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Topological Effects of a Circular Cosmic String

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Abstract

The behaviour of a quantum particle in the spacetime region exterior to a circular cosmic string is studied by constructing a connection one-form in the tetrad formalism. In the weak-field approximation, near the string core, the space exhibits a conical singularity, with an attendant topological phase and distortion of the energy spectrum of a scalar particle determined by the global properties of the spacetime structure of the string loop.

1. Introduction

In general relativity topological defects, such as cosmic strings and domain walls, are associated with phase transitions in the early universe (Vilenkin 1985; Vilenkin and Shellard 1994). In particular, one-dimensional vacuum cosmic strings have received considerable attention in the literature (Kibble 1976, 1980; Vilenkin 1981; Zel'dovich 1980). In this paper we examine the topological effects associated with a circular cosmic string. Loops of cosmic string are formed from closed tubes of false vacuum (Frolov *et al.* 1989). Unlike the straight cosmic string, a string loop has a short lifetime, with the large string tension generating forces that tend to drive the string into relativistic oscillations. These oscillations produce gravitational radiation which results in a rapid collapse of the string loop (Vilenkin 1981). Since isolated string loops are intrinsically unstable, it is necessary to introduce an external radial stress to support a closed string loop against collapse (Hughes *et al.* 1993). The behaviour of a scalar particle in the spacetime region exterior to a circular cosmic string is investigated within the weak-field approximation. In Section 2 we examine the holonomy for a particle transported through a closed circuit around the string loop. The energy spectrum of a particle moving in the vicinity of the string core is evaluated in Section 3. It is found that both the holonomy and the spectrum of the particle depend on the global properties of the spacetime, and are determined by the azimuthal stress density of the string. These global topological phenomena represent the gravitational analogue of the electromagnetic Aharonov–Bohm effect.

2. Gravitational Aharonov–Bohm Effect for a Circular Cosmic String

To analyse the spacetime region in the vicinity of a string loop we start with

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the metric line element defined for an axially symmetric gravitational field (Synge 1960):

$$ds^2 = -e^{2\nu}\,dt^2 + e^{2(\eta-\nu)}(d\rho^2 + dz^2) + e^{2(\zeta-\nu)}\rho^2\,d\phi^2\,, \tag{1}$$

where t is a time-like coordinate $(-\infty \le t \le \infty)$ and ρ, ϕ and z are cylindrical coordinates; the functions η, ν and ζ depend only on the coordinates ρ and z. The signature of the Minkowski metric is diag$(-,+,+,+)$, and natural units $(\hbar = c = 1)$ are used throughout. At this point in the analysis it is useful to transform to toroidal coordinates (σ, ϕ, ψ) (see Frolov *et al.* 1989) via

$$\begin{aligned} z &= aN^{-2}(\sigma,\,\psi)\sin\psi \qquad (-\pi < \psi \le \pi)\,,\\ \rho &= aN^{-2}(\sigma,\,\psi)\sinh\sigma \qquad (0 \le \sigma \le \infty)\,, \end{aligned} \tag{2}$$

where a denotes the radius of the circular string and $N(\sigma,\,\psi)$ is defined by

$$N^2(\sigma,\,\psi) = \cosh\sigma - \cos\psi\,. \tag{3}$$

Using toroidal coordinates the metric (1) becomes (Hughes *et al.* 1993)

$$\begin{aligned} ds^2 = {}& -e^{2\nu}\,dt^2 + a^2N^{-4}(\sigma,\,\psi)\sinh^2\sigma\, e^{2(\zeta-\nu)}\,d\phi^2\\ &+ a^2N^{-4}(\sigma,\,\psi)\,e^{2(\eta-\nu)}(d\sigma^2 + d\psi^2)\,, \end{aligned} \tag{4}$$

where now η, ν and ζ depend only on the toroidal coordinates σ and ψ.

