S. Pac. J. Nat. Sci., (2002), Vol. 20, pp. 56 – 61
Section B: Short Communication

Obtaining Approximate Region of Asymptotic Stability by Computer Algebra: A Case Study

[1]Surya Prakash, [2]Jito Vanualailai and [3]Takashi Soma
[1]*Department of Mathematics, Fiji Institute of Technology, Suva, FIJI*
[2]*Department of Mathematics and Computing Science, University of the South Pacific, Suva, FIJI*
[3]*Computer and Information Division, Institute of Physical and Chemical Research, Saitama 351-0198 JAPAN*

Abstract

One of the classical problems in nonlinear control system analysis and design is to find a region of asymptotic stability by the Direct Method of Lyapunov. This paper tentatively shows, via a numerical example, that this problem can be easily solved using Quantifier Elimination (QE). In particular, if the governing equations are described by differential equations containing only polynomials, then the problem can be conveniently solved by a computer algebra software packages such as Qepcad or Redlog. In our case study, we use a simple Lyapunov function and Qepcad to estimate the stability region, and the results are verified by an optimization method based on Lagrange's method.

Keywords: Region of Stability, Computer Algebra, Quantifier Elimination, Optimization

I. INTRODUCTION

In recent review papers [1–3], it is shown that some control engineering problems of practical importance can be solved by quantifier elimination (QE) software packages such as Qepcad [4] or Redlog [5, 6], which are based on the cylindrical algebraic decomposition. The method constructs a decomposition of $\mathbf{R}^n$ such that a given set of polynomials have constant sign on each component. Such a decomposition is a starting point for the elimination of quantified variables in a statement or a first-order formula. If the formula is not true, the system returns "false" as a result and the system works as a decision maker, while if there is an equivalent quantifier free formula, the system returns the formula as a result and the system works as a quantifier eliminator. As shown in Section III.C, for some problems QE is equivalent to optimization method.

Even though the problem of finding a stability region by the Lyapunov method could be considered as a type of Multivariate Polynomial Inequalities (MPI) problem [2], it seems there have been no attempts made to apply the method of QE to this problem. Nonetheless we intend to show that there are some problems for which QE is effective. Hence the purpose of this paper is to show the applicability of QE, taking an example from Pai [7] as a case study.

This paper consists as follows: in Section II the region of stability for the sample problem is shown as defined by separatrices; in Section III, after reviewing the Lyapunov method, the problem is solved by QE in Section III.A, and by optimization in Section III.B. In Section III.C, the problem of maximizing the region of stability is discussed by introducing parameters in the Lyapunov function.

II. REGION OF STABILITY DEFINED BY SEPARATRICES

We consider a dynamical system defined by a set of differential equations given in [7] as,

$$\begin{aligned} \dot{x} &= -3y - x + x^3 \\ \dot{y} &= -2y + x - x^3. \end{aligned} \tag{1}$$

The behavior of the system is characterized by its equilibrium points and separatrices, among other things. It is easy to see that the point $(-1, 0)$ and $(1, 0)$ are saddle points and the origin $(0, 0)$ is a stable focus point. The separatrices can practically be obtained by integrating the differential equation numerically from a point sufficiently close to the saddle point with both positive and negative time directions. They are shown in Fig. 1; at the saddle point two separatrices cross each other, one is converging and the other is diverging as shown by arrows; at the focus point two separatrices converge, and the region of stability is the shaded region between two converging separatrices including the stable focus point or the origin.

