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A Cyclic Symmetry Principle in Physics

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Abstract

Many areas of modern physics are illuminated by the application of a symmetry principle, requiring the invariance of the relevant laws of physics under a group of transformations. This paper examines the implications and some of the applications of the principle of cyclic symmetry, especially in the areas of statistical mechanics and quantum mechanics, including quantized field theory. This principle requires invariance under the transformations of a finite group, which may be a Sylow π-group, a group of Lie type, or a symmetric group. The utility of the principle of cyclic invariance is demonstrated in finding solutions of the Yang–Baxter equation that include and generalize known solutions. It is shown that the Sylow π-groups have other uses, in providing a basis for a type of generalized quantum statistics, and in parametrising a new generalization of Lie groups, with associated algebras that include quantized algebras.

1. Introduction

Progress in physics in the present century has in many instances followed the discovery of a new type of symmetry in the laws of nature. The understanding of the symmetries of space–time played a vital part in the development of the theory of relativity; the uncovering of dynamical symmetries provided a key to the solution of many problems in quantum mechanics; and quantized field theory has been founded on the recognition of the statistical symmetries implied by Fermi–Dirac and Bose–Einstein statistics, and the symmetries of the underlying gauge transformations. In each instance, an appropriate formulation of the physical laws has been found in terms of an algebraic structure, usually a Lie algebra or superalgebra, and an associated group of transformations under which the physical laws are invariant, depending on the type of symmetry involved.

In this paper we shall examine some of the implications of another symmetry principle that, perhaps because of its apparent simplicity, has not been so well recognized in the scientific literature. Cyclic symmetry, and invariance under cyclic permutations, are important in many areas of physics where there are periodicities in time or space, or some other variable, irrespective of whether the periodic variable has a discrete or continuous spectrum. If the spectrum is discrete, a natural description of cyclic invarance may be given in terms of Sylow π-groups. Thus, electronic spin and parity are usually represented in terms of Dirac matrices, which are generators of a four-dimensional representation of the

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Lorentz group; but there is an alternative representation (see Appendix B) in terms of a π-group, which gives expression to the cyclic invariance of the spin and parity under reflections.

It has been argued (Coish 1959; Shapiro 1960; Plokhotnikov 1991) that the apparent continuity of space and time are consistent with the hypothesis of a finite geometry based on cyclic symmetry, and there are apparent advantages for quantum electrodynamics and other quantized field theories in this hypothesis, since none of the divergences associated with the space–time continuum arize. An important feature of the finite geometries involved is that they are not inconsistent with a kind of rotational, Lorentz and Poincaré invariance. In fact, wherever the spectrum of a variable is continuous, the usual representations are in terms of circular, hyperbolic or elliptic functions; but (as shown in Appendix A to this paper) all such functions can be regarded as furnishing the represention space of a finite π-group.

One application of the π-groups, outlined in Appendix B, is to the quantized theory of particles with 'colour', and was implicit in papers by the author (Green 1975, 1976) and Kleeman (1983, 1985). The commutation coefficients of supersymmetric field theories are elements of one of the simplest of π-groups, and have a natural generalization in the colour superalgebras introduced by Rittenberg and Wyler (1978). In the following, however, we shall be concerned mainly with applications of the principle of cyclic symmetry to some other areas of contemporary physics. In recent years there has been much interest in a new type of algebraic structure, called a 'quantum group' (see Jimbo 1989, 1990) which provides new insight into a class of solutions of physical problems, originally obtained independently by special techniques of considerable ingenuity. Included in this class are problems in scattering theory (McGuire 1964; Yang 1967; Gardner *et al.* 1967), quantized field theory (Thirring 1958; Gross and Neveu 1974; Coleman 1975; Zamolodchikov and Zamolodchikov 1979) and statistical mechanics (Baxter 1972, 1982).

Some of the more recent developments have arizen from studies (e.g. Belavin 1981) of the Yang–Baxter equation

$$\sum_{\mathbf{k}} R(x;k_2,k_3;j_2,j_3)R(y;j_1,k_3;k_1,i_3)R(z;i_1,i_2;k_1,k_2)$$
$$= \sum_{\mathbf{k}} R(z;k_1,k_2;j_1,j_2)R(y;k_1,j_3;i_1,k_3)R(x;i_2,i_3;k_2,k_3)\,, \quad (1)$$

in which x, y and z are continuous variables and the i, j and k are positive integers from 1 to m; the summation is over all such values of k_1, k_2 and k_3. In the application to scattering theory (see Yang 1967), x is a momentum variable and $R(x)$ is a factor of the S-matrix. In the application to statistical mechanics of crystal lattices (Baxter 1972), x is a variable depending on the temperature, and $R(x)$ is a factor of the transfer matrix [see equation (30) below]; there, (1) may be regarded as a formulation of the 'star-triangle' relation.

The known solutions of the Yang–Baxter equation fall into a few categories, sometimes with no obvious common feature. When a special relation, of the form $f(x)+f(z)=f(y)$, is assumed between the variables, there are solutions of the type found by Yang and Baxter, and the generalization found by Belavin. There are also possible solutions corresponding to the semi-simple Lie algebras and superalgebras and the 'q-deformed' generalization of these algebras (often called 'quantum groups' or 'quantized algebras'), which have been extensively investigated by Drinfeld (1985), Jimbo (1989) and Zhang *et al.* (1991*a*, 1991*b*) among others. In some instances the existence of a special relation between the parameters appears to be unnecessary. In this paper we shall attempt a unification of these solutions in the light of cyclic symmetry, and in the process shall uncover the existence of some wider classes of solutions.

In the form (1), the Yang–Baxter equation is somewhat inscrutable, but without restriction we may express the R-matrix in the form

$$R(x;i_1,i_2;j_1,j_2)=\sum_a F_a(x;i_1,j_1)F^a(x;i_2,j_2), \tag{2}$$

with a summation over not more than m^2 terms. The solutions found by Baxter and Belavin can be regarded appropriately as particular instances of this general form, which is also consistent with that adopted by Khoroshkin and Tolstoy (1991) in their derivation of the universal R-matrix for 'quantized algebras'.

