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Self-focusing of Nonlinear Waves in a Relativistic Plasma with Positive and Negative Ions

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Abstract

We have analysed the phenomenon of self-focusing of nonlinear waves in a relativistic plasma consisting of both positive and negative ions, which are assumed to be hot. We also consider the effect of the inertia of the relativistic electron by treating it dynamically. A modified form of reductive perturbation is used to deduce a nonlinear Schrödinger equation describing the purely spatial variation of the nonlinear wave. Self-focusing of the wave can be ascertained by analysing the transversal stability of the solitary wave. It is shown that the zones of stability of the wave may become wider due to the mutual influence of various factors present in the plasma, thus favouring the process of self-focusing.

1. Introduction

Studies of nonlinear processes occupy a central role in the theoretical research on plasmas. One of the most important phenomena is the formation and propagation of nonlinear waves in plasmas called solitons. The whole story started with the pioneering paper by Washimi and Taniuti (1966). Afterwards, people started to take into account more and more physical effects. These included the effects of two-temperature electrons (Murphy *et al.* 1984), ion temperature (Nejoh 1987), external magnetic fields (Kawahara 1970; Kakutani *et al.* 1968), relativistic mass variation and streaming (Das and Paul 1985; Roychowdhury *et al.* 1988). Also to be noted is the discovery of solitons in more than two dimensions (those described by the Kadomstev–Petviashville equation). Several authors applied the approach of Washimi *et al.* (1966) to multidimensional systems (e.g. Tagare 1977) and obtained encouraging results. In this communication we analyse a new phenomenon that has drawn attention very recently. We refer to the phenomenon of self-focusing of nonlinear waves in a plasma. Analysis of such events requires application of a different form of perturbation technique in more than one space and one time dimension. Such a framework has been set up by Sato *et al.* (1990), who considered the phenomenon of ion wave self-focusing in the presence of negative ions. We have studied the self-focusing of nonlinear waves in a relativistic plasma containing both positive and negative ions, considering the ions to be hot. Another important feature of the present analysis is the inclusion of electron inertia. There has already been some important indication that the role of electron inertia is very significant in the study of solitons (Zhang and Kuehl 1992).

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2. Formulation

We consider a relativistic plasma consisting of electrons and two kinds of ion (positive and negative) assumed to be hot. Here the electron is treated relativistically and we have taken into account the full dynamics of the electron component to analyse the effect of its inertia. Furthermore, we have assumed that the usual hydrodynamic description is possible, so that the dynamical equations governing the plasma can be written as follows.

Positive ions

Continuity equation:

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial x}(n_\alpha U_\alpha) + \frac{\partial}{\partial y}(n_\alpha V_\alpha) = 0\,. \tag{1}$$

Momentum equations:

$$n_\alpha\left(\frac{\partial U_\alpha}{\partial t} + U_\alpha\frac{\partial U_\alpha}{\partial x} + V_\alpha\frac{\partial U_\alpha}{\partial y}\right) = -n_\alpha\frac{\partial \phi}{\partial x} - T_\alpha\frac{\partial n_\alpha}{\partial x} - \sigma_\alpha\frac{\partial p_\alpha}{\partial x}\,, \tag{2}$$

$$n_\alpha\left(\frac{\partial V_\alpha}{\partial t} + U_\alpha\frac{\partial V_\alpha}{\partial x} + V_\alpha\frac{\partial V_\alpha}{\partial y}\right) = -n_\alpha\frac{\partial \phi}{\partial y} - T_\alpha\frac{\partial n_\alpha}{\partial y} - \sigma_\alpha\frac{\partial p_\alpha}{\partial y}\,. \tag{3}$$

Pressure equation:

$$\frac{\partial p_\alpha}{\partial t} + U_\alpha\frac{\partial p_\alpha}{\partial x} + V_\alpha\frac{\partial p_\alpha}{\partial y} + 3p_\alpha\frac{\partial U_\alpha}{\partial x} + 3p_\alpha\frac{\partial V_\alpha}{\partial y} = 0\,. \tag{4}$$

Negative ions

Continuity equation:

$$\frac{\partial n_\beta}{\partial t} + \frac{\partial}{\partial x}(n_\beta U_\beta) + \frac{\partial}{\partial y}(n_\beta V_\beta) = 0\,. \tag{5}$$

Momentum equations:

$$n_\beta\left(\frac{\partial U_\beta}{\partial t} + U_\beta\frac{\partial U_\beta}{\partial x} + V_\beta\frac{\partial U_\beta}{\partial y}\right) = \frac{1}{Q}n_\beta\frac{\partial \phi}{\partial x} - \frac{1}{Q}T_\beta\frac{\partial n_\beta}{\partial x} - \sigma_\beta\frac{\partial p_\beta}{\partial x}\,, \tag{6}$$

$$n_\beta\left(\frac{\partial V_\beta}{\partial t} + U_\beta\frac{\partial V_\beta}{\partial x} + V_\beta\frac{\partial V_\beta}{\partial y}\right) = \frac{1}{Q}n_\beta\frac{\partial \phi}{\partial y} - \frac{1}{Q}T_\beta\frac{\partial n_\beta}{\partial y} - \sigma_\beta\frac{\partial p_\beta}{\partial y}\,. \tag{7}$$

Pressure equation:

$$\frac{\partial p_\beta}{\partial t} + U_\beta\frac{\partial p_\beta}{\partial x} + V_\beta\frac{\partial p_\beta}{\partial y} + 3p_\beta\frac{\partial U_\beta}{\partial x} + 3p_\beta\frac{\partial V_\beta}{\partial y} = 0\,. \tag{8}$$

