Geometrical Models of Elementary Particles*

P. K. Smrz

Department of Mathematics, University of Newcastle, Newcastle, NSW 2308, Australia.

Abstract

A new concept of space and time, constructed from a de Sitter structured principal fibre bundle with a connection, is used to discuss a geometrical interpretation for the complex plane of the quantum theory and quantum behaviour of particles. In particular some features of a theory based on a torsion free metric linear connection in a five-dimensional base manifold are described.

1. Introduction

Since its beginning, quantum theory has been associated with a number of conceptual difficulties. Even as a very successful theory capable of explaining most aspects of the behaviour of micro-objects, it never reached a clear standpoint regarding the nature of the wave-particle dualism. Bohr's Copenhagen interpretation played a dominant role for several decades, but discussions concerned with its validity never really stopped, and a number of alternative interpretations were offered. Recently, such discussions intensified mainly due to technological advances allowing observations of individual particles interfering with themselves. Thus more and more researchers are asking again the fundamental question: What is an elementary particle?

It is nothing new to ask such a question in relation to the quantum behaviour of particles. In 1926 Albert Einstein wrote to Max Born: 'I am working hard at deducing the equations of motion of material points regarded as singularities, given the differential equation of general relativity' (Born 1971). Indeed, the Schwarzschild solution of the vacuum Einstein equation with its singularity at the centre and an integration constant having the meaning of mass seems to be a natural choice for a basic geometrical model of an elementary particle. The quantum behaviour, however, did not arise from Einstein equations. If one wants to persevere in such a research direction, one has to try various generalised concepts of space and time that could accommodate geometrical structures needed in quantum descriptions of particle behaviour. There have been many papers

* Refereed paper based on a contribution to the inaugural Australian General Relativity Workshop held at the Australian National University, Canberra, in September 1994. written on the subject and I shall not make any attempt at even a partial listing. Instead, I shall outline briefly the basic ideas of some of the main representative papers.

The fundamental geometrical object of Einstein's theory is a principal fibre bundle with the Lorentz group as its structure group and the four-dimensional space-time manifold as its base manifold. For an explanation of the geometrical terms the reader should consult the book by Kobayashi and Nomizu (1963). One class of generalisation of Einstein's theory keeps the base manifold as the four-dimensional space-time, but changes the structure group. The most prominent is the Poincaré gauge theory of gravity (Hehl *et al.* 1976), where the structure group is enlarged from the Lorentz group to the Poincaré group. One should also mention Carmeli's (1972) SL(2, C) gauge theory which does not extend the structure group as far as the dimension is concerned, but replaces it by its covering group.

Another direction is characterised by enlarging the base manifold, in particular by adding spinor coordinates to the space-time coordinates. This lead in the 1970s to the concept of a superspace (Salam and Strathdee 1974), while suggestions of a similar nature can be found in the literature already in the 1960s (Smrz 1968).

The concept described in the present contribution differs substantially from the above in the following sense. In all the theories mentioned above, as well as in many which are not mentioned, the space-time forms a part of the geometry from the beginning, either as the base manifold, or as a fixed submanifold of the base manifold. According to the concept described here the space-time manifold simply does not exist until a reference cross section (a gauge) is chosen. As the choice of the cross section is directly related to measurements, the very existence of space-time with all its properties including the dimension depends on the method of measurements. Of course, at this stage the connection with actual experiment is quite unclear, but having in mind the importance of measurement in quantum theory one feels encouraged to proceed in such direction.

Returning to the fundamental question about the nature of elementary particles, it should be noted that the standard quantum mechanical approach answers the question in a rather simple way: Elementary particles are irreducible representations of the Poincaré group (Schweber 1961). Such an abstract concept is perfectly suitable for quantum mechanical interpretation of many experiments, but it does not satisfy those who want to really understand the bizzare features of quantum behaviour (Banai 1988; Schommers 1989).

Many researchers feel that one has to look for the true explanation of quantum behaviour in a suitably generalised space-time geometry. One of the more recent contributors is Laurent Nottale, who asks the fundamental question: 'Where does the complex plane of quantum theory lie?' (Nottale 1993). His attempt at answering the question is based on an assumption that the underlying geometry of space and time is of fractal character. I shall stay on a firmer ground, looking for the complex plane within the formalism of a normal differential geometry. At this stage, I am not able to present anything near to a complete theory, but only a collection of not quite clearly related facts which one day may or may not come together. Nevertheless, I trust that the material contains plenty of matter for thought and that it is useful to present it in this unfinished form.