Then, if $F_a(x)$ and $F^a(x)$ denote m-dimensional matrices with elements $F_a(x;i,j)$ and $F^a(x;i,j)$, (1) can be written as

$$\begin{aligned}&\sum_{a,b,c} F_a(x)F_b(y)\otimes F^a(x)F_c(z)\otimes F^b(y)F^c(z)\\ &\qquad=\sum_{a,b,c} F_b(y)F_a(x)\otimes F_c(z)F^a(x)\otimes F^b(z)F^c(y),\end{aligned} \tag{3}$$

where $F_a(x)F_b(y)$ is the matrix product, and $\otimes$ denotes a direct product of the matrices which it separates. There is considerable latitude in the choice of the $F_a(x)$, and also some arbitrariness in the relation assumed between $F_a(x)$ and $F^a(x)$. Here, with little loss of generality, we take

$$F_a=f_a(x)H_a,\qquad F^a=f^a(x)H^a\,, \tag{4}$$

where H_a is an element of a finite group $\mathcal{G}_{\mathcal{H}}$ of matrices, $H^a=(H_a)^{-1}$ is its inverse, and $f_a(x)$ and $f^a(x)$ are scalar coefficients. The latter may be chosen so that $\mid f^a(x)\mid=\mid f_a(x)\mid$. The summations in (2) and (3) are not, in general, over all elements of the group, but over a maximal linearly independent subset forming a basis for the algebra of the group. Then (3) can then be written

$$\begin{aligned}&\sum_{a,b,c}\varphi_a(x)\varphi_b(y)\varphi_c(z)H_aH_b\otimes H^aH_c\otimes H^bH^c\\ &\qquad=\sum_{a,b,c}\varphi_a(x)\varphi_b(y)\varphi_c(z)H_bH_a\otimes H_cH^a\otimes H^cH^b\\ &\qquad=\sum_{a,b,c}\varphi_{\bar a}(x)\varphi_{\bar b}(y)\varphi_{\bar c}(z)H_{\bar b}H_{\bar a}\otimes H_{\bar c}H^{\bar a}\otimes H^{\bar c}H^{\bar b},\end{aligned} \tag{5}$$

where

$$\varphi_a(x) = f_a(x) f^a(x), \tag{6}$$

and $\bar{a} = P_a a$, $\bar{b} = P_b b$, $\bar{c} = P_c c$ represent the results of permutations P_a, P_b, P_c applied to the affixes under the summation. Any of these may of course be an identity, leaving the affixes unchanged.

To satisfy (5), we require that the direct products under the summations in the first and third lines of that equation should be the same, apart from a scalar, unless of course $\varphi_a(x)\varphi_b(y)\varphi_c(z) = 0$. The permutations may be regarded as equivalence relations of the type

$$H_{\bar{a}} = H_p H_a H^p, \qquad H_{\bar{b}} = H_q H_b H^q, \qquad H_{\bar{c}} = H_r H_c H^r \tag{7}$$

on the group. Even then, p, q and r must be chosen to ensure the existence of scalars (S_{ba}, S_c^a, S^{cb}) in the algebra of the group, such that $H_{\bar{b}} H_{\bar{a}} = S_{ba} H_a H_b$, etc., i.e.

$$H_q H_b H^q H_p H_a H^p H^b H^a = U_{qb} U_{bpa} = S_{ba} H_0 ,$$

$$H_r H_c H^r H_p H^a H^p H^c H_a = U_{rc} U_{cp}^a = S_c^a H_0 ,$$

$$H_r H^c H^r H_q H^b H^q H_c H_b = U_r^c U_q^{cb} = S^{cb} H_0 , \tag{8}$$

where $H_0 = 1$ is the identity element of G_H, and

$$U_{ab} = H_a H_b H^a H^b, \qquad U_{abc} = H_a H_b H_c H^b H^a H^c \tag{9}$$

are elements of its derived subgroup D_H. Then (5) reduces to the form

$$\sum_{a,b,c} [\varphi_a(x)\varphi_b(y)\varphi_c(z) - S_{ba} S_c^a S^{cb} \varphi_{\bar{a}}(x)\varphi_{\bar{b}}(y)\varphi_{\bar{c}}(z)]$$

$$\times H_a H_b \otimes H^a H_c \otimes H^b H^c = 0 . \tag{10}$$

In general, all coefficients of the direct product must vanish. However, exceptions are possible if the derived group $\mathcal{D}_{\mathcal{H}}$ is central, so that $U_c^a = V_c^a H_0$, where V_c^a is a scalar, and $H^a H_c$ may be written as $V_c^a H_c H^a$, in the algebra of the group. The exceptions arise because there is then an identical relation $(H^a H_c)(H_a H_b)(H^b H^c) = V_c^a H_0$ between the factors of the direct product, so that it is possible for terms under only two of the summation variables a, b and c to be linearly independent. However, as will be seen in the following Section, this possibility is realized only if there is also an identical relation of the type $f(x) + f(z) = f(y)$ (e.g. $x = z = y + w$) between the variables.

From these considerations it may be concluded that the solutions of the Yang–Baxter equation may be classified according to (a) the type of finite group, assuming that G_H is finite; (b) whether there exists a relation between the variables x, y and z; and (c) the type of permutations P_a, P_b, P_c required to derive (5) or (10). The classes suggested by these criteria are not mutually exclusive, but their relevance for our present purposes may be summarized as follows:

[1] The derived (commutator) group D_H may be central, and G_H is then what is known as a Sylow π-group. If in addition there is a suitable relation between the variables, the terms under the summations in (10) are not linearly independent, and non-trivial solutions are possible. If, in addition, each of the permutations P_a, P_b, P_c is the identity, then $H_p = H_0$ and $\varphi_{\bar{a}}(x) = \varphi_a(x)$, etc., and it follows from (8) that the S_{ab} in (10) can be replaced with V_{ab}. The solutions found by Baxter and Belavin were of this type and are included among the more general solutions obtained in Section 2 below. In Apppendix B it is shown that a group of the same type can be made the basis of a generalized quantum statistics.

[2] If, however, y may be varied independently of x and z, it is necessary in general that each term under the triple summations should vanish. Then $\varphi_{\bar{a}}(x)$ must obviously still be a numerical multiple of, if not identical with, $\varphi_a(x)$, etc. This is indeed a possibility, and some solutions of this type will be also be given in Section 2 of this paper.

[3] There is also a wide variety of finite groups of Lie type. If G_H is of this type, it is a subgroup of a continous group C_H, which may be a Lie group or one of its generalizations. Possibilities of this type will be at least partially explored in Section 3, and here too a use will be found for the Sylow π-groups in deriving a generalization of the known 'quantized' algebras.

[4] The possibility of choosing more general permutations appears to be particularly interesting when G_E itself is isomorphic with a symmetric group or an alternating group. If U_{ab} and U_{abc} are as defined in (9), the general form of (5) is applicable but p, q and r must be chosen so that $U_{qb}U_{pba}$, $U_{rc}U^a_{cp}$ and $U^c_r U^{cb}_q$ are scalar multiples of the identity H_0, whenever $\varphi_a(x)\varphi_b(y)\varphi_c(z)$ is different from zero. It would appear that wherever there are solutions of the Yang–Baxter equation of this type, this should be possible, but the choice of the permutations must depend on the structure of G_H. As a simple example, if $H^a H_b H^c = S^a_c H^c H_b H^a$, with $S^a_c = \pm 1$ as a consequence of the defining relations of the group (as in the quaternion and generalized quaternion groups), we can take $H_p = H_q = H_r = H^b$, since then $S_{ba} = S^{cb} = 1$. It is easy to find other examples, but in the present paper we shall consider in detail only the types [1] to [3] listed above.