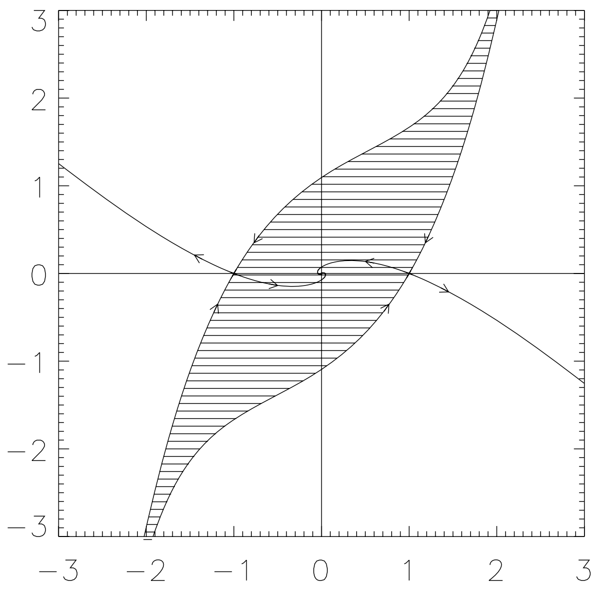

Figure 1: Region of stability defined by separatrices.

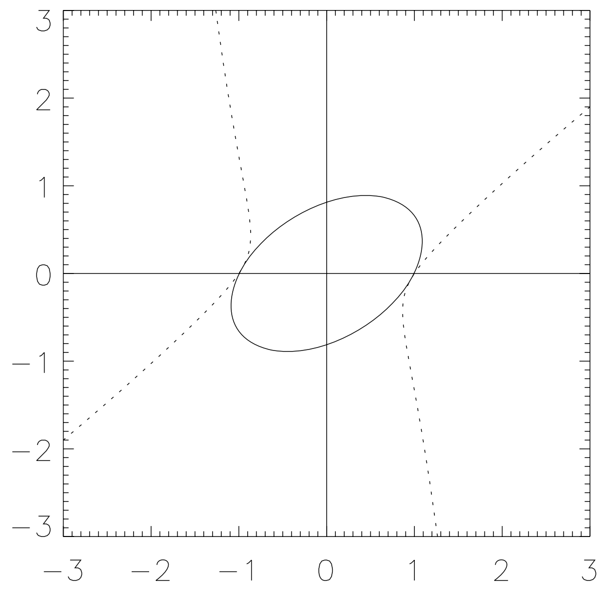

Figure 2: Elliptical region of stability and the dotted curves of $\dot{V}(x, y) = 0$. The area of elliptical region $S = 2.78189\cdots$.

III. REGION OF STABILITY BY LYAPUNOV METHOD, APPROXIMATION BY AN ELLIPSE

The Lyapunov method attempts to make statements on the stability of the equilibrium without any knowledge of the solutions of the differential equations [8–10]. The Lyapunov theorem, in more or less restricted form, says: for a given set of differential equations, such as (1) with equilibrium at the origin, the origin is stable if there exists a positive definite function $V(x, y)$ such that its total time derivative $\dot{V}(x, y)$ for the differential equations (1) is not positive. The function $V(x, y)$ is called a Lyapunov function. It should be noted that the theorem gives only a sufficient condition and the region obtained by the theorem is usually conservative. Since the region is usually defined by a contour of the Lyapunov function and if the function is relatively simple in form, it is easy to know whether a given point belongs to the region of stability.

A. Quantifier Elimination Method

Consider a positive definite function $V(x, y)$ as given in [7],

$$V(x, y) = 2x^2 - 2xy + 3y^2 \tag{2}$$

and its time derivative $\dot{V}(x, y)$ obtained as a total derivative of $V(x, y)$ with respect to time along a particular solution of (1) as,

$$\dot{V}(x, y) = -3x^2 + 3x^4 - 4x^3y - 3y^2. \tag{3}$$

As a QE problem, the Lyapunov condition can be formulated by a statement or a first-order formula: "For any (x, y), $V(x, y) \leq s$ implies $\dot{V}(x, y) \leq 0$," where x and y are quantified variables and s is a free variable. Eliminating the quantified variables from the above statement, the following quartic equation in s is obtained (the listing of an interactive session is given in the APPENDIX),