Electrons

Continuity equation:

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e U_e) + \frac{\partial}{\partial y}(n_e V_e) = 0\,. \tag{9}$$

Momentum equations:

$$n_e\left(\frac{\partial U_e'}{\partial t}+U_e\frac{\partial U_e'}{\partial x}+V_e\frac{\partial U_e'}{\partial y}\right)=-en_e\frac{\partial \phi}{\partial x}+eT_e\frac{\partial n_e}{\partial x}\,, \tag{10}$$

$$n_e\left(\frac{\partial V_e'}{\partial t}+U_e\frac{\partial V_e'}{\partial x}+V_e\frac{\partial V_e'}{\partial y}\right)=-en_e\frac{\partial \phi}{\partial y}+eT_e\frac{\partial n_e}{\partial y}\,. \tag{11}$$

Poisson's equation:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=n_e+n_\beta-n_\alpha\,. \tag{12}$$

Here the subscripts α, β and e stand for the positive ion, negative ion and electron respectively. The symbols n, U, V, T, σ and p respectively denote the density, the x and y component of velocity, the ion temperature, the ratio of ion temperature to electron temperature, and the pressure. Since the ions are nonrelativistic and the electrons are assumed to be relativistic, U_α, V_α, U_β, V_β are nonrelativistic quantities and the electron velocity U_e' along the x-axis is relativistic and is given as

$$U_e'=U_e/(1-U_e^2/c^2)^{\frac{1}{2}}\approx U_e(1+U_e^2/2c^2)\,, \tag{13}$$

whereas V_e, the electron velocity along the y-axis, is nonrelativistic. We have considered here a three-dimensional form of the plasma equations because for the analysis of self-focusing we need to study a purely spatial variation of the nonlinear wave. In the above equations we have normalised all distances by the Debye length $\lambda_{De}=(\epsilon_\theta T_e/n_{e0}e^2)^{\frac{1}{2}}$, time by $\omega_{pi}^{-1}=(n_e e^2/\epsilon_\theta m_\alpha)^{-\frac{1}{2}}$, velocity by $\lambda_{De}\omega_{pi}=(T_e/m_\alpha)^{\frac{1}{2}}$, density by n_{e0}, temperature by T_e and electrostatic potential by $k_B T_e/e$. In these expressionss ϵ_θ, k_B and e denote the dielectric constant, Boltzmann constant and electron charge, respectively.

The computation starts with the expansion of the physical variables (n_α, U_α, V_α, p_α, n_β, U_β, V_β, p_β, n_e, U_e, V_e, ϕ) in the form

$$U=U_0+\sum_{m=1}^{\alpha}\epsilon^m\sum_{l=-\alpha}^{\alpha}U_l^{(m)}(\xi,\eta)\exp[\mathrm{i}l(kx-\omega t)]\,, \tag{14}$$

where $U=(n_\alpha,\ U_\alpha,\ V_\alpha,\ p_\alpha,\ n_\beta,\ U_\beta,\ V_\beta,\ p_\beta,\ n_e,\ U_e,\ V_e,\ \phi)^t\,,$

$$U_0=(n_{\alpha0},\ 0,\ 0,\ 1,\ n_{\beta0},\ 0,0,\ 1,\ 1,\ U_{e0},\ 0,\ 0)^t\,,$$

$$U_l^{(m)}=(n_{\alpha,l}^{(m)},\ U_{\alpha,l}^{(m)},\ V_{\alpha,l}^{(m)},\ p_{\alpha,l}^{(m)},\ n_{\beta,l}^{(m)},\ U_{\beta,l}^{(m)},\ V_{\beta,l}^{(m)},\ p_{\beta,l}^{(m)},$$

$$n_{e,l}^{(m)},\ U_{e,l}^{(m)},\ V_{e,l}^{(m)},\ \phi_l^{(m)})^t\,,$$

so that U, U_0, $U_l^{(m)}$ are each column vectors with twelve components. We also have $n_{\alpha 0} = 1 + n_{\beta 0}$ from the normalised charge neutrality condition. All quantities satisfy the reality condition, e.g. $n_{\alpha,l}^{(m)} = n_{\alpha,l}^{(m)*}$, where the asterisk denotes the complex conjugate. In equations (1)–(12) we have also introduced the scaling variables

$$\xi = \epsilon^2 x\,, \qquad \eta = \epsilon y\,. \tag{15}$$

Now substituting these expansions in equations (1)–(12), and collecting terms of the order of ϵ, we get

$$\mathbf{W}_l^{(1)} U_l^{(1)} = 0\,, \tag{16}$$

where $\mathbf{W}_l$ is a 12×12 matrix with components a_{ij} as follows:

$$a_{11} = -\mathrm{i}l\omega\,, \quad a_{12} = n_{\alpha 0}\mathrm{i}lk\,, \quad a_{21} = T_\alpha \mathrm{i}lk\,,$$

$$a_{22} = -n_{\alpha 0}\mathrm{i}l\omega\,, \quad a_{24} = \sigma_\alpha \mathrm{i}lk\,,$$

$$a_{2,12} = n_{\alpha 0}\mathrm{i}lk\,, \quad a_{33} = -n_{\alpha 0}\mathrm{i}l\omega\,,$$

$$a_{42} = 3\mathrm{i}lk\,, \quad a_{44} = -\mathrm{i}l\omega\,, \quad a_{55} = -\mathrm{i}l\omega\,,$$