2. A New Concept of Space and Time

In 1987 I showed (Smrz 1987) how four-dimensional space-time may be constructed from a de Sitter structured fibre bundle with a connection. Let me go briefly over the construction.

We start with a principal fibre bundle P(M, G), where the base manifold M is of dimension at least four, while the structure group G is a de Sitter group. The bundle manifold P has the required local structure of $M \times G$. The connection in P may be defined by the horizontal lift of $\partial/\partial x^{\mu} \in T_x(M)$, $x \in M$, to $u \in P$, $\pi(u) = x$, in the form of

$$X^{(h)}_{\mu} = \frac{\partial}{\partial x^{\mu}} - \frac{1}{2} A^{ij}_{\mu}(x) Y_{ji}(g) , \qquad (1)$$

where x^{μ} are the local coordinates in M and Y_{ji} are the right-invariant vector fields in G. The above form depends on the selected local reference cross section in P needed to define the local coordinates (x,g) in $P \simeq M \times G$. In the local coordinates the reference cross section corresponds to (x,e), where e is the identity element of G. When the change of the local cross section (a gauge transformation) is characterised by a variable group element $g(x) \in G$ with the matrix elements $a_j^i(x)$, the gauge transformation of the connection components reads as

$$\bar{A}^{ij}_{\mu} = b^{i}_{k} A^{kl}_{\mu} b^{j}_{l} + (\partial_{\mu} b^{j}_{k}) b^{i}_{l} g^{kl} \,, \tag{2}$$

where we denote by b_j^i the matrix elements of $g^{-1}(x)$.

For $a_5^i = a_i^5 = 0$, i = 1, ..., 4, and $a_5^5 = 1$, the gauge transformation belongs to the Lorentz subgroup of G.

Separating the fifth component from the rest, the horizontal lift (1) may be written as

$$X_{\mu}^{(h)} = \frac{\partial}{\partial x^{\mu}} - \frac{1}{2} A_{\mu}^{ij}(x) Y_{ji}(g) - A_{\mu}^{i5} Y_{5i}(g) , \qquad (3)$$

where the range of the summation is only 1 to 4. Under the Lorentz gauge transformations the components A_{μ}^{ij} , i, j = 1, ..., 4, transform as in (2), while the components A_{μ}^{i5} transform according to

$$\bar{A}^{i5}_{\mu} = b^i_k A^{k5}_{\mu} \,. \tag{4}$$

Thus if $\dim(M) = 4$ and the 4×4 matrix $[A_{\mu}^{k5}]$ is invertible, it is possible to identify it with the inverse of the tetrads corresponding to a cross section $\{h_i^{\mu}(x)\partial/\partial x^{\mu}; i = 1, ..., 4\}$ in the bundle of frames of M.

If the coordinates in M carry a physical dimension, then we need a constant with the dimension of length (or time) to make the identification:

$$A^{i5}_{\mu} = \frac{1}{l} h^i_{\mu} \,. \tag{5}$$

In this way a connection plus a partially fixed cross section (up to an arbitrary Lorentz gauge transformation) define the bundle of frames of M as well as a Lorentzian connection in that bundle.

I propose that the physical interpretation is as follows. There exists a fundamental gauge invariant theory characterised by a connection in P. The base manifold M is not the observed space-time and is inaccessible to direct observations. The geometry of M may be explored only via the action of the structure group G. The reference cross section (gauge) in P depends on the method of geometrical observations. Thus as far as the subgroup of G consisting of spatial rotations is concerned, selecting the cross section means setting up a frame of three orthogonal vectors at each point of the base manifold. This may be done in many ways, while the frames at different points of M may be compared via the parallel transport defined by the connection. Similarly, extending the spatial rotations by adding the Lorentz boosts, the reference cross section is a selection of 4-frames. In principle, any arbitrary orthogonal 4-frame may be selected at each point, but in practical observations the choice of frames is severely limited by the parameter v/c being very close to zero. The remaining de Sitter transformations outside the Lorentz subgroup are to be interpreted as translations. However, the choice of the reference cross section is assumed to be entirely fixed by the macroscopical methods of space-time measurements, so that one cannot use these rotations and pseudo-rotations while remaining at a given point of the base manifold. At different points of the base manifold the frames may be compared again using the connection: space-time translations along a given curve in the base manifold are measured by the discrepancy between the horizontal lift of the curve and its lift to the reference cross section. If the reference cross section is either horizontal in a given direction or at least its vertical component is within the Lorentz subgroup, no translation is observable in that direction. In this way the base manifold M may be even of dimension greater than 4.