Following Bezerra (1990), a holonomy transformation is calculated using the tetradic connection. We begin by defining the one-forms ω^a $(a = 1, 2, 3, 4)$:

$$\begin{aligned} \omega^1 &= aN^{-2}e^{(\eta-\nu)}\cos\phi\,d\sigma - aN^{-2}e^{(\zeta-\nu)}\sinh\sigma\,\sin\phi\,d\phi\,,\\ \omega^2 &= aN^{-2}e^{(\eta-\nu)}\sin\phi\,d\sigma + aN^{-2}e^{(\zeta-\nu)}\sinh\sigma\,\cos\phi\,d\phi\,,\\ \omega^3 &= aN^{-2}e^{(\eta-\nu)}\sinh\sigma\,d\psi\,,\\ \omega^4 &= e^{\nu}\,dt\,. \end{aligned} \tag{5}$$

A tetrad coordinate system $\{e^{\mu}_{(a)}\}$ is related to the one-forms by $\omega^a = e^{(a)}_{\mu}\,dx^{\mu}$, where the tetrads are defined so that $e^{\mu}_{(a)}\,e^{(b)}_{\mu} = \delta^b_a$; thus

$$\begin{aligned} e^{(1)}_1 &= aN^{-2}e^{(\eta-\nu)}\cos\phi, & e^{(1)}_2 &= -aN^{-2}\sinh\sigma\, e^{(\zeta-\nu)}\sin\phi\,,\\ e^{(2)}_1 &= aN^{-2}e^{(\eta-\nu)}\sin\phi, & e^{(2)}_2 &= aN^{-2}\sinh\sigma\, e^{(\zeta-\nu)}\cos\phi\,,\\ e^{(3)}_3 &= aN^{-2}e^{(\eta-\nu)}, & e^{(4)}_4 &= e^{\nu}\,. \end{aligned} \tag{6}$$

Near the string loop $(\sigma \to \infty)$ the spacetime resembles that of the straight cosmic string; it exhibits a conical singularity with deficit angle $8\pi G\mu$, where μ is the linear mass density of the string. In the limit $\sigma \to \infty$, the asymptotic form of the potentials η, ν and ζ are readily determined (Hughes *et al.* 1993); i.e. $\eta \to -4Gk\sigma$, $\nu \to -2G(\mu+k)\sigma$ and $\zeta \to 4Gkb$, where G is Newton's constant, k denotes the azimuthal stress density of the string loop (for 'string' matter $k=-\mu$), and $b=\ln(a/R_0)$ is a constant involving the loop radius a and the radius of the string core R_0. We have utilised a thin string approximation in order to avoid infinities in the potentials as the string core is approached.

Using Cartan's structure equations $\mathrm{d}\omega^a = -\omega^a_b \wedge \omega^b = -\Gamma^a_{b\mu}\,\omega_\mu \wedge \omega^b$, we obtain the following expressions for the connection one-forms in the tetrad formalism:

$$\begin{aligned}
\Gamma^1_{2\mu}\,\omega^\mu &= -\,\Gamma^2_{1\mu}\,\omega^\mu = N^{-2}\mathrm{e}^{4Gk(\sigma+b)}(\cosh\sigma\cos\psi - 1)\mathrm{d}\phi\,,\\
\Gamma^1_{3\mu}\,\omega^\mu &= -\,\Gamma^3_{1\mu}\,\omega^\mu = -N^{-2}\sin\psi\,\mathrm{d}\sigma + N^{-2}[\sinh\sigma + 4Gk(\cosh\sigma - \cos\phi)]\mathrm{d}\psi\,,\\
\Gamma^2_{3\mu}\,\omega^\mu &= -\,\Gamma^3_{2\mu}\,\omega^\mu = -N^{-2}\mathrm{e}^{4Gk(\sigma+b)}\sinh\sigma\sin\psi\,\mathrm{d}\phi\,.
\end{aligned} \tag{7}$$

The holonomy transformation $U(C)$, for a vector that is parallel transported along a closed curve in spacetime, is calculated from the tetradic connection (7) according to

$$U(C) = \mathcal{P}\exp\left(-\frac{\mathrm{i}}{2}\int_C \Gamma^{ab}_\mu\, J_{ab}\,\mathrm{d}x^\mu\right), \tag{8}$$

where J_{ab} are the generators of the Lorentz group SO(3,1), and $\mathcal{P}$ denotes an ordered product along the curve C. We can obtain an explicit expression for $U(C)$ in the asymptotic limit, $\sigma\to\infty$, where the particle approaches the 'core' of the circular cosmic string. In the weak-field limit the metric (4) reduces to

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2\mathrm{e}^{8Gkb}\,\mathrm{d}\phi^2 + a^2\mathrm{e}^{-2(1+4Gk)\sigma}(\mathrm{d}\sigma^2 + \mathrm{d}\psi^2)\,. \tag{9}$$