2. Pi-group Symmetries

In this Section we intend to demonstrate the utility of cyclic symmetry by obtaining general solutions of the Yang–Baxter equation, classified as type [1] in the Introduction, where the derived group D_H of G_H is not only abelian but central. Then, within the algebra of G_H, the commutator U_{ab} is a numerical multiple V_{ab} of the identity H_0, and the first relation of (9) may be written

$$H_a H_b = V_{ab} H_b H_a \,. \tag{11}$$

As in (3), we suppose that, as matrices, the H_a are linearly independent; they may also be regarded as forming the factor group $\mathrm{G}'_H = \mathrm{G}_H/\mathrm{D}_H$. The group

G_H is finite, so that any element can be written in the form $\omega^K H_a$, where K is an integer (mod m) and ω is an mth root of unity, in an m-dimensional representation. We may write the group multiplication law for G'_H as

$$H_a H_b = W_{ab} H_{a+b}, \qquad W_{ab}/W_{ba} = V_{ab}, \tag{12}$$

where the W and V are mth roots of unity. The addition of subscripts defined in (12) is easily seen to be commutative and associative; the subscripts are therefore in a vector space S over the integers (mod m), called a grading vector space. The elements of G_H that are numerical multiples of H_a form a normal subgroup G_a of G_H and are said to be a-graded. We note that $H_{-a} = W_{a,-a} H^a$ may be different from the inverse H^a of H_a, except when $m = 2$, though the latter also belongs to G_{-a} and is $(-a)$-graded.

The group G_H characterized by (11) is a Sylow π-group in general, so-called because its structure depends on the set π of relatively prime factors of m; it is called a p-group if m is some power of a single prime p. If m is not prime, we shall therefore seek a solution of the Yang–Baxter equation corresponding to some factorization $m = m_1 m_2 \dots m_n$ of m, not necessarily complete or in terms of relatively prime factors. The factors can be ordered so that $m_1 \geq m_2 \geq \dots \geq m_n$; the jth factor corresponds to a factor F_j in a chain of normal subgroups in the composition series of D_H, and a grading vector a can be expressed in terms of n linearly independent components a_j, thus:

$$a = a_1 + m_1\big(a_2 + m_2(\dots + m_{n-1} a_n)\dots\big). \tag{13}$$

If the m_j are relatively prime, the a_j are uniquely determined by a relation of this type; otherwise, it will provide essential information concerning the dimensionality of the grading vector space. In any event, there is a representation as a direct product of n factors, thus:

$$H_a = H_{a_1} \otimes H_{a_2} \otimes \dots \otimes H_{a_n}, \tag{14}$$

where the jth factor is an m_j-dimensional matrix. But as such a matrix has m_j^2 linearly independent elements, a_j is itself a two-dimensional vector in a subspace S_j of S, with components a_j^1 and a_j^2 that are integers (mod m_j) with a suitably chosen basis. If Ω_{j1} and Ω_{j2} are the linearly independent vectors of this basis, a_j is expressed in the form

$$a_j = a_j^1 \Omega_{j1} + a_j^2 \Omega_{j2}. \tag{15}$$

In general, the integers a_j^1 and a_j^2 form a subset of the $2n$ components a^i of a. It is possible, and usually most convenient, to interpret a_j as a complex number, and Ω_{j1} and Ω_{j2} are then any pair of complex numbers whose ratio is not real.

If, in particular, $n = 1$, a is expressed in the form $a^1\Omega_1 + a^2\Omega_2$, where Ω_1 and Ω_2 may be identified with the complex moduli of Weierstrass elliptic functions (Whittaker and Watson 1940); as shown in Appendix A here, these functions furnish a natural representation space for the type of π-group we are considering. We remark that, though the basis (Ω_1, Ω_2) can be chosen arbitrarily, the special choice $\Omega_1 = \pi$ and $\Omega_2 = \pi\tau$, where τ is pure imaginary, allows Weierstrass' functions to be expressed in terms of Jacobi's theta-functions. The solutions of the Yang–Baxter equation found by Baxter and Belavin were in fact expressed in terms of theta-functions.

We shall now find explicit matrix representations for the H_a. Because of the decomposition (14), it will be found possible to obtain a general result by assuming to begin with that m is prime or unfactorized. The matrix elements can be expressed in terms of the W in (11): if a and b are grading vectors, a representation for W_{ab} can be written down corresponding to any integral bilinear form $\langle a, b\rangle$ in a and b, thus:

$$W_{ab} = \exp(2\pi \mathrm{i}\langle a, b\rangle/m)\,. \tag{16}$$

According to (12), the commutation coefficients are given by

$$V_{ab} = \omega^{(a,b)}, \qquad (a,b) = (\langle a, b\rangle - \langle b, a\rangle), \tag{17}$$

where $\omega = \exp(2\pi\mathrm{i}/m)$ is the primitive mth root of unity. The conditions $V_{ab}V_{ba} = 1$ and $V_{a+b,c} = V_{ac}V_{bc}$ required by (8) are obviously satisfied, and $V_{a0} = V_{0a} = 1$. The corresponding matrix representation for the H_a is

$$(H_a)_{bc} = W_{ac}\delta_{b,a+c}, \tag{18}$$

and it is easy to verify that (12) is satisfied. There are also irreducible m-dimensional representations for G_H, obtained from (18) by replacing vector subscripts with their m-valued components; in a representation of this type, the entire π-group is generated by three matrices whose elements are $\omega\delta_{j,k}$, $\omega^j\delta_{j,k}$ and $\delta_{j,k+1}$, where j and k are integers (mod m); the order of the group is therefore m^3. We note that in all these representations, the inverse is also the hermitean conjugate.

When m is not prime, (18) is still valid, but we may suppose that the grading vectors are expressed as in (13), with linearly independent components corresponding to factors of m. Then H_a is given by (14), and the the grading vectors a_j may be expressed as in (15), with a possibly different basis for each value of j. The matrix elements are $W_{a_jb_j} = \exp[2\pi\mathrm{i}\langle a, b\rangle_j/m_j]$, with the following relation between the $\langle a, b\rangle_j$ and $\langle a, b\rangle$:

$$\langle a, b\rangle = \langle a, b\rangle_1 + m_1\big(\langle a, b\rangle_2 + m_2(\ldots + m_{n-1}\langle a, b\rangle_n) \ldots\big)\,. \tag{19}$$

The representations obtained from (14) are analogous to those obtained by the Hopf construction to generate new representations of quantum algebras from two or more known representations.

Apart from the matrix representations shown in (18) there are representations in terms of Weierstrass' elliptic functions (and therefore also in terms of Jacobian ϑ-functions), as shown in Appendix A.

Sylow π-groups also have an application to a generalized quantum statistics for particles with colour, and a matrix factor group of this type (with elements denoted by ϵ_a) was introduced by the author (Green 1975, 1976; see also Kleeman 1983, 1985) to form the basis of a field theory of particles (quarks and gluons) with colour. This application, which could also have implications for statistical mechanics, is outlined, in a somewhat more general form, in Appendix B. Our principal interest in this Section is in the application to solutions of type [1] of the Yang–Baxter equation, and here it appropriate to recognize that a generalization of Baxter's solution in the form

$$R(x) = \sum_a \theta(x - w + v) H_a \otimes H^a / \theta(v) \,,$$

in which $\theta(x)$ is a modified Jacobian ϑ-function, was first conjectured by Belavin (1981); this implicitly made use of a π-group of the type considered in this Section, with a complex grading vector and a special bilinear $\langle a, b\rangle$. The same π-group was also introduced, implicitly or explicitly, by Bovier (1983), Richey and Tracy (1986) and Quano and Fujii (1991), in confirming Belavin's solution of the Yang–Baxter equation; Richey and Tracy called it the Heisenberg group. The published verifications of Belavin's solution are by no means transparent, however, and here we shall show that, in the light of the properties and representations of the π-group, our generalized solution of type [1] amounts to no more than an identity satisfied by Weierstrass' σ-function.