Let $\dim(M) = n > 4$ and the rank of the $4 \times n$ matrix $A^{i5}_{\mu}(x)$ be 4 at each $x \in M$. If the 4-dimensional distribution on M defined naturally by such a matrix is involutive, then there exists a coordinate system $\{x^{\mu}; \mu = 1, ..., n\}$ on M such that $A^{i5}_{\mu}(x)$ for $\mu = 1, ..., 4$ is invertible, while $A^{i5}_{\mu}(x) = 0$ for i = 1, ..., 4 and $\mu = 5, ..., n$. The identification (5) for $\mu = 1, ..., 4$ defines the bundle of frames and a Lorentzian connection on the 4-dimensional submanifold of M defined by fixing coordinates $x^5, ..., x^n$.

In a similar fashion, while one reference cross section may lead to a 4-dimensional space-time, another reference cross section (gauge) may for the same connection lead to $A_{\mu}^{i5}(x)$ of rank lower than four, thus generating a space of lower dimension. A natural question arises from such considerations: If a connection is capable of generating the flat Minkowski space-time in some gauge and one is allowed to use any arbitrary de Sitter gauge transformation, how much can the dimension of the generated space be reduced? This question has been answered by Smrz (1987) with the following result. Working in Minkowski coordinates one can solve the equation $\bar{A}_{\mu}^{i5} = 0$ with \bar{A}_{μ}^{i5} given by (2), where $A_{\mu}^{kl} = 0$ and $A_{\mu}^{k5} = (1/l)\delta_{\mu}^{i}$, $\mu, k, l = 1, ..., 4$, for three values of the index μ , thus reducing the dimension of the generated space to 1. The character of the remaining dimension depends on the type of the de Sitter group used in the construction. The group of type

Geometrical Models of Elementary Particles

(4,1) leads to a space-like dimension, while the type (3,2) corresponds to the remaining dimension being time-like.

The derivation by Smrz (1987) may be repeated for a more general case. Let x^i , i = 1, ..., 4 be Minkowski coordinates and x^{μ} , $\mu = 1, ..., 4$, an arbitrary coordinate system. Assume that the de Sitter group is of type (3, 2), i.e.

$$g^{ij} = \text{diag}(1, 1, 1, -1, -1).$$

We want to solve $\bar{A}^{i5}_{\mu} = 0$ for i = 1, ..., 4 and $\mu = 1, ..., 3$. Equation (2) yields

$$\frac{1}{l}b_k^i h_\mu^k b_5^5 - \frac{1}{l}b_5^i h_\mu^k b_k^5 + (\partial_\mu b_5^5) b_5^i g^{55} + (\partial_\mu b_j^5) b_k^i g^{jk} = 0\,.$$

Noting that the above expression is identically zero when index i is replaced by 5 and multiplying by matrix $[a_i^j]$ with j = 1, ..., 5 we obtain for j = 5

$$-\frac{1}{l}h^k_{\mu}b^5_k + (\partial_{\mu}b^5_5)g^{55} = 0 \tag{6}$$

and for j = 1, ..., 4

$$\frac{1}{l}h^{j}_{\mu}b^{5}_{5} + (\partial_{\mu}b^{5}_{k})g^{kj} = 0.$$
⁽⁷⁾

Equations (6) and (7) reduce to equation $(5 \cdot 10)$ of Smrz (1987) when Minkowski coordinates are used. It is easy to see that the same solution, namely

$$b_1^1 = b_2^2 = b_3^3 = b_5^4 = b_4^5 = 1$$
(8)

with the remaining matrix elements zero, solves also (6) and (7) for $\mu = 1, 2, 3$ as long as $h_{\mu}^4 = 0$ for $\mu = 1, 2, 3$. Thus, given any time-like curve in M, we may choose a coordinate system in such a way that the remaining observable coordinate is measured along that curve. Let functions $x^i(x^4), i = 1, ..., 3$, characterise a time-like curve, $u^2 = \sum_{i=1}^3 (dx^i/dx^4)^2$, and τ be the proper time measured along the curve. A Lorentz invariant process of dimension reduction due to the gauge transformation (2) may then be considered as a map from M onto the bundle manifold Q of a principal fibre bundle Q(T, U(1)), where T is a one-dimensional manifold with local coordinate τ , and U(1) is the group of de Sitter rotations within the (4,5)-plane. For each time-like curve in M we have