$$a_{56} = n_{\beta 0}\mathrm{i}lk\,, \quad a_{65} = T_\beta \mathrm{i}lk/Q\,,$$

$$a_{66} = -n_{\beta 0}\mathrm{i}l\omega\,, \quad a_{68} = \sigma_\beta \mathrm{i}lk\,,$$

$$a_{6,12} = -n_{\beta 0}\mathrm{i}lk/Q\,, \quad a_{77} = -n_{\beta 0}\mathrm{i}l\omega\,,$$

$$a_{86} = 3\mathrm{i}lk\,, \quad a_{88} = -\mathrm{i}l\omega\,,$$

$$a_{99} = U_{\mathrm{e}0}(\mathrm{i}lk - \mathrm{i}l\omega)\left(1 + \frac{U_{\mathrm{e}0}^2}{2c^2}\right),$$

$$a_{9,10} = \left(1 + \frac{3U_{\mathrm{e}0}^2}{2c^2}\right)\mathrm{i}lk\,, \quad a_{10,9} = eT_\mathrm{e}\mathrm{i}lk\,,$$

$$a_{10,10} = \left(1 + \frac{3U_{\mathrm{e}0}^2}{2c^2}\right)(-\mathrm{i}l\omega + U_{\mathrm{e}0}\mathrm{i}lk)\,,$$

$$a_{10,12} = e\mathrm{i}lk\,, \quad a_{11,11} = -\mathrm{i}l\omega + U_{\mathrm{e}0}\mathrm{i}lk\,,$$

$$a_{12,1} = 1\,, \quad a_{12,9} = -1\,, \quad a_{12,12} - -l^2k^2\,, \tag{17}$$

with all other $a_{ij} = 0$. The vector $U_l^{(1)}$ is given as

$$(n_{\alpha,l}^{(1)}, U_{\alpha,l}^{(1)}, V_{\alpha,l}^{(1)}, \ldots, \phi_l^{(1)})\,.$$

The components of this vector for $l = \pm 1$ are

$$
\begin{aligned}
n^{(1)}_{\alpha,\pm1} &= \frac{n^2_{\alpha0}k^2}{\Omega_\alpha}\phi^{(1)}_{\pm1}\,, \\
U^{(1)}_{\alpha,\pm1} &= \frac{n^2_{\alpha0}k\omega}{\Omega_\alpha}\phi^{(1)}_{\pm1}\,, \\
V^{(1)}_{\alpha,\pm1} &= 0\,; \quad V^{(1)}_{\beta,\pm1} = 0\,, \\
p^{(1)}_{\alpha,\pm1} &= \frac{3n_{\alpha0}k^2}{\Omega_\alpha}\phi^{(1)}_{\pm1}\,, \\
n^{(1)}_{\beta,\pm1} &= -\frac{n^2_{\beta0}k^2}{\Omega_\beta}\phi^{(1)}_{\pm1}\,, \\
U^{(1)}_{\beta,\pm1} &= -\frac{n_{\beta0}k\omega}{\Omega_\beta}\phi^{(1)}_{\pm1}\,, \\
p^{(1)}_{\beta,\pm1} &= -\frac{3n_{\beta0}k^2}{\Omega_\beta}\phi^{(1)}_{\pm1}\,, \quad n^{(1)}_{\mathrm{e},\pm1} = \frac{ek^2}{\Omega_\mathrm{e}}\phi^{(1)}_{\pm1}\,, \\
U^{(1)}_{\mathrm{e},\pm1} &= \frac{\Omega_{\mathrm{ce}}ek}{\Omega_\mathrm{e}}\phi^{(1)}_{\pm1}\,, \quad V^{(1)}_{\mathrm{e},\pm1} = 0\,.
\end{aligned}
\tag{18}
$$

The dispersion relation is given by

$$
\det \mathbf{W}_{\pm1} = 0\,, \tag{19}
$$

with

$$
\begin{aligned}
\Omega_\alpha &= n_{\alpha0}\omega^2 - T_\alpha n_{\alpha0}k^2 - 3\sigma_\alpha k^2\,, \\
\Omega_\beta &= n_{\beta0}\omega^2 Q - T_\beta n_{\beta0}k^2 - 3\sigma_\beta k^2 Q\,, \qquad (20) \\
\Omega_\mathrm{e} &= (\omega - U_{\mathrm{e}0}k)\{\omega - (1 + V^2_{\mathrm{e}0}/2c^2)kU_{\mathrm{e}0}\} + eT_\mathrm{e}k^2\,,
\end{aligned}
$$

$$
\Omega_{\mathrm{ce}} = \frac{\omega - (1 + U^2_{\mathrm{e}0}/2c^2)kU_{\mathrm{e}0}}{1 + 3U^2_{\mathrm{e}0}/2c^2}\,. \tag{21}
$$

Next we proceed to terms of second order in ϵ, for $l = 1$, whence we get

$$
\begin{aligned}
n^{(2)}_{\alpha,1} &= \frac{n^2_{\alpha0}\,k^2}{\Omega_\alpha}\phi^{(2)}_1\,, \qquad n^{(2)}_{\beta,1} = -\frac{n^2_{\beta0}\,k^2}{\Omega_\beta}\phi^{(2)}_1\,, \\
U^{(2)}_{\alpha,1} &= \frac{n^2_{\alpha0}\,k\omega}{\Omega_\alpha}\phi^{(2)}_1\,, \qquad U^{(2)}_{\beta,1} = -\frac{n^2_{\beta0}\,k\omega}{\Omega_\beta}\phi^{(2)}_1\,,
\end{aligned}
$$