$$1296s^4 + 17368s^3 - 50655s^2 + 23850s - 3375 = 0 \tag{4}$$

As a polynomial in s this equation defines algebraic numbers as its root. One of the roots, which is positive and real, is $s = 1.98005\cdots$. This is the level of the Lyapunov function defining the elliptical region of stability as its contour. The elliptical region of stability is shown in Fig. 2 together with the curves $\dot{V}(x, y) = 0$ in dotted line. It can be shown that in the region between two curves of $\dot{V}(x, y) = 0$, $\dot{V}(x, y) < 0$ holds except for the origin where it takes the peak value of zero. The area of the elliptical region is found to be $S = 2.78189\cdots$. It should be noted that the contour line does not pass through the saddle points.(This contradicts the statement in [7] that $V(x, y) = 2$ defines the region of stability. The incorrectness of this statement is easily seen by the fact that the slopes of the ellipse and the separatrix at the saddle point are not equal, 2.0 for the former and $1.72075\cdots$ for the latter.)

B. Optimization Method

Anai [2] points out that some QE problems are equivalent to the optimization problem and QE is more effective for non-convex type optimization problem. Pai [7] suggests that the problem of finding a region of stability is formulated as a minimization problem of a Lyapunov function. As seen from Fig. 2, the ellipse defining the region of stability touches the curve $\dot{V}(x,y)=0$ at, say (x_s, y_s), and the problem of finding the region of stability is formulated as an optimization problem: *given a set of differential equations as (1) and a positive definite function as (2), we want to find a point (x_s, y_s) such that the function $V(x,y)$ has a minimum value s, such that $V(x_s, y_s) = s$, along the curve $\dot{V}(x,y) = 0$.* This is a kind of optimization problem with constraints and can easily be solved by the Lagrange's method of indeterminate coefficients.

It should be noted that although QE and the optimization method gives the same result for s, the former gives it directly, while the latter indirectly via the point (x_s, y_s). It should also be noted that the constraint in Lagrange's method can have a wider interpretation. It includes the point at which the two contour curves cross or touch each other where any directions have zero derivative. It may also include the stationary point where $V(x,y)$ has opposite sign on both sides of the point (x_s, y_s) along the curve $\dot{V}(x,y) = 0$. In this case $V(x,y)$ does not satisfy the Lyapunov condition but it may define a stability region. These points will be elaborated elsewhere [11]. Since the optimization method has no restriction on the form of the governing equations and the Lyapunov function, it has a wider applicability than the method by QE.

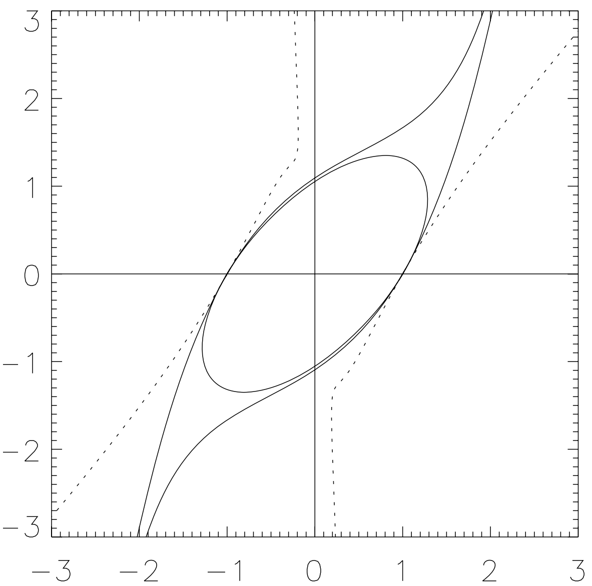

Figure 3: Maximum elliptical region, $a = 0.904\cdots$, $b = 0.5955\cdots$ in (5) and the area $S = 4.2333\cdots$, separatrices defining the stability region, and the dotted curves of $\dot{V}(x,y) = 0$.

