psi_{N,X+Y/2}(\boldsymbol{r}_1)\,\psi_{M,X-Y/2}(\boldsymbol{r}_2) = \sum_{j=0}^{N+M} B_j^{N,M}\,\psi^R_{j,X}(\boldsymbol{R})\,\psi^r_{N+M-j,Y}(\boldsymbol{r})\,, \tag{11}$$

where

$$B_j^{N,M} = \left(\frac{j!(N+M-j)!N!M!}{2^{N+M}}\right)^{\frac{1}{2}} \sum_{m=0}^{j} \frac{(-)^{M-m}}{(j-m)!(N+m-j)!(M-m)!m!}. \tag{12}$$

Note that both left- and right-hand sides of (11) are manifestly eigenstates of h with eigenvalue $\hbar\omega_c(N+M+1)$. The coefficients $B_j^{N,M}$ give the amplitude for having KE $\hbar\omega_c(j+1/2)$ in the COM motion [and $\hbar\omega_c(N+M-j+1/2)$ in the relative motion] when the individual particles have definite KEs $\hbar\omega_c(N+1/2)$ and $\hbar\omega_c(M+1/2)$.

The coefficients appearing in the unitary transformation between the two sets of two-particle eigenstates, $\{B_j^{N,M}\}$, will play a central role in the discussion below. Note that the transformation is block-diagonal, with no mixing between eigenstates of different total kinetic energy. The completeness of either set of eigenstates implies the following identities:

$$\sum_{N=0}^{K} B_{j'}^{N,K-N} B_j^{N,K-N} = \delta_{j',j}, \tag{13}$$

$$\sum_{j=0}^{K} B_j^{N',K-N'} B_j^{N,K-N} = \delta_{N',N}. \tag{14}$$

Since the COM kinetic energy does not commute with the individual particle kinetic energies, the COM kinetic energy is necessarily uncertain if the individual particle states are known precisely. Conversely, for given COM and relative state kinetic energies, the individual particle kinetic energies are necessarily uncertain. Given j and the relative motion eigenstate, or equivalently j and the total kinetic energy index K, $|B_j^{N,K-N}|^2$ gives the normalised probability distribution for the individual electron states with the same total kinetic energy. Explicit expressions for small j are easily obtained from (12):

$$|B_0^{N,K-N}|^2 = \frac{1}{2^K}\binom{K}{N}, \tag{15}$$

$$|B_1^{N,K-N}|^2 = \frac{1}{2^K}\binom{K}{N}\frac{(K-2N)^2}{K}, \tag{16}$$

$$|B_2^{N,K-N}|^2 = \frac{1}{2^K}\binom{K}{N}\frac{((K-2N)^2-K)^2}{2K(K-1)}. \tag{17}$$

For $K \gg |k|$ ($k \equiv N-M$) it can be shown (MacDonald 1992, unpublished) that

$$B_j^{(K+k)/2,(K-k)/2} \sim \left(\frac{2}{K\pi}\right)^{\frac{1}{4}} \left(\frac{1}{2^j j!}\right)^{\frac{1}{2}} H_j(k/\sqrt{2K})\exp(-k^2/4K), \tag{18}$$

where H_j is a Hermite polynomial.

4. T_c in a Magnetic Field

The implicit T_c equation in a magnetic field* is completely analogous (MacDonald *et al.* 1992) to the $B = 0$ equation (1) cited at the beginning of this article:

$$1 = \frac{V}{4\pi\ell^2} \sum_{N,M}' \left[\frac{1 - f(\varepsilon_N) - f(\varepsilon_M)}{\varepsilon_N + \varepsilon_M} \right] |B_j^{N,M}|^2 . \tag{19}$$