$$V_{\alpha,1}^{(2)} = -\mathrm{i}\frac{n_{\alpha 0}\,\omega}{\Omega_\alpha}\,\frac{\partial \phi_1^{(1)}}{\partial \eta}\,, \qquad V_{\beta,1}^{(2)} = \mathrm{i}\frac{n_{\beta 0}\,\omega}{\Omega_\beta}\,\frac{\partial \phi_1^{(1)}}{\partial \eta}\,,$$

$$p_{\alpha,1}^{(2)} = \frac{3n_{\alpha 0}\,k^2}{\Omega_\alpha}\phi_1^{(2)}\,, \qquad p_{\beta,1}^{(2)} = -\frac{3n_{\beta 0}\,k^2}{\Omega_\beta}\phi_1^{(2)}\,,$$

$$n_{\mathrm{e},1}^{(2)} = \frac{ek^2}{\Omega_\mathrm{e}}\phi_{12}^{(2)}\,, \qquad U_{\mathrm{e},1}^{(2)} = \frac{\Omega_{\mathrm{ce}}\,ke}{\Omega_\mathrm{e}}\phi_1^{(2)}\,,$$

$$V_{\mathrm{e},1}^{(2)} = -\mathrm{i}e\left(1+\frac{3U_{\mathrm{e}0}^2}{2c^2}\right)\frac{\Omega_{\mathrm{ce}}}{\Omega_\mathrm{e}}\,\frac{\partial \phi_1^{(1)}}{\partial \eta}\,. \tag{22}$$

Also, from the $l = 2$ components of ϵ^2 terms,

$$U_2^{(2)} = U_{20}^{(2)}\{\phi_1^{(1)}\}^2\,,$$

where

$$U_{20}^{(2)} = (N_{\alpha,2}^{(2)},\, U_{\alpha,2}^{(2)},\, V_{\alpha,2}^{(2)},\, P_{\alpha,2}^{(2)},\, N_{\beta,2}^{(2)},\, U_{\beta,2}^{(2)},\, V_{\beta,2}^{(2)},\, P_{\beta,2}^{(2)},\, N_{\mathrm{e},2}^{(2)},\, V_{\mathrm{e},2}^{(2)},\, \Phi_2^{(2)})^{\mathrm{t}}$$

represents a vector whose components are functions of plasma parameters only:

$$N_{\alpha,2}^{(2)} = \frac{n_{\alpha 0}\,k}{2\Omega_\alpha}\left(\frac{2n_{\alpha 0}^3\,k^3\,\omega^2}{\Omega_\alpha^2} + \frac{6\sigma_\alpha\,n_{\alpha 0}^2\,k^5}{\Omega_\alpha^2} + \frac{n_{\alpha 0}^2\,k^3}{\Omega_\alpha}\right) + \frac{k^2\,n_{\alpha 0}^2}{\Omega_\alpha}\Phi_2^{(2)}\,,$$

$$U_{\alpha,2}^{(2)} = \frac{\omega}{n_{\alpha 0}\,k}N_{\alpha,2}^{(2)} - \frac{n_{\alpha 0}^2\,k^3\,\omega}{\Omega_\alpha^2}\,,$$

$$P_{\alpha,2}^{(2)} = \frac{3k}{\omega}U_{\alpha,2}^{(2)} + \frac{6n_{\alpha 0}^2\,k^4}{\Omega_\alpha^2}\,,$$

$$N_{\beta,2}^{(2)} = \frac{kQ_{\beta 0}}{2\Omega_\beta}\left(\frac{2n_{\beta 0}^3\,\omega^2\,k^3}{\Omega_\beta^2} + \frac{6k^5\,\sigma_\beta\,n_{\beta 0}^2}{\Omega_\beta^2} - \frac{k^3\,n_{\beta 0}^2}{Q\Omega_\beta}\right) - \frac{n_{\beta 0}^2\,k^2}{\Omega_\beta}\Phi_2^{(2)}\,,$$

$$U_{\beta,2}^{(2)} = \frac{\omega}{kn_{\beta 0}}N_{\beta,2}^{(2)} - \frac{n_{\beta 0}^2\,k^3\,\omega^2}{\Omega_\beta^2}\,,$$

$$P_{\beta,2}^{(2)} = \frac{3k}{\omega}U_{\beta,2}^{(2)} + \frac{6n_{\beta 0}^2\,k^4}{\Omega_\beta^2}\,,$$

$$V_{\alpha,2}^{(2)} = V_{\beta,2}^{(2)} = V_{\mathrm{e},2}^{(2)} = 0\,,$$

$$N_{\mathrm{e},2}^{(2)} = \frac{k}{2\Omega_{\mathrm{e}}}\left\{\frac{k^3 e^2 (kU_{\mathrm{e}0} - \omega)(1 + 3U_{\mathrm{e}0}^2/2c^2)\Omega_{\mathrm{ce}}}{\Omega_{\mathrm{e}}^2} + (kU_{\mathrm{e}0} - \omega)\frac{3U_{\mathrm{e}0}}{c^2}\right.$$
$$\left. - \left(\frac{9U_{\mathrm{e}0}^2 k}{2c^2} - \frac{3U_{\mathrm{e}0}\,\omega}{c^2}\right)\frac{\Omega_{\mathrm{ce}}^2 e^2 k^2}{\Omega_{\mathrm{e}}^2} - \frac{e^2 k^3}{\Omega_{\mathrm{e}}}\right\} - \frac{k^2 e}{\Omega_{\mathrm{e}}}\Phi_2^{(2)},$$