With the group G_H defined as in (11) and (12), solutions fall into two classes, listed under [1] and [2] in Section 1. We consider first those of type [1], which require a relation of the type $f(x) + f(z) = f(y)$ between x, y and z; following a change of variable $f(x) \to x - w$ if necessary, this may be written $x + z = y + w$. From (10) with $\bar{a} = a$, and $S_{ab} = V_{ab}$, etc., we infer that the Yang–Baxter equation may be written

$$\sum_{a,b,c} (V_c^a - V_{ba} V^{cb}) \varphi_a(x) \varphi_b(y) \varphi_c(z) H_a H_b \otimes H_c H^a \otimes H^b H^c \,. \tag{20}$$

We now substitute $H_a H_b = W_{ab} H_{a+b}$ and $H_c H_b = W_{cb} H_{b+c}$, so that $H_c H^a = W_{cb} H_{b+c} H^{a+b} / W_{ab}$ and $H^b H^c = H^{b+c} / W_{cb}$, and then change two of the summation variables a and c to $d = a + b$ and $e = b + c$. After substituting from (17) for the V and discarding the summations over d and e, we find that (20) is satisfied, provided

$$(\omega^{(e,d)} - 1) \sum_b \omega^{(b, e-d)} \varphi_{d-b}(x) \varphi_b(y) \varphi_{e-b}(z) = 0. \tag{21}$$

If we proceed from (14), it becomes clear that (20) may also be satisfied by

$$\varphi_a(x) = \varphi_{a_1}^{(1)}(x) \varphi_{a_2}^{(2)}(x) \ldots \varphi_{a_n}^{(n)}(x) \,, \tag{22}$$

where the jth factor is the solution of an equation identical to (21), but with dimension equal to m_j instead of m, if $n > 1$. To obtain the most general solution of this type, it is therefore sufficient to obtain solutions of (20) when $m = m_j$, and this will be done. However, we first write down a solution for any m that, with the choice of a Jacobian basis, is equivalent to the solution obtained by Belavin. To verify the solution, we shall need to know only that if $g = g^1\Omega_1 + g^2\Omega_2$, where g^1 and g^2 are integers, then Weierstrass' σ-function satisfies

$$\sigma(u + 2g) = s_g \exp[2g'(u + g)]\sigma(u)\,,$$

$$s_g = \exp[\pi \mathrm{i}(g^1 + g^2 + g^1 g^2)], \qquad g' = g_1\eta_1 + g_2\eta_2\,, \tag{23}$$

where η_1 and η_2 are constants given by $\eta_1 = \sigma'(\Omega_1)/\sigma(\Omega_1)$ and $\eta_2 = \sigma'(\Omega_2)/\sigma(\Omega_2)$.

As (21) is obviously satisfied identically when $\omega^{(e,d)} = 1$, the possibilities $d = 0$, $e = 0$ and $d = e$ need be given no further consideration. A consideration of other pairs of values shows that a solution, in terms of Weierstrass' σ-function, when $x + z = y + w$, is

$$\varphi_a(x) = \sigma(x - w + v + 2a)/\sigma(v + 2a). \tag{24}$$

To prove this, we examine the effect of a permutation $b \to \bar{b} = b + g$ of terms under the summation in (21) and note that, according to (23), $\varphi_{d-b}(x)$ acquires a factor $\exp[-2g'(x - w)]$, $\varphi_b(y)$ acquires a factor $\exp[2g'(y - w)]$ and $\varphi_{e-b}(z)$ acquires a factor $\exp[-2g'(z - w)]$, so that the product of the φ is invariant. However, if we choose $g = d$ or $-e$, the commutator $\omega^{(b,e-d)}$ acquires an additional factor $\omega^{(d,e)}$ which is different from 1. As the sum in (21) cannot change under a permutation of terms, it can only be zero, and the equation is satisfied identically by (24) for all values of d and e. Special instances of identities of this type, satisfied by the theta-functions, were found by Jacobi (see para. 21·22, Whittaker and Watson 1940).

We consider next factorized solutions of (21); because of (14), only $m = m_j$ need be considered. When $m_j = 2$, $\omega = -1$, and subscripts like b take only the four values 0, Ω_1, Ω_2 and $\Omega_3 = \Omega_1 + \Omega_2$, so if (d, e) does not vanish, d, e and $d + e$ (mod Ω_1, Ω_2) must be a permutation of the last three of these. When $m_j > 2$, ω is complex, so that E_{-a} is distinct from E_a, except when $a = 0$. The required modification of (24), for fixed d and e, but any a, is

$$\varphi_a(x) = \prod_{j=0}^{m-1} \sigma(x - w + v_j + 2a/m)/\sigma(v_j + 2a/m)\,,$$

$$v_j = v + 2jg/m\,. \tag{25}$$

To show that this makes (21) an identity, we again exclude from consideration the possibilities $d = 0$, $e = 0$ and $d = e$ for which $(d, e) = 1$, and notice that the permutation $b \to \bar{b} = b + g$ (where $g = d$ or $g = -e$) changes the factor $\omega^{(b,e-d)}$ by an additional factor $\omega^{(d,e)}$ different from 1. This is sufficient to ensure the required vanishing of the left side, provided that the product of the φ in (21) is invariant under the same permutation. This invariance is again guaranteed by the condition $x + z = y + w$. For when the arguments of the σ-functions are changed by $2g/m$, the factors of the numerator and the denominator in (25) are permuted, but one of them acquires a multiplicative factor; thus $\varphi_{d-b}(x)$ again acquires the factor $\exp[-2g'(x - w)]$. But, when $(x - w) + (z - w) = (y - w)$, the product of these multiplicative factors is 1 and the product of the φ in (21) is left invariant under the permutation. But as $\omega^{(b,e-d)}$ acquires a factor different from 1, it follows that the left side of (21) must vanish identically.

Using (15) and (25), the solutions of type [1] of the Yang–Baxter equation can be written down for any integer m, and any factorization. Factorization yields not only more general solutions (because of the variety of possible values of Ω_{j1} and Ω_{j2}), but different matrix representations when a power of a prime is factorized. The solution in the form (24) is a true analogue of Belavin's solution: with the choice $(1,\tau)$ of basis it can be seen that they are equivalent, since the periodicities are the same. The significance of the appearance of the σ-functions is further elucidated in Appendix A.