$$A^{45}_{\mu}dx^{\mu} = A^{45}_{\tau}d\tau = rac{1}{l}rac{d au}{\sqrt{1-u^2}}\,,$$

which defines a cross section in Q with tangent vector

$$\frac{\partial}{\partial \tau} + A_{\tau}^{45} Y_{54} \quad . \tag{9}$$

Thus identifying points in M characterised by $x^4 = 0$ with the point $(\tau, g) \in Q$ where $\tau = 0$ and g = e, every time-like curve in M maps into a cross section in Q with the tangent vector given by the above formula. The map is clearly not one-to-one. In general a class of curves in M will map into a single cross section, even when they have the same initial point. Only the x^4 -axis itself is the unique curve passing through the origin and corresponding to the cross section with tangent $\partial/\partial \tau + (1/l)Y_{54}$.

Before we discuss the possible relationship of the above with the geometric origin of the complex plane of quantum mechanics, I would like to make an important comment. Many papers were written on the subject of replacing the Poincaré group by one of the de Sitter groups in the context of the structure of the Universe, where the de Sitter radius is usually very large (Weinberg 1972), as well as the structure of particles and quantisation, where the radius is usually very small (Drechsler and Prugovečki 1991; Drechsler 1993). This should not be confused with what is proposed above. In all these papers the main geometrical structure is the de Sitter space, i.e. a four-dimensional space-time with constant curvature, on which the de Sitter group acts in a way which is analogous to the action of the Poincaré group on a flat Minkowski space, namely as a group of isometries. In our case the de Sitter group does not act on any space-time manifold, and its relationship with the translations is due to the restriction of the gauge outside the Lorentz subgroup. One should also notice that the flat Minkowski space is generated from a non-flat connection, and that while the gauge outside the Lorentz subgroup remains fixed, the fundamental length l is entirely unobservable and may be of any size, small or large.

3. The Complex Plane of Quantum Theory

Let me now point out some facts which connect the abstract construction of Section 1 with the quantum behaviour of particles and a possible geometrical place for the complex plane.

One of the simplest manifestations of the quantum complex plane is the fact that particles correspond to a fast rotating vector in that plane even when they are at rest. It is as if each particle measured its progress along the time axis by such a rotation. Could this rotation be simply the rotation in the 4–5 plane of the (3, 2) de Sitter group that corresponds to the time translations according to the scheme of Section 1? Classically, we do not perceive the time translations as rotations due to the gauge restriction. However, particles may not be bound by such restrictions. Perhaps, the natural gauge for particles is the gauge that corresponds to the minimum dimension. This could go together with another strange feature of quantum behaviour: Particles seem to exist 'all over the place' unless they are observed to occupy a definite region in space. This particular property is well described mathematically by Feynman's (1948) 'path integral' formulation. According to Feynman's quantum mechanics the probability amplitude of a particle travelling from point A at time t_A to a point B at time t_B is proportional to

$$\sum \exp\left(\frac{i}{\hbar} \int_{t_A}^{t_B} L(\dot{x}(t), x(t)) dt\right), \qquad (10)$$

Geometrical Models of Elementary Particles

where each integral is taken along a path from A to B and the sum is over all such possible paths. Let us see at least some threads connecting this with the dimension reduction of Section 1.

An observer working in the one-dimensional 'time-only' gauge describes all possible time-like paths as different cross sections of Q(T, U(1)) above an interval in T, connected by U(1) gauge transformations. The gauge transformation connecting the cross section corresponding to the time axis (the path of a particle at rest) with the one corresponding to a general time-like path is of the form

$$\exp\left[i(\theta(\tau) - \theta_0(\tau))\right] \in U(1), \tag{11}$$

where

$$heta(au) = rac{1}{l} \int_{ au_A}^ au rac{d au}{\sqrt{1-u^2}} \;\;,$$

and $\theta_0(\tau)$ is given by the same formula with u = 0. The coordinate θ in the U(1) group of 4–5 rotations was selected in such a way that $Y_{54} = \partial/\partial\theta$. The two cross sections were also selected to coincide at $\tau = \tau_A$. With l replaced by \hbar/mc^2 we have