$$\Phi_2^{(2)} = \frac{1}{A}\left[-\frac{n_{\alpha 0} k}{2\Omega_\alpha}\left(\frac{2k^3 \omega^2 n_{\alpha 0}^3}{\Omega_\alpha^2} + \frac{6\sigma_\alpha n_{\alpha 0}^2 k^5}{\Omega_\alpha^2} + \frac{n_{\alpha 0}^2 k^3}{\Omega_\alpha}\right)\right.$$
$$+ \frac{kQn_{\beta 0}}{2\Omega_\beta}\left(\frac{2k^2 \omega^2 n_{\beta 0}^3}{\Omega_\beta^2} + \frac{6k^5 \sigma_\beta n_{\beta 0}^2}{\Omega_\beta^2} - \frac{n_{\beta 0}^2 k^3}{\Omega_\beta Q}\right)$$
$$- \frac{k}{2\Omega}\left\{\frac{k^3 e^2 (kU_{\mathrm{e}0} - \omega)(1 + 3U_{\mathrm{e}0}^2/2c^2)\Omega_{\mathrm{ce}}}{\Omega_{\mathrm{e}}^2} + (kU_{\mathrm{e}0} - \omega)\frac{3U_{\mathrm{e}0}}{c^2}\right.$$
$$\left.\left. - \left(1 + \frac{9U_{\mathrm{e}0}^2}{2c^2}\right)\frac{\Omega_{\mathrm{ce}}^2 e^2 k^3}{\Omega_{\mathrm{e}}^2} + \frac{3U_{\mathrm{e}0}\,\omega\Omega_{\mathrm{ce}}^2 e^2 k^2}{c^2 \Omega_{\mathrm{e}}^2} - \frac{e^2 k^3}{\Omega_{\mathrm{e}}}\right\}\right], \qquad (23)$$

where we have set

$$A = \frac{1}{\Omega_\alpha \Omega_\beta \Omega_{\mathrm{e}}}[k^2 n_\beta^2 \Omega_\alpha \Omega_{\mathrm{e}} - k^2 e\Omega_\beta \Omega_\alpha + k^2 n_\alpha^2 \Omega_\beta \Omega_{\mathrm{e}} - 4k^2 \Omega_\beta \Omega_\alpha \Omega_{\mathrm{e}}].$$

The $l = 0$ components in the terms of order ϵ^2 yield

$$n_{\alpha,0}^{(2)} = N_{\alpha,0}^{(2)} |\phi_1^{(1)}|^2,$$
$$n_{\beta,0}^{(2)} = N_{\beta,0}^{(2)} |\phi_1^{(1)}|^2,$$
$$n_{\mathrm{e},0}^{(2)} = N_{\mathrm{e},0}^{(2)} |\phi_1^{(1)}|^2,$$
$$\phi_0^{(2)} = \Phi_0^{(2)} |\phi_1^{(1)}|^2,$$
$$U_{\alpha,0}^{(2)} = V_{\alpha,0}^{(2)} = p_{\alpha,0}^{(2)} = U_{\beta,0}^{(2)} = V_{\beta,0}^{(2)} = p_{\beta,0}^{(2)} = U_{\mathrm{e},0}^{(2)} = V_{\mathrm{e},0}^{(2)} = 0. \qquad (24)$$

In equations (24) the coefficients are as follows:

$$\Phi_0^{(2)} = -k^2(n_{\alpha 0}^2 T_{\mathrm{e}} \Omega_{\mathrm{e}} T_\beta \Omega_\beta - n_{\beta 0}^2 T_{\mathrm{e}} \Omega_{\mathrm{e}} T_\alpha \Omega_\alpha + k^2 eT_\alpha \Omega_\alpha T_\beta \Omega_\beta)\Delta^{-1},$$

$$\Delta = \Omega_\alpha \Omega_\beta \Omega_{\mathrm{e}}(n_{\alpha 0}T_{\mathrm{e}}T_\beta + n_{\beta 0}T_{\mathrm{e}}T_\alpha + T_\alpha T_\beta),$$

$$N_{\alpha,0}^{(2)} = -\frac{n_{\alpha 0}}{T_\alpha}\Phi_0^{(2)} - \frac{n_{\alpha 0}^2 k^2}{\Omega_\alpha T_\alpha},$$

$$N_{\beta,0}^{(2)} = \frac{n_{\beta 0}}{T_\beta}\Phi_0^{(2)} - \frac{n_{\beta 0}^2 k^2}{\Omega_\beta T_\beta},$$

$$N_{\mathrm{e},0}^{(2)} = \frac{\Phi_0^{(2)}}{T_{\mathrm{e}}} + \frac{ek^2}{T_{\mathrm{e}} \Omega_{\mathrm{e}}},$$

so that eliminating all the variables in favour of $\phi_1^{(1)}$ in the equations with $l = 1$, we obtain

$$\frac{\partial \phi_1^{(1)}}{\partial \xi} + \frac{B}{\alpha_1} \frac{\partial^2 \phi_1^{(1)}}{\partial \eta^2} + \frac{C}{\alpha_1} |\phi_1^{(1)}|^2 \phi_1^{(1)} = 0 \,, \tag{25}$$

which is the nonlinear Schrödinger equation describing the space variation of the envelope soliton. Note that both ξ and η are scaled space variables. The coefficients B, C and α_1 are very long expressions and are given in the Appendix.