It is appropriate to mention here some solutions of type [2], which require no relation between the variables. It is clear from (19) that such solutions are possible only if

$$V_{ab}V_c^a V^{bc} = \omega^{(a,b)+(c,a)+(a,b)} = 1 \tag{26}$$

for non-vanishing φ_a, φ_b and φ_c. This relation is not satisfied if (a,b) is a general antisymmetic bilinear form, but it clearly is if the vectors a, b and c satisfy

$$(a,b) = (a-b,g), \tag{27}$$

etc., with a fixed value of g. This degenerate form results if a and b are both on a fixed line (mod $2\Omega_1, 2\Omega_2$) in the complex plane, and (27) is thus seen to be an equivalence relation satisfied by a set of grading vectors whose range is singly periodic, instead of doubly periodic as usual. If such a relation holds whenever φ_a and φ_b are both different from zero, the Yang–Baxter equation of this type is again an identity. There is also a more general and less restrictive solution, obtained by writing

$$H_a = H_{a_1} \otimes H_{a_2}\,, \tag{28}$$

where a_1 and a_2 are unrestricted in the grading vector space, but have a sum $a = a_1 + a_2$ which is singly periodic as implied by (27). One of the representations of the direct product may be defined on a space of Weierstrassian elliptic functions, as shown in Appendix A. A hierarchy of solutions of this type can be expressed in the form (14), where the a_j are unrestricted except that their sum is singly periodic.

We conclude this Section with a brief and generalized discussion of the applications of cyclic symmetry to various problems in statistical mechanics and quantum mechanics. These are problems where the Hamiltonian energy of a system is invariant, or approximately so, under the cyclic permutation of the set of dynamical variables (spins or momenta) which determine the state of the system. In the statistical mechanics of a two-dimensional lattice, this is true if edge effects are neglected so that the lattice may be thought of as wound on a torus. In the motion of a set of similar particles on a line, the invariance is exact when a periodic boundary condition is applied. Since the Hamiltonian is a cyclic invariant, its eigenvalues can be found by expressing it as the sum of a set of commuting cyclic invariants. This was essentially the method adopted by Onsager (1944) in his brilliant but difficult solution of the two-dimensional Ising problem; this corresponds to the choice $m = 2$ and assumes the relation $\varphi_0^2 = \varphi_1^2 - \varphi_2^2 + \varphi_3^2$, in our notation. This relation can be exhibited as an identity satisfied by the Jacobian theta-functions.

In statistical mechanics, the free energy is proportional to $-\log(Z)$, where Z is the partition function. As in Belavin's model, we consider a two-dimensional rectangular lattice, with interactions depending on configurations on adjacent sites; toroidal boundary conditions are assumed, so that the ends of all rows and columns are adjacent. For a lattice with k rows and l columns, the partition function is given by

$$Z = \mathrm{tr}\,(T^k)\,, \tag{29}$$

where T is the transfer matrix connecting configurations on adjacent rows, and tr denotes the trace. The transfer matrix T may in turn be expressed as a partial trace of a direct product $R_1 \otimes R_2 \otimes \ldots \otimes R_l$ of l similar m^2-dimensional R-matrices, thus:

$$T = \sum_a \varphi_{a_1}\varphi_{a_2}\ldots\varphi_{a_l} H_{a_1} \otimes H_{a_2} \otimes \ldots \otimes H_{a_l} \mathrm{tr}(H^{a_1}H^{a_2}\ldots H^{a_l})\,,$$

$$R_j = \sum_{a_j} \varphi_{a_j} H_{a_j} \otimes H^{a_j}\,, \tag{30}$$

where the summation in the first line is over all of the a_j. According to (14) and (22), we may also write

$$T = \sum_a \varphi_a H_a \mathrm{tr}(H^{a_1}\ldots H^{a_l})\,, \tag{31}$$

where H_a is now an m^l-dimensional matrix. This expression best exhibits the fact that the T-matrix is itself a partial trace of an m^{2l}-dimensional R-matrix, and that two-dimensional lattice models of this type exhibit two types of cyclic symmetry, which are, however, closely related. There is the rather obvious symmetry associated with cyclic permutations of lattice sites in the same row or column. For planar space lattices, the assumed toroidal boundary conditions represent an approximation, though a good one, and the cyclic symmetry is actually broken by edge effects. Quite apart from this, however, there is the symmetry of the R_j-matrices. The latter are not actually cyclic invariants but, according to (23)–(25), transform very simply, like $R(x) \to \exp[-2g'(x-w)]R(x)$, under a cyclic transformation $a \to a + g$; therefore, the transformation leaves the Yang–Baxter equation invariant. As we have seen above, this invariance is associated with a local symmetry which, however, is important in allowing the R-matrix, and hence the T-matrix of statistical mechanics or the S-matrix of quantum mechanics to be determined explicitly.

3. Cyclic Symmetry and Quantum Algebras

In this Section, we extend the application of the principle of cyclic symmetry to obtain representations of continuous groups and their associated 'quantum algebras', which will be found to include 'quantum groups' or 'quantized algebras', colour algebras, and Lie algebras and superalgebras, all as particular examples. Since the general method adopted is well known in its appplication to Lie algebras and superalgebras, our principal interest is in the colour and quantum algebras. This approach also suggests a possible classification of solutions of type [3] of the Yang–Baxter equation.

We regard the group G_H as finite, but as a subgroup of a continuous group C_H of transformations with parameters that are elements of the algebra A of a π-group similar to that considered in the last Section. We here denote the generators of this algebra by h_a instead of H_a; if $x_a = \lambda_a h_a$ is a numerical multiple of h_a, the group C_H is generated by elements of the form

$$H_a = H(x_a). \tag{32}$$

As in the previous Section, the subscript a may be regarded as a grading vector, so that if x_a and x_b are arbitrary a- and b-graded elements of A, their product $x_a x_b$ is $(a+b)$-graded. The group-theoretical product of $H(x_a)$ and $H(x_b)$ is expressed thus:

$$H(x_a)H(x_b) = H[z(x_a, x_b)]\,, \tag{33}$$

where $z(x_a, x_b)$ is a function of x_a and x_b which completes the specification of C_H. The unit element is denoted by $H(h_0) = 1$, so that $z(x_a, h_0) = x_a$ and $z(h_0, x_b) = x_b$. The inverse of $H(x_a)$ is here denoted by $H(x^a)$, so that $z(x_a, x^a) = h_0$, and $H[z(x^a, x^b)]$ is the inverse of $H[z(x_b, x_a)]$. Of course the grading vector of h_0 is the null vector, and the 0-graded elements of A form a subalgebra A_0 which is also the centre of A. Obviously the grading vector of x^a is $(-a)$.

According to (12), the h_a satisfy relations like $h_a h_b = w_{ab} h_{a+b}$, so that, if $v_{ab} = w_{ab}/w_{ba}$, we have

$$x_a x_b = v_{ab} x_b x_a, \tag{34}$$

where the v_{ab} may be regarded as (numerical) commutation coefficients of a colour algebra, satisfying

$$v_{ab} v_{ba} = 1, \qquad v_{a+b,c} = v_{ac} v_{bc}. \tag{35}$$

There is the possibility that $H_a = h_a = x_a/\lambda_a$, so that the normalized parameters are elements of the group G_H; thus, the present considerations are a true generalization of those of the previous Section. Here, however, we follow the well-known procedure for deriving Lie algebras (where the parameters x_a commute) and superalgebras (where the 'odd' parameters are elements of a Clifford algebra), from the corresponding groups. We therefore give λ_a and λ_b, and consequently x_a and x_b, infinitesimal values, and write

$$H(x_a) = 1 + x_a e_a\,,$$

$$z(x_a, x_b) = x_a e_a + x_b e_b + x_a x_b z_{ab} e_{a+b}. \tag{36}$$

The z_{ab} are not necessarily central, but they belong to a commutative ring R; moreover, if

$$e_b x_a = x_a y_{ab} e_b, \tag{37}$$

then the y_{ab} are also elements of R.