As in the $B = 0$ case T_c depends on the Cooper pair state. As we discussed previously, the superconducting T_c decreases with $|\boldsymbol{P}|$ for $B = 0$. For $B \neq 0$, T_c is independent of the guiding centre quantum number X for the Cooper pair. The fact that instabilities occur simultaneously in a macroscopic number of channels is responsible for the dimensional reduction (Bergmann 1969; Lee and Shenoy 1972; Thouless 1975; Brézin *et al.* 1985, 1990) which causes superconducting fluctuations to be qualitatively altered by a magnetic field. The superconducting instability still depends, however, on the Landau level index of the Cooper pair. We first examine the weak field limit where $k_B T \gg \hbar\omega_c$. In this limit the sums over Landau levels may be replaced by integrals and (19) becomes

$$\begin{aligned} 1 = \lambda \int_{2N_F}^{K^+} \frac{dK}{K - 2N_F} \int_0^\infty dk \, [1 - f(\varepsilon_{(K+k)/2}) \\ - f(\varepsilon_{(K-k)/2})] \, |B_j^{(K+k)/2,(K-k)/2}|^2 . \end{aligned} \tag{20}$$

[Here we have noted that $\nu(0) = 1/(2\pi\ell^2\hbar\omega_c)$ and K^+ is the maximum kinetic energy index allowed by the high-energy cutoff.] To understand why superconductivity is suppressed by weak magnetic fields it is sufficient to consider the $T = 0$ limit. The Landau levels with indices $(K + k)/2$ and $(K - k)/2$ are on the same side of the Fermi level and can contribute to the pairing only if $|k| < |K - 2N_F|$ (see Fig. 3). For a given COM index j of the Cooper pair and a given total kinetic energy, the probability of finding both members of a Cooper pair on the same side of the Fermi energy $[\varepsilon_F \equiv \mu(T = 0)]$ is necessarily less than unity. In Fig. 4 we plot

$$\begin{aligned} P_j(K) \equiv \sum_k [1 - \theta(2N_F - K - k) \\ - \theta(2N_F - K + k)] |B_j^{(K+k)/2,(K-k)/2}|^2 \end{aligned} \tag{21}$$

for $j = 0$ and $N_F = 12 \cdot 5$ against K. From (18) we see that most of the contribution to $P_j(K)$ comes from $|k| <\sim \sqrt{(j + 1/2)K}$. The logarithmic divergence of the

* This is the mean-field T_c equation for two spatial dimensions. In three dimensions (3D) it is necessary to integrate over momenta along the field direction in addition to summing over Landau level indices. For notational simplicity we restrict our attention here to the 2D case. In 3D T_c depends on the Cooper pair momentum along the field direction as well as on the Landau level of the Cooper pair.