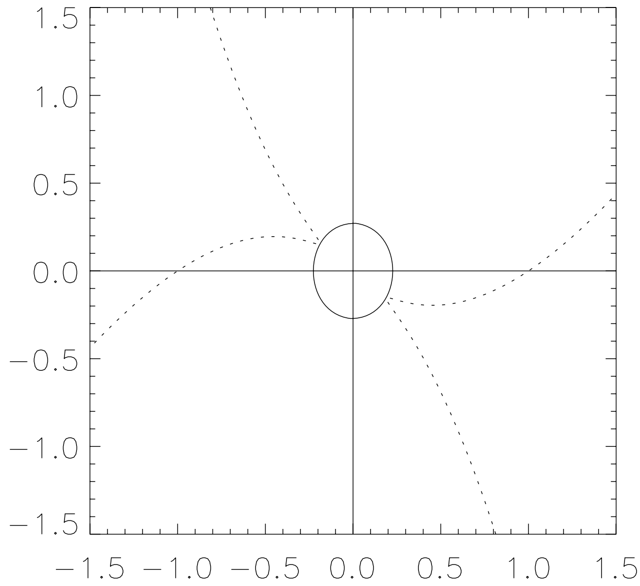

Figure 4: $a = 0.7$, $b = 0$, in (5) and the area $S = 0.1940995\cdots$.

C. Maximizing the Elliptical Region of Stability

Next consider the problem of maximizing the elliptical stability region by considering the following quadratic form with parameters a and b as,

$$V(x,y;a,b) = x^2 - 2bxy + ay^2 \tag{5}$$

with the positive definite condition of $a \geq b^2$. For the following *prenex* formula (see APPENDIX) describing the Lyapunov condition,

```
(A x)(A y)[ [ [ (b^2 - a) <= 0 ]/\ [(x^2 - 2 b x y + a y^2) <= s ] ] ==>
[ ( (2 x - 2 b y)(-3 y - x + x^3) + (-2 b x + 2 a y )(-2 y + x - x^3) ) <= 0 ] ].
```

Qepcad is supposed to give the equation relating three free variables a, b and s. It is found that the present version Qepcad does not give the answer in reasonable

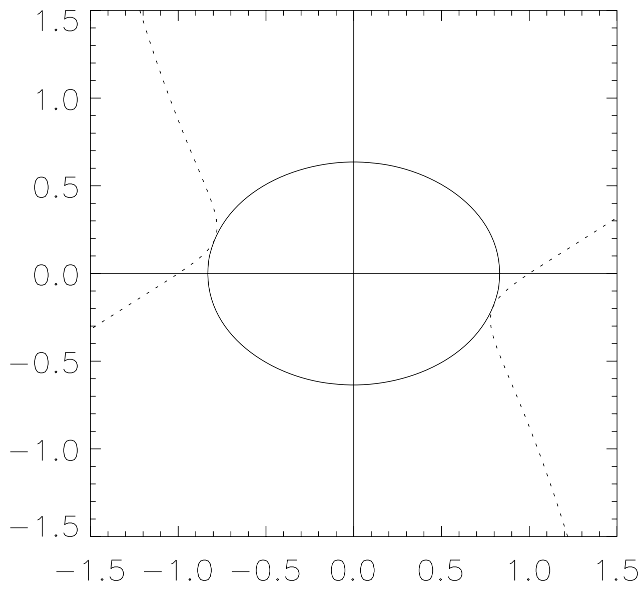

Figure 5: $a = 1.71604\cdots$, $b = 0$, in (5) and the area $S = 1.66010\cdots$.