To obtain the generalized commutation relations of the e, we equate coefficients of $x_a x_b y_{ba}$ in the group commutator

$$U_{ab} = H(x_a)H(x_b)H(x^a)H(x^b) = H[z(x_a, x_b)]H[z(x^a, x^b)], \tag{38}$$

and obtain

$$[e_a, e_b] \equiv e_{ab} - u_{ab} e_b e_a = C_{ab} e_{a+b},$$

$$u_{ab} = v_{ab} y_{ab} / y_{ba},$$

$$C_{ab} = z_{ab} - u_{ab} z_{ba}. \tag{39}$$

The commutation factors u_{ab} and 'structure coefficients' C_{ab} are all elements of the commutative ring R; but need commute with only a subset of the e. Thus (39) presents a significant generalization of Lie algebras and superalgebras (reduced to Cartan form), in which the u have values 1 and -1, and also of the somewhat more general colour algebras defined by Rittenberg and Wyler (1978), in which the u are still numerical constants of modulus 1. In these applications, and that outlined in Appendix B, the u satisfy the same relations as the v in (35). In general, however, the u satisfy $u_{ab}u_{ba} = 1$ but not the second of the relations (35). Obviously there is a grading vector (a) associated with any element e_a of the algebra, determined by the subscript of x_a in the first line of (36). All elements of the ring R are 0-graded. The defining relations of the generalized algebra are (39), together with an *ansatz* for u_{abc}, as defined in

$$e_a u_{bc} = u_{ca} u_{abc} e_a, \tag{40}$$

with v_{ab} and y_{ab} defined in (34) and (37). We note that $e_a e_b e_c = u_{abc} e_c e_a e_b$, so the u are analogues of the U defined in (9).

As the *ansatz* for u_{abc} in (40) is not very obvious, we first note some general results. It follows by cyclic permutation of subscripts and from $u_{bc}u_{cb} = 1$ that

$$u_{abc} u_{bca} u_{cab} = 1,$$

$$u_{abc} u_{acb} = u_{ab} u_{ac}. \tag{41}$$

From (39) it follows also that

$$C_{ab} = u_{ab} C_{ba},$$

$$C_{ab} C_{a+b,c} + u_{abc} C_{ca} C_{c+a,b} + u_{abc} u_{cab} C_{bc} C_{b+c,a} = 0. \tag{42}$$

The second of these relations is clearly a generalized form of Jacobi's identity, which reduces to the known generalization for colour algebras [equation (45) below] only when $u_{abc} = u_{ac}u_{bc}$ or $y_{ab} = y_{ba}$. But even if y_{ab} is not symmetric, we can define

$$D_{ab} = C_{ab}/y_{ab} \tag{43}$$

and verify, with the help of the *ansatz*

$$u_{abc} = v_{bc}y_{ab}y_{a+b,c}/(v_{ca}y_{ca}y_{c+a,b}), \tag{44}$$

that

$$D_{ab} = -v_{ab}D_{ba},$$

$$v_{ca}D_{ab}D_{a+b,c} + v_{bc}D_{ca}D_{c+a,b} + v_{ab}D_{bc}D_{b+c,a} = 0, \tag{45}$$

which are precisely the relations satisfied by the structure constants of a corresponding colour algebra. The elements c_a of the colour algebra satisfy

$$c_a c_b - v_{ab}c_b c_a = D_{ab}c_{a+b}, \tag{46}$$

and the importance of the generalized Jacobi identities (45) is that they guarantee the existence of an adjoint representation in which the matrix elements $(c_a)_{bc}$ are $D_{ba}\delta_{a+b,c}$. When D_{ab} and y_{ab} are given, $C_{ab} = y_{ab}D_{ab}$ will satisfy (42). Thus, the *ansatz* of (44) has the virtue is that it is sufficient to ensure the existence of a solution of (42). There may be other possibilities, but it seems likely that any solution of the Yang–Baxter equation in terms of continuous groups is associated with some type of 'quantum algebra' defined in this way.

It is noteworthy that the generalized algebras of Lie type defined above do in fact include quantized algebras such as $U_q[gl(n)]$ which have been studied intensively in recent years, and also a related but somewhat simpler algebra which (Green *et al.* 1993) we call $gl_Q(n)$. We adopt a tensor notation, in which the affixes denoted by a, b, ... above are replaced by pairs of affixes $\binom{i}{j}$, $\binom{k}{l}$, ..., each of which takes values 1, ... n. The commutation relations of (39) for both algebras may be written

$$[e^i_j, e^k_l] \equiv e^i_j e^k_l - u^{ik}_{jl} e^k_l e^i_j = a^{ik}_{jl}(\delta^k_j e^i_l - \delta^i_l e^k_j). \tag{47}$$

The grading vectors of e^i_j and e^j_i are equal and opposite, so that the e^i_i are 0-graded and are elements of the commutative ring R.

We first give the commutation and structure coefficients for $gl_Q(n)$; these are not merely elements of R, but numerical multiples of the identity:

$$u^{ik}_{jl} = Qa^{ik}_{jl} = Q \quad \text{if } j = k \text{ and } i \neq l\,,$$

$$u^{ik}_{jl} = a^{ik}_{jl} = Q^{-1} \quad \text{if } i = l \text{ and } j \neq k\,,$$

$$u^{ik}_{jl} = 1, \quad a^{ik}_{jl} = \delta^i_l \delta^k_j \quad \text{otherwise}\,, \tag{48}$$

and it can be inferred from $e^i_i e^i_j = e^i_j(Qe^i_i + 1)$ that

$$e^i_i = (Q^{d_i} - 1)/(Q-1)\,,$$

$$[e^i_j, e^j_i] = (Q^{d_i} - Q^{d_j})/(Q-1)\,, \tag{49}$$

where the d_i take non-negative integral eigenvalues.