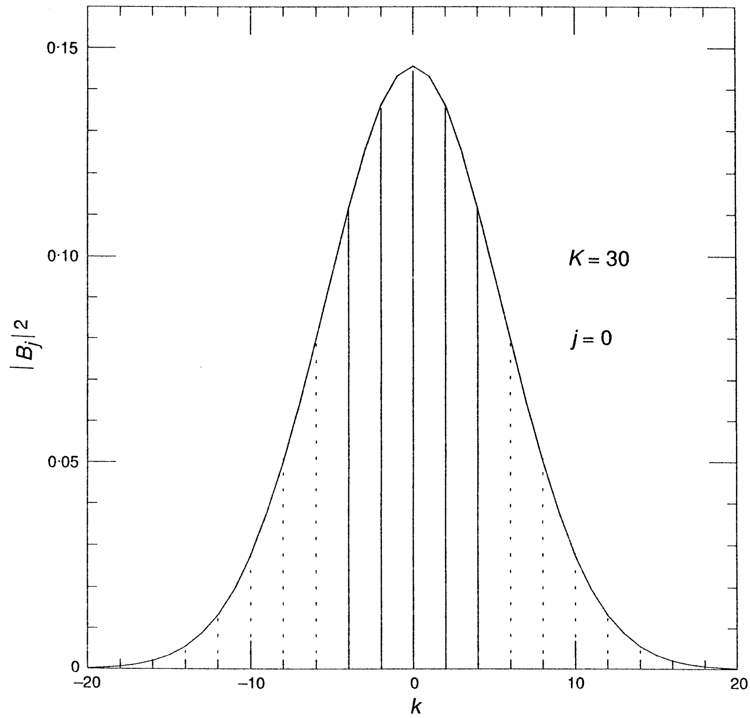

Fig. 3. Probability of having individual electron kinetic energies $\hbar\omega_c(K+k+1)/2$ and $\hbar\omega_c(K-k+1)/2$, given COM kinetic energies $\hbar\omega_c(j+1/2)$ and total kinetic energies $\hbar\omega_c(K+1/2)$. The probabilities are represented by the vertical lines at even integer values of k (k must be even when K is even and odd when K is odd). The results shown here are for $K=30$ and $j=0$. For $N_F = 12{\cdot}5$, i.e. for the first 12 Landau levels occupied, the two single-particle states are both occupied or both empty only for $k=0$, $k=\pm 2$ and $k=\pm 4$. Larger values of k, for which the probability is indicated by a dashed line, are Pauli blocked and cannot contribute to pairing in a $j=0$ COM state. For this case the probability that the two single-particle states will be on the same side of the Fermi energy is $P_0(\mathrm{K{=}30}) = 0{\cdot}6384$. The solid curve which envelopes the probabilities is the large K expression (18).

integral over K in (20) which guarantees a solution is therefore cut off, since $P_j(K)$ will fall to zero for $|K-2N_F| <\!\sim \sqrt{(2j+1)N_F}$. It follows that solutions at $T=0$ exist only if

$$\hbar\omega_c <\!\sim (k_B T_c)^2/(2j+1)\varepsilon_F\,. \tag{22}$$

The superconducting instability is suppressed most weakly for Cooper pairs with $j=0$, i.e. for COM in the lowest Landau level, in agreement with Ginzburg–Landau theory.

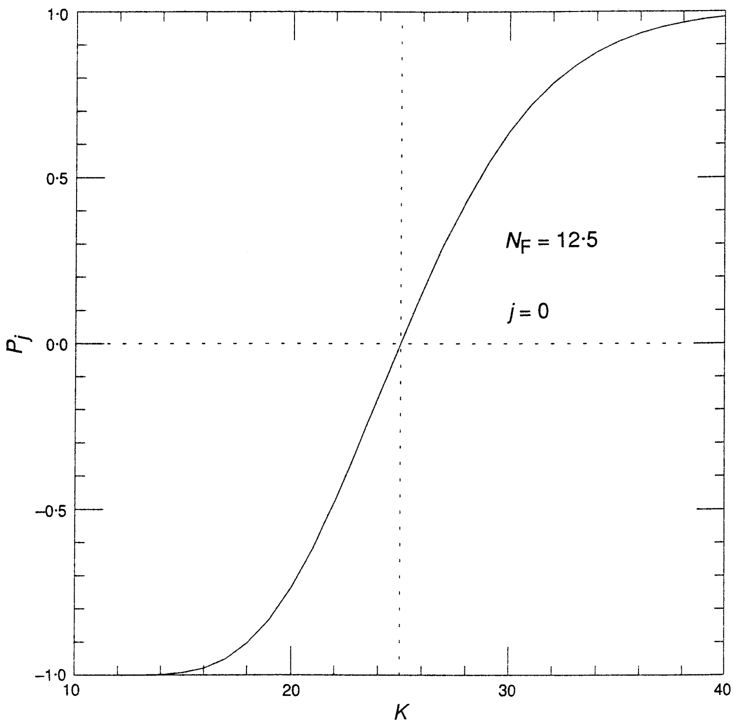

Fig. 4. Plot of P_j versus K for $j = 0$ and $N_{\mathrm{F}} = 12\cdot5$, i.e. for the first twelve Landau levels occupied. For the case $K = 30$, P_j is given by the sum of the probabilities indicated by the solid lines in Fig. 3.