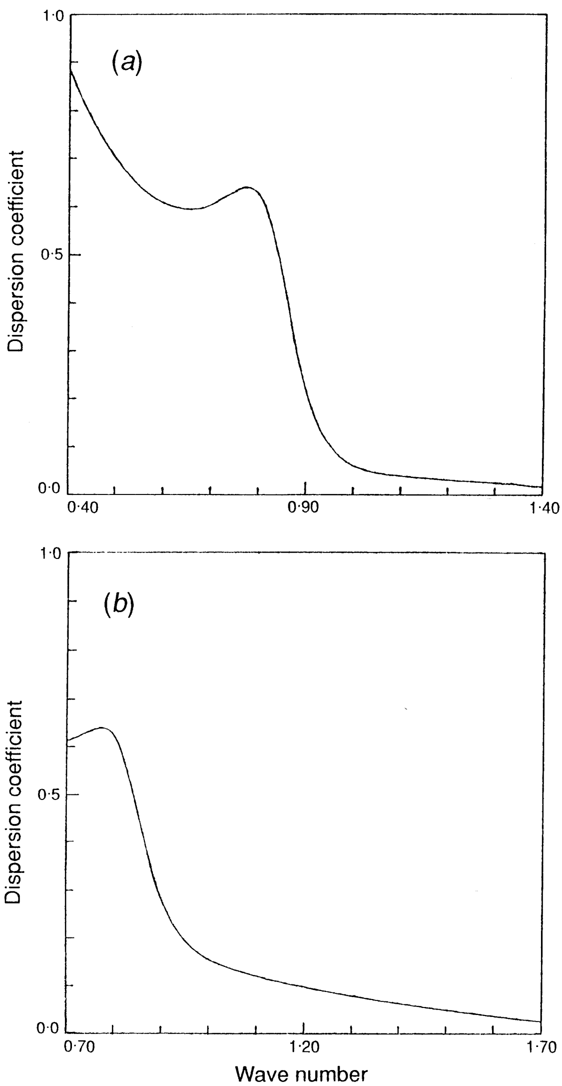

Fig. 1. Positive nature of the dispersion coefficient B/α_1 in equation (25) throughout the wave number range for $U_{e0}/c = 0{\cdot}8$, $T_\alpha = 0{\cdot}1$, $T_\beta = 0{\cdot}1$, $T_e = 0{\cdot}1$, $\sigma_\alpha = 0{\cdot}1$, $\sigma_\beta = 0{\cdot}1$, $\omega = 0{\cdot}5$, $Q = 0{\cdot}476$ and (a) $n_{\beta 0}/n_{\alpha 0} = 0{\cdot}15$, (b) $n_{\beta 0}/n_{\alpha 0} = 0{\cdot}55$.