We denote the corresponding elements, commutation coefficients and structure coefficients of the quantized algebra $U_q[gl(n)]$ by E^i_j, U^{ik}_{jl} and A^{ik}_{jl}, and set

$$Q = q^2, \quad f_i = \prod_{k=1}^{i} q^{d_k - 1}\,. \tag{50}$$

Then, if

$$E^i_j = q^{-1} f_i^{-1} e^i_j f_j \; (i \geq j), \quad E^i_j = q^{-1} f_{i-1} e^i_j f^{-1}_{j-1} \; (i \leq j)\,, \tag{51}$$

it follows from (48) and (49) that

$$U^{ik}_{jl} = q^{-1} u^{ik}_{jl} \text{ and } A^{ik}_{jl} = q^{-1} a^{ik}_{jl} \text{ if } i < j = k < l \text{ or } i > j = k > l\,,$$

$$U^{ik}_{jl} = q^{-1} u^{ik}_{jl} \text{ and } A^{ik}_{jl} = 0 \text{ if } i = k < j < l \text{ or } i = k > j > l\,,$$

$$U^{ik}_{jl} = u^{ik}_{jl} \text{ and } A^{ik}_{jl} = f_j f_{j-1}(f_i f_{i-1})^{-1} \text{ if } i = l > j = k\,. \tag{52}$$

Since $U^{ik}_{jl} U^{ki}_{lj} = 1$ and $A^{ik}_{jl} = U^{ik}_{jl} A^{ki}_{lj}$, these relations are sufficient to ensure that the essential commutation relations

$$[E^i_{i+1}, E^{i+1}_i] = (q^{h_i} - q^{-h_i})/(q - q^{-1}), \quad h_i = d_i - d_{i+1}\,,$$

$$[E^i_{i+1}, E^i_{i+2}] = 0, \quad E^i_{i+2} = [E^i_{i+1}, E^{i+1}_{i+2}]\,, \tag{53}$$

of $U_q[gl(n)]$, with the above commutation coefficients (cf. Gould *et al.* 1992), are satisfied.

The results of the present Section provide an alternative demonstration of the relevance of quantized algebras to the solution of the Yang–Baxter equation, as well as a more general definition of such algebras than has hitherto appeared in the literature. It has also been shown that quantized algebras result from the use of generalized parameters expressed in terms of elements of π-groups instead of the numerical or anticommuting parameters of Lie algebras or superalgebras.

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References

Baxter, R. N. (1972). *Ann. Phys. (New York)* **70**, 193–228.
Baxter, R. N. (1973). *Ann. Phys. (New York)* **76**, 1–71.
Baxter, R. N. (1982). 'Exactly Solved Models in Statistical Mechanics' (Academic: New York).
Belavin, A. A. (1981). *Nucl. Phys.* **180**, 189–200.
Bovier, A. (1983). *J. Math. Phys.* **24**, 631–41.
Coish, H. R. (1959). *Phys. Rev.* **114**, 383–8.
Coleman, S. (1975). *Phys. Rev.* D **11**, 2088–97.
Gardner, C. S., Greene, J. M., Kruskal, M. D., and Miura, M. (1967). *Phys. Rev. Lett.* **19**, 1095–7.
Gould, M. D., Links, J., and Bracken, A. J. (1992) *J. Math. Phys.* **33**, 1008–22.
Green, H. S. (1975). *Aust. J. Phys.* **28**, 115–25.
Green, H. S. (1976). *Aust. J. Phys.* **29**, 483–8.
Green, H. S., Bracken, A. J., and Gould, M. D. (1993). University of Adelaide Preprint.
Greenberg, O. W. (1989). University of Maryland Preprint.
Gross, D., and Neveu, A. (1974). *Phys. Rev.* D **10**, 3235–53.
Jimbo, M. (1989). *Int. J. Mod. Phys.* A **4**, 3759–77.
Jimbo, M. (Ed.) (1990). 'Yang–Baxter Equation in Integrable Systems' (World Scientific: Singapore).
Khoroshkin, S. M., and Tolstoy, V. N. (1991). *Commun. Math. Phys.* **141**, 599–617.
Kleeman, R. (1983). *J. Math. Phys.* **24**, 166–72.
Kleeman, R. (1985). *J. Math. Phys.* **26**, 2405–12.
McGuire, J. B. (1964). *J. Math. Phys.* **5**, 622–36.
Mohapatra, R. N. (1990). *Phys. Lett.* B **242**, 407–11.
Onsager, L. (1944). *Phys. Rev.* **65**, 117–49.
Plokhotnikov, K. E. (1991). *Sov. Phys., Dok.* **36**, 38–40.
Polychronakos, A. P. (1990). *Mod. Phys. Lett.* A **5**, 2325–33.
Quano, Y., and Fujii, A. (1991). University of Tokyo Preprint 585.
Richey, M. P., and Tracy, C. A. (1986). *J. Stat. Phys.* **42**, 311–48.
Rittenberg, V., and Wyler, D. (1978). *Nucl. Phys.* B **139**, 189–202.
Shapiro, I. S. (1960). *Nucl. Phys.* **21**, 474–91.
Thirring, W. (1958). *Ann. Phys. (New York)* **3**, 91–112.
Yang, C. N. (1967). *Phys. Rev. Lett.* **19**, 1312–15.
Zamolodchikov, A. B., and Zamolodchikov, A. B. (1979). *Ann. Phys. (New York)* **120**, 253–91.

Appendix A

In Section 2 we have made implicit use of the irreducible representation of a Sylow π-group of order m^3 in a linear space S of regular functions of the complex variable x, that are quasi-periodic for displacements $2\Omega_1$ and $2\Omega_2$ in two different directions in the complex plane; these are Weierstrassian σ-functions. Here we shall show that, when the derived group D_E of G_E is central, the Yang–Baxter equation can be rewritten as a functional equation satisfied identically by functions of that type.

If $\varphi(x)$ is any of the functions forming the basis of S, we denote by T_a any translation between the points of an $m \times m$ sublattice of a periodic lattice, with x as a vertex, in the complex plane:

$$T_a \varphi(x) = \varphi(x + 2a/m)\,,$$

$$a = a^1\Omega_1 + a^2\Omega_2, \quad (a^1, a^2 \epsilon \mathbf{Z}_m)\,, \tag{A1}$$

where τ is pure imaginary. The T_a commute, but let us define E_a by

$$E_a = s_a \exp[-2a'(x + a/m)]T_a\,, \tag{A2}$$

where s_a and a' are defined as in (23). Then $E_a E_b = V_{ab} E_b E_a = \omega^{(a,b)} E_b E_a$, as in (11) and (17), with (a, b) given by

$$(a, b) = 2i(ab' - a'b)/\pi = a^1 b^2 - a^2 b^1\,, \tag{A3}$$

because of the identity $\eta_1\Omega_2 - \eta_2\Omega_1 = \mathrm{i}\pi/2$. This is not the most general antisymmetric bilinear form: even when m is prime, it may be multiplied by any integer relatively prime to m; but it is adopted for the sake of simplicity.

All elements of the group G_E are of order m, so it is necessary that $E_a^m = E_0 = 1$, and since $E_a^m \varphi(x) = s_a^m \exp[-2ma'(x + a)]\varphi(x + 2a)$, we require that

$$\varphi(x + 2a) = s_a^m \exp[2ma'(x + a)]\varphi(x)\,. \tag{A4}$$

Because of the identity (23), this functional equation is satisfied by

$$\varphi(x) = \prod_{j=0}^{m-1} \sigma(x + v_j)\,, \tag{A5}$$

where the v_j are disposable constants. This result is consistent with the x-dependence of the solution of the Yang–Baxter equation found in (24), and in retrospect can be seen to motivate that solution. The considerations of Section 2 show that when m is not a prime, representations can be found in which $\varphi(x)$ is replaced by

$$\varphi(x) = \prod_{k=1}^{n} \prod_{j=0}^{m(k)-1} \sigma(x + v_{jk} \mid \Omega_{k1}, \Omega_{k2})\,, \tag{A6}$$

with factors corresponding to k and a set of factors $m(k)$ of m.