At zero magnetic field the superconducting instability occurs first for COM momentum $\boldsymbol{P} = 0$; the pairing of time-reversed single-particle states guarantees that all pairs are allowed by the Pauli exclusion principle at $T = 0$ even if their energies are very close to the Fermi energy. In a magnetic field, time-reversal symmetry is broken so that time-reversed pairs of single-particle states no longer exist. The kinetic energy eigenstates in a magnetic field are usefully thought of as having a definite magnitude of momentum corresponding to the quantised kinetic energy, but a completely uncertain direction of momentum since they are executing circular orbits. For definite COM and relative kinetic energies ε_R and ε_r, the mean squared difference between the kinetic energies of individual electrons is

$$\langle(\varepsilon_1 - \varepsilon_2)^2\rangle_\theta = 2\varepsilon_R\varepsilon_r\,. \tag{23}$$

The average here is over the angle θ between the COM and relative momenta, which is completely uncertain in a magnetic field. This classical root-mean-squared energy difference agrees with the energy width of the quantum-mechanical distribution function discussed above. When the mean energy of the pair is within $\sim 2\varepsilon_R\varepsilon_r$ of the Fermi energy, contributions to pair formation are suppressed by the Pauli exclusion principle. For $\varepsilon_R = \hbar\omega_c(j + 1/2) \ll \varepsilon_r \sim 2\varepsilon_{\mathrm{F}}$, the resulting low-energy cutoff is

$$E^- =\sim 2\sqrt{\hbar\omega_c\,(j+1/2)\varepsilon_{\rm F}} = V_{\rm F}P_j\,. \tag{24}$$

In (24) $\hbar P_j^2/4m = \hbar\omega_c\,(j+1/2)$ so that P_j is the 'quantised' magnitude of the COM momentum. We see from this discussion that pairing in COM Landau level j in a magnetic field is very similar to pairing at COM momentum P_j in the absence of a magnetic field.

The above discussion explains from a quantum-mechanical point of view the familiar suppression of superconductivity by a magnetic field in the weak field regime where the discretisation of allowed kinetic energies transverse to the magnetic field is washed out by either temperature or disorder. In clean 2D systems the Landau level structure becomes important in the thermally averaged density of states for $\hbar\omega_c \gtrsim k_{\rm B}T$; in 3D systems the free motion along the magnetic field partly obscures the Landau level structure and the strong-field regime is reached only for $\hbar\omega_c \gtrsim \sqrt{N_{\rm F}}k_{\rm B}T$. In the strong-field regime the density of states has strong peaks and the chemical potential tends to be pinned to these peaks. It is these peaks in the density of states that can reverse the decrease of $T_{\rm c}$ with field and lead to a peculiar regime where $T_{\rm c}$ increases with field. As the strong-field limit is approached the Landau level at the Fermi energy contributes more strongly to the sum in (19). One immediate effect apparent even at comparatively weak fields (MacDonald *et al.* 1992) is the decrease in $T_{\rm c}$ for odd j. (For COM j odd the probability of pairs occupying the same Landau level is zero.) Magneto-oscillations (Rajagopal and Vasudevan 1966; Gruenberg and Gunther 1968; Rasolt 1987; Tešanović and Rasolt 1989; Tešanović *et al.* 1991; Norman 1990; Rieck *et al.* 1990; MacDonald *et al.* 1992) in $T_{\rm c}$, and in all properties of the mixed state (Norman *et al.* 1992) of the superconductor occur as Landau levels pass through the Fermi level. These oscillations have been observed experimentally (Graebner and Robbins 1976; Kido *et al.* 1991; Smith *et al.* 1991) and are not yet understood in complete detail.