$$\theta(\tau_B) - \theta_0(\tau_B) = \frac{mc^2}{\hbar} \int_{\tau_A}^{\tau_B} \left(\frac{1}{\sqrt{1-u^2}} - 1\right) d\tau = \frac{1}{\hbar} \int_{\tau_A}^{\tau_B} E_{\rm kin} d\tau \,,$$

establishing a link with (10) for a free particle of mass m. We have a hint of a geometrical representation for a single contribution to the Feynman path integral, but it remains unclear why adding the contributions from all paths yields the probability amplitude, and also where the probability is coming from in the first place. It is also not clear why l should be replaced by \hbar/mc^2 . One expects that l is a universal length (or time interval) present in the underlying geometrical gauge invariant theory, independent of m. All this brings us to the first question asked in the Introduction: What is an elementary particle ? It must be expected that to answer such a question one must have first of all a clearer idea about the structure of the underlying geometry. A few uncertain steps in that direction are sketched in the next section.

4. A de Sitter Gauge Invariant Theory of Space-Time

In this section we shall see a particular example of the fundamental gauge invariant theory mentioned in Section 1, which contains a description of the usual relativistic space-time when the gauge is partially fixed.

Since de Sitter groups act naturally in five-dimensional vector spaces, the first geometrical system suitable for the construction described in Section 1 that comes to mind is a five-dimensional manifold with a linear connection. Let P(M,G) be the bundle of orthonormal frames of a five-dimensional manifold M, and the connection in P be a torsion-free linear metric connection. Consider

$$R_{\alpha\beta} = \frac{4}{l^2} g_{\alpha\beta}, \quad \alpha, \beta = 1, ..., 5$$
(12)

(---

as the fundamental equation of the underlying theory. Here $R_{\alpha\beta}$ is the Ricci tensor, and

$$g_{\alpha\beta} = H^i_{\alpha} H^j_{\beta} g_{ij}, \quad \alpha, \beta, i, j = 1, ..., 5$$
⁽¹³⁾

is the metric tensor on M. A particular cross section in P is characterised by H^i_{α} , and g_{ij} is the diagonal (1, 1, 1, -1, -1) metric.

Consider solutions of equation (12) in the form

$$g_{\mu\nu} = e^{-x^5/l} g_{\mu\nu}^{(0)}, \qquad g_{\mu5} = g_{5\mu} = 0, \qquad g_{55} = -1,$$
 (14)

realised by the cross section satisfying

$$H^i_{\mu}=e^{x^5/l}h^i_{\mu}, \ \ H^i_5=H^5_{\mu}=0, \ \ H^5_5=1,$$

where the range of the indices μ , ν , and *i* is only 1 to 4, and

$$g^{(0)}_{\mu\nu} = g_{\mu\nu}|_{x^5=0} = h^i_{\mu}h^j_{\nu}g_{ij}, \quad \mu,\nu,i,j=1,...,4$$

A straightforward calculation gives the Christoffel symbols

$$\Gamma^5_{\mu
u} = -rac{1}{l}g_{\mu
u}, \ \ \Gamma^{
u}_{\mu5} = \Gamma^{
u}_{5\mu} = rac{1}{l}\delta^{
u}_{\mu},$$

and the Ricci tensor

$$R_{\mu
u}=R^{(0)}_{\mu
u}+rac{4}{l^2}g_{\mu
u}, \qquad R_{\mu5}=R_{5\mu}=0, \qquad R_{55}=-rac{4}{l^2}=rac{4}{l^2}g_{55},$$

where $R^{(0)}_{\mu\nu}$ is the Ricci tensor calculated from $g^{(0)}_{\mu\nu}$. Equation (12) thus implies that

$$R^{(0)}_{\mu\nu} = 0.$$

The components of the connection with respect to the chosen cross section are given by

$$A^{ij}_{\alpha} = H^i_{\gamma} \Gamma^{\gamma}_{\alpha\beta} H^{\beta}_k g^{kj} + (\partial_{\alpha} H^{\gamma}_k) H^i_{\gamma} g^{kj} ,$$

which yields in particular

$$A_5^{ij} = 0, \ A_{\mu}^{i5} = \frac{1}{l} H_{\mu}^i.$$
 (15)

Notice that the second equation in (15) is invariant with respect to an arbitrary Lorentz group gauge transformation as both sides transform according to (4), but it is in general broken by de Sitter gauge transformations outside the Lorentz subgroup. A four-dimensional space-time characterised by $x^5 = \text{const}$ in the chosen coordinates is thus generated by the partially fixed cross section in P, illustrating the construction of Section 1.