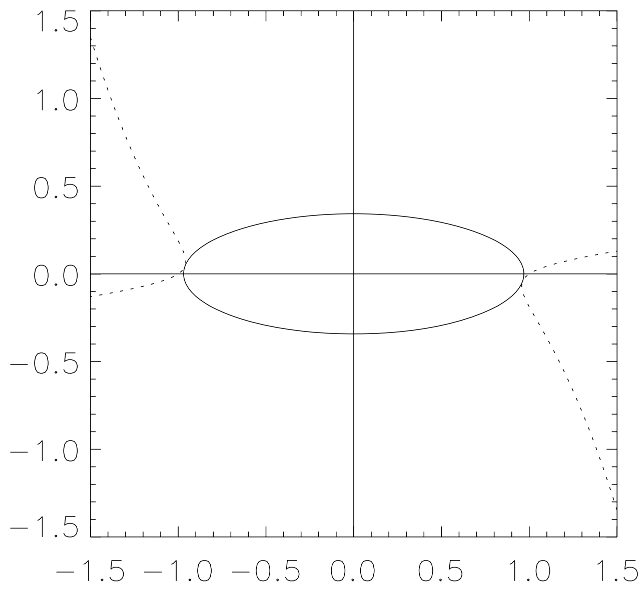

Figure 6: $a = 8$, $b = 0$, in (5) and the area $S = 1.04353\cdots$.

time. But as shown in Sections III.A and III.B, both methods can be used to solve the numerical case where a and b have particular values. Finding s and putting it in (5), we can obtain the major and minor axes, and the area S of the ellipse. By numerical maximization we obtain the maximum elliptical region $S = 4.2333\cdots$, at $a = 0.904\cdots$ and $b = 0.5955\cdots$ (S and s are related as $S = \pi s/\sqrt{a - b^2}$). The elliptical region is shown in Fig. 3 together with the separatrices defining the stability

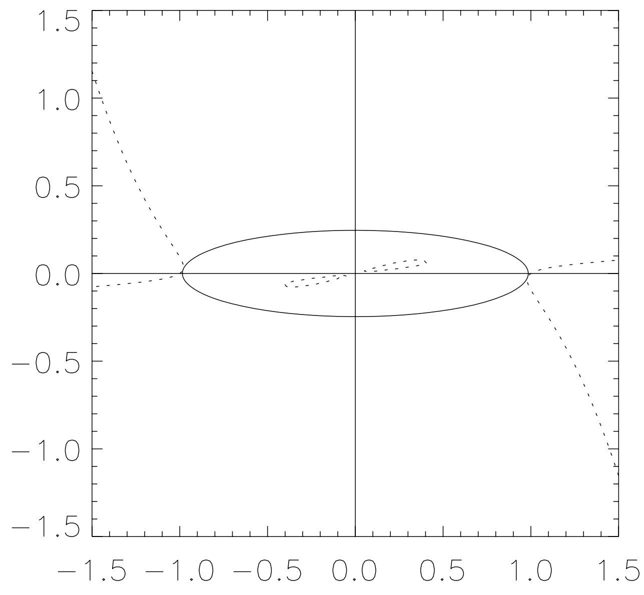

Figure 7: $a = 16$, $b = 0$, in (5) and the area $S = 0.761893\cdots$.

region as given in Fig. 1 and the curves $\dot{V}(x, y) = 0$ in dotted line.

It is interesting to note that the change in the shape of the stability region with the change of parameters a and b. Let us consider the case $b = 0$, for example. Increasing a from 0, we can show that at $a = 0.675444\cdots$, the origin changes from the saddle point to the peak point of the surface defined by the quadratic form for $\dot{V}(x, y)$ near the origin, and the stability region appears. The area increases as a increases (Fig. 4) and at $a = 1.71604\cdots$, the maximum area of $S = 1.66010\cdots$, is attained (Fig. 5). Increasing a further, the area decreases steadily (Fig. 6) and at $a = 13.3245\cdots$, the origin becomes the saddle point again and the positive region for the function $\dot{V}(x, y)$ appears inside the elliptical region. Fig. 7 shows such a positive region. The region no longer satisfies the Lyapunov condition though it defines the stability region.(It can be shown that all the trajectories are inward going along the ellipse boundary which guarantees the stability in this case.) It can be shown that similar situation occurs for $b \neq 0$. These properties can be derived from the characteristics of the quadratic form for $\dot{V}(x, y)$ near the origin obtained by neglecting the higher order terms.