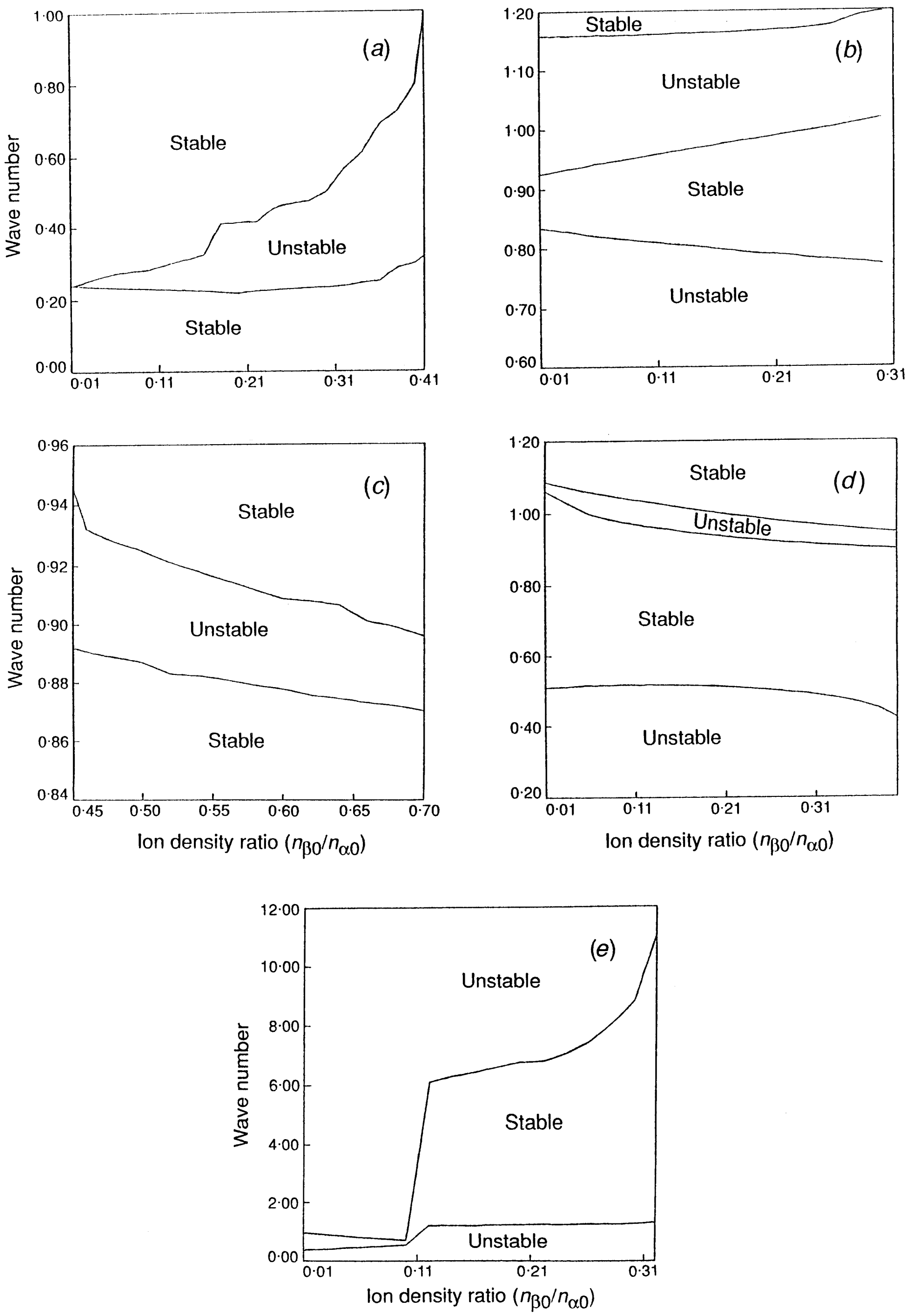

Fig. 2. Stability regions for the self-focusing problem under various conditions ($Q = 0{\cdot}476$, $\omega = 0{\cdot}5$ in all cases): (*a*) $U_{e0}/c = 0{\cdot}05$, $T_\alpha = 0{\cdot}01$, $T_\beta = 0{\cdot}01$, $T_e = 0{\cdot}1$, $\sigma_\alpha = 0$, $\sigma_\beta = 0$; (*b*) $U_{e0}/c = 0{\cdot}8$, $T_\alpha = 0{\cdot}01$, $T_\beta = 0{\cdot}01$, $T_e = 0{\cdot}1$, $\sigma_\alpha = 0{\cdot}1$, $\sigma_\beta = 0{\cdot}1$; (*c*) $U_{e0} = 0{\cdot}8$, $T_\alpha = 0{\cdot}1$, $T_\beta = 0{\cdot}1$, $T_e = 0{\cdot}1$, $\sigma_\alpha = 0{\cdot}1$, $\sigma_\beta = 0{\cdot}1$, with large $n_{\beta 0}$; (*d*) $U_{e0}/c = 0{\cdot}8$, $T_\alpha = 0{\cdot}1$, $T_\beta = 0{\cdot}1$, $T_e = 0{\cdot}1$, $\sigma_\alpha = 0{\cdot}1$, $\sigma_\beta = 0{\cdot}1$; and (*e*) $U_{e0}/c = 0{\cdot}8$, $T_\alpha = 0{\cdot}1$, $T_\beta = 0{\cdot}1$, $T_e = 0{\cdot}1$, $\sigma_\alpha = 0$, $\sigma_\beta = 0$.

3. Self-focusing of the Nonlinear Waves

In the previous section we have derived a new type of nonlinear Schrödinger equation (25), which depends only on space variables. One of these represents the longitudinal direction and the other one gives the direction transverse to it. We now proceed to extract the condition under which the self-focusing of the nonlinear wave may take place, using equation (25). In equation (25), B/α_1 is called the dispersion coefficient and C/α_1 is called the nonlinear coefficient. The phenomenon of self-focusing is studied by analysing the transversal stability of the system. To determine the transversal stability, we have to consider the system under perturbations perpendicular to the propagation direction. It can be shown that if the sign of the product of the dispersion and the nonlinear coefficients in equation (25) is positive, then the ion wave becomes unstable with respect to such transverse perturbation. In the present case the dispersion coefficient B/α_1 in (25) is positive in the whole wave number range which we have displayed in Fig. 1 corresponding to different plasma parameters. Hence the analysis of stable and unstable zones depends solely on the nature of the nonlinear coefficient C/α_1 in (25). The stability and instability zones are depicted in Fig. 2. In each case the figure shows the variation of the wave number against the ion density ratio $n_{\beta 0}/n_{\alpha 0}$ for different values of the ion temperature and electron–ion temperature ratio and a fixed value of Q. Fig. $2a$ depicts the nonrelativistic region, that is, $U_{e0}/c = 0{\cdot}05$, while for the others $U_{e0}/c = 0{\cdot}8$. It should be noted that the region of stability changes drastically due to this change in the streaming velocity: while Fig. $2b$ shows a resemblance to that of Sato *et al.* (1990), the relativistic situation is totally different from their result. Also, these regions have a quite different nature when $\sigma_\alpha = \sigma_\beta = 0$, and decrease in number with increasing T_α and T_β.

We conclude by discussing some points about the necessity of the negative ions for the whole process of self-focusing. In fact this is a consequence of the ponderomotive force, which is proportional to grad$\langle (\phi_1^{(1)})^2 \rangle$, with $\langle\,\rangle$ denoting the time average. The force drives the plasma out of the wave potential. The electrons hardly respond to this force because of their high pressure, whereas the ions, whose temperature is lower than that of the electrons, are easily subjected to it. It should be pointed out that in the absence of negative ions the self-focusing cannot take place, because the ions cannot move out of the wave potential as then charge neutrality would be violated. In the presence of negative ions, however, the ions can move outwards from the wave potential without violating charge neutrality, because not only the positively charged ions but also the negatively charged ones move together outwards. As a result, the ion plasma frequency is lower and the dielectric constant is higher inside the potential than outside. The plasma itself then acts as a convex lens, and makes the ion wave focused. When the ion temperature of the ions rises, their response to the ponderomotive force changes and hence the region of stability changes also.