The non-trivial connection one-form is calculated from equation (7). In the limit $\sigma\to\infty$ we have

$$\Gamma^1_{3\mu}\,\omega^\mu = -\Gamma^3_{1\mu}\,\omega^\mu = (1+4Gk)\mathrm{d}\psi\,. \tag{10}$$

For a closed circuit in the poloidal angle ψ, the holonomy transformation is given by

$$\begin{aligned}
U_\psi(C) &= \exp\left(\int_{-\pi}^{\pi} -\mathrm{i}J_{13}(1+4Gk)\,\mathrm{d}\psi\right) = \exp[-8\pi\mathrm{i}GkJ_{13}]\\
&= \begin{pmatrix} \cos(8\pi Gk) & 0 & \sin(8\pi Gk) & 0\\ 0 & 1 & 0 & 0\\ -\sin(8\pi Gk) & 0 & \cos(8\pi Gk) & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned} \tag{11}$$

As expected the holonomy is identical to that produced by an infinitely long cosmic string. The holonomy transformation for a closed circuit in the azimuthal angle ϕ, that does not thread the cosmic loop, is also readily calculated to give $U_\phi(C) = \boldsymbol{I}$ (identity matrix). This result is consistent with the observation that this closed path is homotopic to a point, since the background spacetime region is simply connected.

3. Scalar Particle in the Spacetime of a Circular Cosmic String

To establish the energy level shifts of a relativistic scalar particle of rest mass M moving in the vicincity of a circular cosmic string, we start with the Klein–Gordon equation in covariant form:

$$\left(\frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}\,g^{\mu\nu}\partial_\nu) - M^2\right)\Phi = 0\,. \tag{12}$$

In the spacetime region near the string loop ($\sigma \to \infty$) the Klein–Gordon equation (12) takes the form

$$\left[-\,\partial_{tt} + \frac{1}{a^2\mathrm{e}^{8Gkb}}\,\partial_{\phi\phi} + \frac{1}{a^2}\,\mathrm{e}^{2(1+4Gk)\sigma}(\partial_{\sigma\sigma} + \partial_{\psi\psi}) - M^2\right]\Phi = 0\,. \tag{13}$$

Once again we have substituted the asymptotic forms for the potentials η, ν and ζ. We adopt the following ansatz for the relativistic wavefunction,

$$\Phi(t,\,\phi,\,\sigma,\,\psi) = \exp[-\mathrm{i}(Et - l\phi)]\Theta(\sigma,\,\psi)\,, \tag{14}$$

where E is a constant and single-valuedness of the wavefunction requires l to be an integer. Using this ansatz the Klein–Gordon equation (13) becomes

$$\left(E^2 - \frac{l^2}{a^2\mathrm{e}^{8Gkb}} - M^2\right)\Theta(\sigma,\,\psi) + \frac{1}{a^2}\mathrm{e}^{2(1+4Gk)\sigma}[\partial_{\sigma\sigma}\,\Theta(\sigma,\,\psi) + \partial_{\psi\psi}\,\Theta(\sigma,\,\psi)] = 0\,. \tag{15}$$

Separation of variables, with $\Theta(\sigma,\psi) = F(\sigma)\,G(\psi)$, produces two equations of the form

$$\partial_{\psi\psi}\,G(\psi) + \omega^2 G(\psi) = 0\,, \tag{16}$$

$$\partial_{\sigma\sigma}\,F(\sigma) + (\lambda^2\mathrm{e}^{-\kappa\sigma} - \omega^2)F(\sigma) = 0\,, \tag{17}$$

where ω is a separation constant, and

$$\lambda = a\sqrt{E^2 - \frac{l^2}{a^2\mathrm{e}^{8Gkb}} - M^2} \quad \text{and} \quad \kappa = 2(1 + 4Gk)\,. \tag{18}$$

The solution to equation (16) is easily found, i.e.

$$G_\omega(\psi) = D_\omega^{(1)}\sin\omega\psi + D_\omega^{(2)}\cos\omega\psi\,, \tag{19}$$

where $D_\omega^{(1)}$ and $D_\omega^{(2)}$ denote normalisation constants. Equation (17) can be transformed into a Bessel differential equation with the substitution $s = \mathrm{e}^{-\kappa\sigma/2}$; consequently, the solution to equation (17) can be written as a linear combination of Bessel functions of the first and second kinds, i.e.