IV. CONCLUDING REMARKS

It is shown that Qepcad can be applied to obtain the region of stability for a sample problem and the results are verified with that by the optimization method by Lagrange multiplier. For the sample problem including parameters, it is found that the present version Qepcad fails to give the result, but the shape of the stability region is investigated by finding solutions at the different values of parameter applying both QE and the optimization methods. It would be interesting to apply both methods to non-autonomous systems, especially the optimization method, which has wider applicability, taking into account the averaging method for differential equations.

Acknowledgments

The authors are grateful to Dr. Joseph Schicho at RISC and Dr. Hoon Hong at NCU for providing them with a copy of computer algebra system, Qepcad. The symbolic computation except QE and a part of numerical computations are done by REDUCE [6], the trajectory and contour plottings by IDL [12], and the original manuscript by LyX [13].

[1] H. Anai, *Solving LMI and BMI Problems by Quantifier Elimination*, IMACS Conf. on ACA, August 1998.
[2] H. Anai, *Control Theory and Computer Algebra*, Journal of AICE, **37**, 863-869, 1998, (in Japanese).
[3] M. Jirstrand, *Nonlinear Control System Design by Quantifier Elimination*, J. Symbolic Computation **24**, 137-152, 1997.
[4] H. Hoon, *Simple Solution Formula Construction in Cylindrical Algebraic Decomposition Based Quantifier Elimination*, ISSAC'92, ACM Press, 177-188, 1992.
[5] A. Dolzmann and T. Strum, *Redlog User's Manual, Version 2.0*, Universität Passau, 1999.
[6] A. C. Hearn, *REDUCE User's Manual, Version 3.7*, March 1999.
[7] M. A. Pai, *Power System Stability: Analysis by the Direct Method of Lyapunov*, North-Holland, Amsterdam, The Netherlands, 1981.
[8] J. LaSalle and S. Lefschets, *Stability by Liapunov's Direct Method with Applications*, Academic Press, New York, 1961.
[9] W. Hahn, T*heory and Application of Liapunov's Direct Method*, English Edition by S. H. Lehnigk, Prentice-Hall, Inc. Englewood Cliffs, NJ, USA, 1963.
[10] W. Hahn, *Stability of Motion*, Springer-Verlag, New York, 1967.
[11] S. Prakash, J. Vanualailai and T. Soma, *Obtaining Approximate Limit Cycle by Computer Algebra: A Case Study*, in preparation.
[12] Research Systems, Kodak Co., *IDL User's Manual, Version 5.1*, March 1998.
[13] LyX home pages around the world, http://www.lyx.org/

APPENDIX: Qepcad session for a sample problem from Pai's book; lines in italic indicate the input.

```
======================================================
                 Quantifier Elimination
                           by
          Partial Cylindrical Algebraic Decomposition

                 Version 19 (Interactive)

                           by
                        Hoon Hong
                  (hhong@math.ncsu.edu)
               North Carolina State University
======================================================
Enter an informal description  between '[' and ']':
[A sample problem from Pai's book]
Enter a variable list:
(s,x,y)
Enter the number of free variables:
1
Enter a prenex formula:
(A x) (A y) [ [(2 x^2 - 2 x y + 3 y^2)<=s ]==>
        (-3 x^2 + 3 x^4 - 4 x^3 y - 3 y^2) <= 0 ].
fin
======================================================
An equivalent quantifier-free formula:
```

```
[ 1296 s^4 + 17368 s^3 - 50655 s^2 + 23850 s - 3375 <= 0
\/ 4 s - 5 <= 0 ]
In other words,
[ P_1,2 <= \/ P_1,3 <= 0 ]
where
P_1,2 = 1296 s^4 + 17368 s^3 - 50655 s^2 + 23850 s - 3375
P_1,3 = 4 s - 5
=====================  The End  =======================
```