Geometrical Models of Elementary Particles

The solutions of the form (14) include all solutions of the vacuum Einstein equations in the generated space-time. In all such solutions l plays a rather formal role within the exponential scaling coefficients. One expects that l would be some kind of a universal constant like the Planck length

$$l = \sqrt{G\hbar/c^3} \,.$$

Solutions representing elementary particles should be expected to have the form (14) with the Schwarzschild $g^{(0)}_{\mu\nu}$ only for $r \gg l$, while in the region $r \sim l$ an interplay between l and the Schwarzschild radius

$$r_0 = 2Gm/c^2$$

should lead to the required time unit \hbar/mc^2 . All this is quite hazy at the moment, but it might be worth mentioning that the Schwarzschild geometry naturally contains the expression

$$r_0 c/2r^2$$
,

which at r = l yields mc^2/\hbar . In any case, finding more general 'particle-like' solutions of equation (12) and studying their properties under the dimension reducing gauge transformations seems to be a promising direction for further research.

5. Conclusions

Let me recapitulate the main points. To find the true geometrical model of elementary particles, one has to find the geometrical role of the quantum complex plane together with the reasons for its disappearance in classical The (3,2) de Sitter group provides a very simple possibility of physics. the complex plane being the plane in which the de Sitter transformations generated by Y_{54} act as rotations. The use of the classical gauge, fixed up to arbitrary Lorentz gauge transformations, gives us the opportunity to interpret the transformations generated by Y_{5i} , i = 1, ..., 4, as space-time translations, their rotational and pseudo-rotational character being entirely hidden to a One may say that the space-time is an illusion created classical observer. by the limited methods of classical observations. Once the gauge is free, the four-dimensional space-time radically changes its character and may even This is where some contact with the lose some of its spatial dimensions. strange quantum behaviour of particles is found. The classical and quantum aspects are associated with different gauges, different ways of making geometrical observations. In the case that such an approach proves correct, an important by-product will be the non-existence of a quantum theory of gravitation in the usual sense. Einstein's theory is meaningful only in the setting connected with the classical gauge, while quantum behaviour requires going outside such a gauge. Just this last point should provide enough reason for investigating the scheme further, before rushing forward in the canonical quantisation of Einstein's theory and spending years in search of something that may not exist.

- Banai, M. (1988). 'Problems in Quantum Physics, Gdansk 1987' (Eds L. Kostro et al.), p. 375 (World Scientific: Singapore).
- Born, M. (1971). 'The Born–Einstein Letters', p. 91 (Macmillan: London).
- Drechsler, W. (1993). Found. Phys. 23, 261.
- Drechsler, W., and Prugovečki, E. (1991). Found. Phys. 21, 513.
- Feynman, R. P. (1948). Rev. Mod. Phys. 20, 367.
- Hehl, F. W., von der Heyde, P., Kerlick, G. D., and Nester, J. M. (1976) *Rev. Mod. Phys.* 48, 393.
- Kobayashi, S., and Nomizu, K. (1963). 'Foundations of Differential Geometry', Vol. 1 (Interscience: New York).

Nottale, L. (1993). 'Fractal Space-Time and Microphysics', p. 14 (World Scientific: Singapore).

Salam, A., and Strathdee, J. (1974). Nucl. Phys. B 76, 477.

- Schommers, W. (1989). 'Quantum Theory and Pictures of Reality' (Ed. W. Schommers), p. 217 (Springer: Berlin).
- Schweber, S. (1961). 'An Introduction to Relativistic Quantum Field Theory', p. 58 (Row, Peterson and Co.: Elmsford, N.Y.).
- Smrz, P. K. (1968). Can. J. Phys. 46, 261.
- Smrz, P. K. (1987). J. Math. Phys. (NY) 28, 2824.
- Weinberg, S. (1972). 'Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity', p. 615 (John Wiley: New York).

Manuscript received 31 January, accepted 17 May 1995