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Appendix

Here we give the detailed forms of the nonlinear and dispersion coefficients appearing in the nonlinear Schrödinger equation (25). These expressions have been used in our stability analysis.

$$B = \frac{n_{\alpha 0}^2 \omega^2}{\Omega_\alpha^2} + \frac{n_{\alpha 0}^3 Q \omega^2}{\Omega_\beta^2} + \frac{e(kU_{e0} - \omega)(1 + 3U_{e0}^2/2c^2)\Omega_{ce}}{\Omega_e} - 1,$$

$$\begin{aligned} C = \frac{kn_{\alpha 0}}{\Omega_\alpha}\Bigg[&\left(\frac{n_{\alpha 0} k\omega^2}{\Omega_\alpha} - \frac{3\sigma_\alpha k^3}{\Omega_\alpha} + 1\right)N_{\alpha,0}^{(2)} \\ &+ \left(\frac{2n_{\alpha 0} k\omega^2}{\Omega_\alpha} - \frac{3k^3\sigma_\alpha}{\Omega_\alpha} - 1\right)N_{\alpha,2}^{(2)} + \left(\frac{2\omega n_{\alpha 0}^2 k^2}{\Omega_\alpha} - \frac{6k^4\sigma_\alpha n_{\alpha 0}}{\omega\Omega_\alpha}\right)U_{\alpha,2}^{(2)} \\ &- \frac{3k^3\sigma_\alpha n_{\alpha 0}}{\Omega_\alpha}P_{\alpha,2}^{(2)} - \frac{n_{\alpha 0}^4 k^5\omega^2}{\Omega_\alpha^3} + \frac{n_{\alpha 0}^2 k^2}{\Omega_\alpha}\Phi_0^{(2)}\Bigg] \\ &+ \frac{kn_{\beta 0} Q}{\Omega_\beta}\Bigg[\left(\frac{n_{\beta 0} k\omega^2}{\Omega_\beta} - \frac{3\sigma_\beta k^3}{\Omega_\beta} + \frac{1}{Q}\right)N_{\beta,0}^{(2)} \\ &+ \left(-\frac{n_{\beta 0} k\omega^2}{\Omega_\beta} + \frac{3k^3\sigma_\beta}{\Omega_\beta} - \frac{1}{Q}\right)N_{\beta,2}^{(2)} + \left(kn_{\beta 0} - \frac{\omega n_{\beta 0}^2 k^2}{\Omega_\beta}\right)U_{\beta,2}^{(2)} \\ &- \frac{3k^3\sigma_\beta n_{\beta 0}}{\Omega_\beta}P_{\beta,2}^{(2)} - \frac{n_{\beta 0}^4 k^5\omega^2}{\Omega_\beta^3} - \frac{n_{\beta 0}^2 k^2}{\Omega_\beta Q}\Phi_0^{(2)}\Bigg] \\ &+ \frac{k}{\Omega_e}\Bigg[\Bigg\{\frac{(kU_{e0} - \omega)(1 + 3U_{e0}^2/2c^2)\Omega_{ce} ke}{\Omega_e} + \frac{\Omega_{ce} ek\omega}{\Omega_e} \\ &+ \frac{ek(1 + 3U_{e0}^2/2c^2)\Omega_{ce}\omega}{\Omega_e}\Bigg\}N_{e,0}^{(2)} + \Bigg\{U_{e0}\left(\frac{ek^2(1 + 3U_{e0}^2/2c^2)\Omega_{ce}}{\Omega_e}\right) \\ &- \frac{k^2 e(kU_{e0} - \omega)(1 + 3U_{e0}^2/2c^2)\Omega_{ce}}{\Omega_e} - \frac{ek(1 + 3U_{e0}^2/2c^2)\Omega_{ce}\omega}{\Omega_e}\Bigg\}N_{e,2}^{(2)} \\ &- \Bigg\{\frac{(kU_{e0} - \omega)(1 + 3U_{e0}^2/2c^2)k^3 e}{\Omega_e} - \frac{3U_{e0}^2\Omega_{ce} ek^2}{2c^2\Omega_{ce}}\Bigg\}U_{e,2}^{(2)} \end{aligned}$$

$$
- (kU_{e0} - \omega)\frac{3\Omega_{ce}\, e^2\, k^3\, U_{e0}}{\Omega_e^2\, 2c^2} + \frac{\Omega_{ce}^3\, e^3\, \omega\, k^3}{2c^2\, \Omega_e^3}
$$

$$
- \frac{\Omega_{ce}^3\, e^3\, k^4 U_{e0}}{\Omega_e^3\, 2c^2} + \frac{\Omega_{ce}^2\, e^3\, k^5}{\Omega_e^3} - \left.\frac{e^2\, k^3}{\Omega_e}\right],
$$

and

$$
\alpha_1 = T_e\frac{e^2\, k^2}{\Omega_e} + \frac{kn_{\alpha 0}^3\, \omega^2}{\Omega_\alpha^2} + \frac{kn_{\alpha 0}^2}{\Omega_\alpha} + \frac{n_{\alpha 0}^3\, k^3\, T_\alpha}{\Omega_\alpha^2} + \frac{3n_{\alpha 0}^2\, k^3\, \sigma_\alpha}{\Omega_\alpha^2}
$$

$$
+ \frac{kQn_{\beta 0}^3\, \omega^2}{\Omega_\beta^2} + \frac{kn_{\beta 0}^2}{\Omega_\beta} + \frac{n_{\beta 0}^3\, k^3\, T_\beta}{\Omega_\beta^2} + \frac{3n_{\beta 0}^2\, k^3\, \sigma_\beta}{\Omega_\beta^2}
$$

$$
- \frac{(kU_{e0} - \omega)(1 + U_{e0}^2/2c^2)k\, eU_{e0}}{\Omega_e} + U_{e0}\frac{ek(1 + 3U_{e0}^2/2c^2)\Omega_{ce}}{\Omega_e} - e\,.
$$

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