$$F_{\omega*}(s) = C_{\omega*}^{(1)} J_{|\omega*|}(\lambda^* s) + C_{\omega*}^{(2)} Y_{|\omega*|}(\lambda^* s) , \tag{20}$$

where

$$\omega^* = \omega(1+4Gk)^{-1} \quad \text{and} \quad \lambda^* = a(1+4Gk)^{-1}\sqrt{E^2 - \frac{l^2}{a^2 \mathrm{e}^{8Gkb}} - M^2} , \tag{21}$$

and $C_{\omega*}^{(1)}$ and $C_{\omega*}^{(2)}$ are normalisation constants. Suppose we assume that the particle is constrained to a tubular region, in toroidal coordinates, between $\sigma_1 < \sigma < \sigma_2$, then the requirement that the wavefunction vanishes on the surfaces $\sigma = \sigma_1$ and $\sigma = \sigma_2$ gives rise to the following boundary conditions:

$$F_{\omega*}(s_1) = F_{\omega*}(s_2) = 0 . \tag{22}$$

These boundary conditions imply

$$J_{|\omega*|}(\lambda^* s_1)\, Y_{|\omega*|}(\lambda^* s_2) - J_{|\omega*|}(\lambda^* s_2)\, Y_{|\omega*|}(\lambda^* s_1) = 0 . \tag{23}$$

From equation (23) we can determine the energy spectrum of the particle. To apply the boundary conditions we utilise Hankel's asymptoptic expansion with ω^* fixed and for a situation were $\lambda^* s_{1,2} \gg 1$, in which case we get

$$\lambda^{*2} \sim \left(\frac{n\pi}{s_2 - s_1}\right)^2 + \frac{4\omega^{*2} - 1}{4 s_1 s_2} , \tag{24}$$

where $|n| = 0, 1, 2, \ldots$. The energy spectrum of the particle is found by substituting the asymptotic form for λ^* into equation (21), whence we obtain

$$E = +\sqrt{M^2 + \frac{l^2}{a^2 \mathrm{e}^{8Gkb}} + \left(\frac{1+4Gk}{a}\right)^2 \left[\frac{n^2\pi^2}{(s_2 - s_1)^2} + \frac{4\omega^{*2} - 1}{4 s_1 s_2}\right]} . \tag{25}$$

To ensure that E is constant when $s_2 \to s_1$ we must introduce an attractive potential in the tubular region $s_1 < s < s_2$ to compensate for the increasing energy of the 'toroidal' modes. Thus the energy becomes

$$E = +\sqrt{M^2 + \frac{l^2}{a^2 \mathrm{e}^{8Gkb}} + \left(\frac{1+4Gk}{a}\right)^2 \left[\frac{\omega^{*2}}{s^2} - \frac{1}{4s^2}\right]} . \tag{26}$$

From this result it is evident that although the scalar quantum particle moves in a region of spacetime that is locally flat, at least near the string core, the energy spectrum of the particle depends on the 'global' properties of the spacetime, through the azimuthal stress density k. This is the gravitational analogue of the electromagnetic bound state Aharonov–Bohm effect. As one might expect the energy spectrum (26) is similar to that of a scalar particle moving in the spacetime of a straight cosmic string (see Bezerra 1991); the correspondence becomes apparent when the parameter sa in equation (26) for the string loop is identified with the radial distance r from the straight string.

4. Conclusion

In summary, near the string loop the spacetime resembles that of a straight cosmic string. However, in the present weak-field model, the circular cosmic string is externally supported against collapse with additional radial stresses. Consequently, far from the string core a circular cosmic string acts like an ordinary gravitational source. The non-vanishing components (to order $G\mu$) of the Riemann tensor imply that the weak-field metric is not strictly globally flat. This is to be contrasted with Vilenkin's weak-field analysis of the straight string, which exhibits non-zero components of the Riemann tensor only at order $(G\mu)^2$. Since $G\mu \sim 10^{-6}$ (for a typical GUT), this suggests that the exact metric is flat, a conjecture that was subsequently confirmed by Hiscock (1985). However, it is an open question as to whether an exact solution to an axially symmetric circular cosmic string can give rise to a globally flat spacetime, with a concomitant gravitational Aharonov–Bohn effect.

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