It is interesting to note that, assuming the group underlying a solution of the Yang–Baxter equation is a Sylow π-group, the equation itself can be written as

$$\sum_{a,b,c} E_a^{(x)} E_b^{(x)} E_{(y)}^a E_c^{(y)} E_{(z)}^b E_{(z)}^c$$

$$\times\, \varphi_a(x - 2d/m)\varphi_b(y + 2(d - e)/m)\varphi_c(z + 2e/m)$$

$$= \sum_{a,b,c} E_b^{(x)} E_a^{(x)} E_c^{(y)} E_{(y)}^a E_{(z)}^c E_{(z)}^b$$

$$\times \varphi_a(x - 2d/m)\varphi_b\big(y + 2(d-e)/m\big)\varphi_c(z + 2e/m)\,, \tag{A7}$$

instead of (20), where $d = a+b$ and $e = b+c$ as before, thus replacing the direct products by difference operators. This is a functional equation, whose derivation shows that it is no more than an identity satisfied by the σ-function.

In retrospect, the σ-function may be regarded as the most general of a class of functions with periodic or quasi-periodic properties. When $\Omega_1 = \pi$ and $\Omega_2 = \pi\tau$, it can be expressed in terms of the Jacobian ϑ-functions, and when $\tau \to \infty$, the latter may be expressed in terms of circular functions. Wherever these functions appear in the physical literature, it is likely that a cyclic symmetry and an application of the principle of cyclic invariance may be found.

Appendix B

Here we outline some applications of cyclic invariance to particle physics, and especially to the formulation of a kind of generalized quantum statistics, called 'modular statistics' (Green 1975, 1976), different from parastatistics, though it includes parafermi and parabose statistics of order 2, as well as Fermi and Bose statistics in a new combination. As with other types of generalized statistics, the creation and annihilation operators for unobserved particles with 'colour' do not commute or anti-commute; however, they may be used to construct 'colourless' products or modules which do commute or anti-commute and may therefore be used as the basis of a field theory of observable composite particles. Field theories based on modular statistics are consistent with the correspondence principle.

Modular statistics makes use of a π-group G_H of the type introduced in Section 2. In its simplest form, it may be generated by three elements, ω, H_a and H_b, of order m; for convenience, we choose a and b so that the antisymmetric bilinear form (a,b) has the value 1 (mod m); then $H_a H_b = \omega H_b H_a$. The linearly independent elements in the algebra of the group are

$$H_{qa+rb} = (\omega^{qr})^{\frac{1}{2}} (H_a)^q (H_b)^r\,, \tag{B1}$$

where q and r are integers (mod m) which may be interpreted as a particle creation or annihilation number and a colour index, respectively. A composite with a particle number which is a multiple of m is colourless. If m is odd, the square root of ω^{qr} is always an integral power of ω, but if m is even it may include an additional imaginary factor.

For the simplest choice of $m = 2$, $\omega = -1$ and we can set $H_a = \sigma_3$, $H_b = \sigma_1$ and $H_{a+b} = \sigma_2 = \mathrm{i}\sigma_1\sigma_3$, where the σ_k $(k = 1,2,3)$ are Pauli matrices. For $m = 4$, there are two different π-groups, one of which has elements, expressible as in (14), with the properties of Dirac matrices. However, in the application to colour, the simplest choice is $m = 3$.

In general, to construct a field for elementary particles with colour, we use as a colour basis the operators

$$\epsilon_r = H_a (H_b)^r, \quad \epsilon_r^* = \omega^{r-1}\epsilon_{-r}\,, \tag{B2}$$

where ϵ_r^* is the inverse, which is also the hermitean conjugate of ϵ_r. It is easy to verify with the help of (B1) and (B2) that $\epsilon_r\epsilon_s = \omega^{r-s}\epsilon_s\epsilon_r$, $\epsilon_s^*\epsilon_r = \omega^{s-r}\epsilon_r\epsilon_s^*$ and, more generally,

$$\epsilon_r(\epsilon_t \dots \epsilon_w)\epsilon_s = \omega^{p(r-s)}\epsilon_s(\epsilon_t \dots \epsilon_w)\epsilon_r ,$$

$$(\epsilon_t \dots \epsilon_w)\epsilon_s^*\epsilon_r = \omega^{p(s-r)}\epsilon_r\epsilon_s^*(\epsilon_t \dots \epsilon_w) , \tag{B3}$$

where $(\epsilon_t \dots \epsilon_w)$ is a product of $(p-1)$ unstarred factors. From (B1) and (B2) it also follows that products of m ϵ-factors (or the hermitian conjugates of such products), and products of the types $\epsilon_r\epsilon_s^*$ and $\epsilon_t^*\epsilon_u$ form a set of 'modules' in the abelian subgoup of G_H generated by ω and H_a.

The generalized quantum statistics is formulated in the first instance in terms of a finite set $(\alpha_j,\ \alpha^j;\ j = 1 \dots n)$ of modular creation and annihilation operators of order m. These satisfy the generalized commutation relations given in (B6) below, but may be constructed from mn_b ordinary boson and mn_f ordinary fermion operators $(\alpha_{jr},\ \alpha_r^j;\ j = 1 \dots n,\ n = n_b + n_f,\ r = 1 \dots m)$, where

$$\alpha_{jr}\alpha_{ks} - v_{jk}\alpha_{ks}\alpha_{jr} = \alpha_r^j\alpha_s^k - v^{jk}\alpha_s^k\alpha_r^j = 0 ,$$

$$\alpha_{jr}\alpha_s^k - v_j^k\alpha_s^k\alpha_{jr} = \delta_{rs}\delta_j^k \tag{B4}$$

and v_{jk}, v^{jk} and v_j^k are -1 if both affixes are graded odd (as for fermions), and 1 otherwise (when at least one boson is involved). The modular operators are given by

$$\alpha_j = \sum_{r=0}^{m-1} \alpha_{jr}\epsilon_r/m^{\frac{1}{2}}, \quad \alpha^j = \sum_{r=0}^{m-1} \alpha_r^j\epsilon_r^*/m^{\frac{1}{2}} . \tag{B5}$$

There is no particular difficulty in extending the numbers (n_b and n_f) of modular boson and fermion states to infinity, keeping m finite.

From (B2) it is now easily verified that the modular operators defined in (B5) satisfy the relations

$$\alpha_j(\alpha_i \dots \alpha_l)\alpha_k - v_{jk}\alpha_k(\alpha_i \dots \alpha_l)\alpha_j = 0 ,$$

$$(\alpha_i \dots \alpha_l)\alpha_j\alpha^k - v_j^k\alpha^k\alpha_j(\alpha_i \dots \alpha_l) = \delta_j^k(\alpha_i \dots \alpha_l), \tag{B6}$$

if the product $(\alpha_i \dots \alpha_l)$ has $m-1$ factors. With their hermitean conjugates, these relations are obviously those of Fermi–Bose statistics when $m = 1$; they are also the same as for parastatistics when $m = 2$, but are different otherwise. They generalize in a natural way the relations of modular statistics for any number of bosons and fermions with m colour states. The variety of colour states may obviously be expanded indefinitely by increasing the dimensions of the grading vectors in (B1).

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