At extremely strong fields a regime can be reached where only electrons in the Landau level at the Fermi energy contribute importantly to the pairing. In this limit (for 2D systems) $T_{\rm c}$ reaches a maximum when the Landau level is half full (Akera *et al.* 1991) and (19) reduces to

$$T_{cj} = \frac{\hbar\omega_c\lambda}{8}\,|B_j^{N_{\rm F},N_{\rm F}}|^2\,. \tag{25}$$

Note that T_{cj} is proportional to the magnetic field strength. In the extreme quantum limit all electrons are in the lowest Landau level and $N_{\rm F}=0$. Since the maximum value of j is $2N_{\rm F}$ it happens that superconductivity occurs in the $j=0$ channel just as in the weak magnetic field limit. This similarity in the nature of the superconducting order in the weak and infinitely strong field regimes suggests that no novel behaviour can occur at intermediate fields. This suggestion is misleading as we can see by looking at the case where $N_{\rm F}\neq 0$. The maximum COM kinetic energy channel for the Cooper pair is $2N_{\rm F}$ and pairing can occur in any even-j channel. From the expression for $B_j^{N_{\rm F},N_{\rm F}}$ we find that in this case $T_{\rm c}$ tends to be larger for j close to either its minimum or maximum values and is always the same for $j = 0$ and $j = 2N_{\rm F}$ (see Table 1). This result can be understood by calculating the probability that two electrons of the same energy $\varepsilon_{\rm F}$ but with completely uncertain relative orientations of momentum will have a given COM kinetic energy, ε_R. Averaging over angles it is easy to show that

$$P(\varepsilon_R) = \frac{1}{\pi}[\varepsilon_R(2\varepsilon_{\rm F} - \varepsilon_R)]^{-\frac{1}{2}}\,, \tag{26}$$

which is peaked near the minimum and maximum possible values for ε_R. Thus in the extremely strong field regime there is the possibility of unusual superconducting states in which Cooper pairs are in states with elevated kinetic energies. In mean-field theory the vortex–lattice state is found (Akera *et al.* 1991) to have $j > 0$ and to have associated unusual properties including the possibility of having several vortices per period of the lattice.

Table 1. $|B_j^{N_{\rm F},N_{\rm F}}|^2$ **for** $N_{\rm F} = 0,1,2,3$

$j = 0$	$j = 2$	$j = 4$	$j = 6$
1	0	0	0
1/2	1/2	0	0
3/8	1/4	3/8	0
15/48	3/16	3/16	15/48

5. Concluding Remarks

In this article we have discussed how the suppression of superconductivity by a magnetic field can be understood completely microscopically in terms of the quantum mechanics of pairs of particles in a magnetic field. The results obtained in this way are equivalent to those obtained by Ginzburg–Landau theory and Gor'kov theory in their ranges of validity. The suppression is related to the quantum uncertainty in the kinetic energies of the individual electrons making up a Cooper pair of definite COM kinetic energy. We have also discussed how the suppression can be overcome by the enhancement of the density of states near the Fermi energy which occurs for sufficiently strong magnetic fields in clean samples, and we have explained why the Cooper pair wavefunction can be unusual in this regime. Ginzburg–Landau theory is not valid in the regime of strong-field superconductivity except for the case where pairing occurs entirely within the $N = 0$ Landau level.

We have restricted our attention here to aspects which follow directly from the quantum mechanics of pairs of charged particles in a magnetic field and the reader should be aware that many other issues arise, some parasitically, especially when considering superconductivity in extremely strong magnetic fields. For example, in our discussion we have, for the sake of definiteness, taken the electron g-factor to be zero; a nonzero g-factor will affect results at strong fields (Rasolt 1987; Tešanović and Rasolt 1989; Tešanović *et al.* 1989, 1991; Norman 1990; Rieck *et al.* 1990). For the sake of our discussion here we have also assumed that the standard mean-field theory of superconductivity, which leads to the expressions for $T_{\rm c}$ we have employed and which is known to be reliable at weak fields, can also be used in the strong-field regime. It is certain (Rasolt and Tešanović 1992) that this is not entirely correct, especially in the 2D case (MacDonald *et al.*, to be published), although we believe that the considerations discussed here are still essential for the physics in that regime.

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