

INTERACTION OF SELECTION AND RECOMBINATION

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Summary

Each of two separate loci on a chromosome may be occupied by one of two alternative alleles. The genotypic frequencies depend on both selection and recombination. The crossover frequency will itself be modified by selection. If this latter process is slow compared with the former, we may envisage the population as passing through a succession of equilibrium states. The stability of these states is analysed and it is shown that changes in crossover frequency may lead to (1) sudden shifts in the frequencies of the various chromosomal types, and (2) a split into two different populations. This latter case provides a new model for incipient speciation.

I. INTRODUCTION

Linkage of genes is known to affect genotypic frequencies which are brought about by the action of selection. Lewontin and Kojima (1960) and Lewontin (1964) have written on this point, deriving the basic equations and submitting them to numerical analysis in certain cases. It is also recognized, however, that selection may act to modify the linkage between genes. In many cases this leads to a decrease in the recombination coefficient (Sheppard 1955; Kimura 1956; Bodmer and Parsons 1962).

The present paper analyses some of the expected effects arising from the interaction of these two processes. It is shown that this interaction can account for phenomena which would appear at first sight to require separate and special examination.

II. BASIC EQUATIONS

Suppose A, a to be alternative alleles at one locus, and B, b to be alternative alleles at another locus on the same chromosome. Let the crossover frequency (recombination coefficient) be R . Four chromosomal types are possible: AB, Ab, aB , and ab . Following Lewontin and Kojima (1960) we may consider these formally as four alternative alleles at a single locus, calling them g_1, g_2, g_3 , and g_4 respectively.

Now let x_i be the frequency of the "gene" g_i and suppose w_{ij} to be the fitness of the genotype $g_i g_j$. We note that

$$w_{ij} = w_{ji},$$

and

$$w_{23} = w_{14} = w \text{ (say).}$$

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Define

$$w_i = \sum_{j=1}^4 w_{ij}x_j,$$

and

$$\bar{w} = \sum_{i=1}^4 w_i x_i.$$

Further, set

$$D = x_1x_4 - x_2x_3.$$

In this notation, the equations given by Lewontin and Kojima (1960) become:

$$\begin{aligned}\Delta x_1 &= [x_1(w_1 - \bar{w}) - R w D] / \bar{w}, \\ \Delta x_2 &= [x_2(w_2 - \bar{w}) + R w D] / \bar{w}, \\ \Delta x_3 &= [x_3(w_3 - \bar{w}) + R w D] / \bar{w}, \\ \Delta x_4 &= [x_4(w_4 - \bar{w}) - R w D] / \bar{w}.\end{aligned}\tag{1}$$

These equations are not independent in that

$$\sum_{i=1}^4 x_i = 1,$$

and hence

$$\sum_{i=1}^4 \Delta x_i = 0.$$

Thus x_4 may be eliminated from the system (1) to give three equations

$$\Delta x_i = f_i(x_1, x_2, x_3; R) \quad (i = 1, 2, 3)\tag{2}$$

which may be written as the single vector equation

$$\Delta \mathbf{x} = \mathbf{f}(\mathbf{x}; R).\tag{3}$$

R is supposed to be a slowly varying function of time, and we note that \mathbf{f} depends linearly on R . Hence to a first approximation (cf. Lotka 1956), equilibrium is attained when

$$\mathbf{f}(\mathbf{x}; R) = \mathbf{0}.\tag{4}$$

Suppose \mathbf{x}_0 to be a solution of (4). Then

$$\mathbf{x}_0 = \mathbf{x}_0(R),\tag{5}$$

and this solution will be stable if the solutions $\delta \mathbf{x}$ of

$$\Delta \delta \mathbf{x} = \nabla \mathbf{f}(\mathbf{x}_0; R) \cdot \delta \mathbf{x}\tag{6}$$

tend to zero as the number of generations tends to infinity. This may be achieved in two ways (see Bellman 1953):

- (1) the three eigenvalues of $\nabla f(\mathbf{x}_0; R)$ are all negative.
- (2) $\nabla f(\mathbf{x}_0; R)$ has one negative eigenvalue and two complex conjugate eigenvalues with negative real part.

As only stable solutions are realized in practice, we restrict discussion to these. The stability of a solution will, in general, depend on the value of R . Consequently, change in the value of R may force a stable solution to become unstable. In case (1) this means that one of the eigenvalues becomes positive and in case (2), either (a) the negative eigenvalue becomes positive, or (b) the real part common to the two complex eigenvalues becomes positive.

Cases (1) and (2a) exhibit a transition point at which one of the eigenvalues is zero, and consequently

$$\det\{\nabla f(\mathbf{x}_0; R)\} = 0, \quad (7)$$

and thus examination of equation (7) would yield all losses of stability of these two types. Transitions of type (2b) would not, in general, be discovered by this test (Cherry 1961).

III. BEHAVIOUR OF SOLUTIONS OF EQUATION (4)

We may envisage the solutions (5) of equation (4) as curves in x_1 - x_2 - x_3 - R -space. These will be termed the solution curves. It has been found in hydrodynamic studies that a change in the stability of a solution often coincides with a point at which two solution curves intersect. This phenomenon was first noticed by Poincaré (1885) in the course of studies on rotating masses of fluid. The account to be presented here essentially follows that of Lamb (1932), but see also Cherry (1961).

Lewontin and Kojima (1960) have pointed out that the solution of equation (4) involves solving three simultaneous cubics in three unknowns. Such a system may be reduced to a single polynomial equation of ninth degree, which necessarily possesses at least one real solution.

Actually, on biological grounds, such a solution must exist, and indeed such reasoning assures us of the existence of at least one stable solution of equation (4) for any given value of R . Such a solution may represent fixation. Lewontin (1964) uses a slightly different terminology, and does not apply the word "solution" to such a case.

Suppose now that $\mathbf{x}_0(R)$ is a stable solution of equation (4), and we wish to extend the corresponding solution curve to neighbouring values of R . This is possible if the vector $d\mathbf{x}_0/dR$ can be determined.

Since equation (5) represents the solution of equation (4) we may assert

$$f(\mathbf{x}_0; R) = 0 \quad (8)$$

and hence

$$(\partial f / \partial R) + \nabla f(\mathbf{x}_0; R) \cdot (d\mathbf{x}_0/dR) = 0. \quad (9)$$

Equations (9) have a unique solution for $(d\mathbf{x}_0/dR)$ provided equation (7) does not hold. Where equation (7) is satisfied, either $(d\mathbf{x}_0/dR)$ has an infinite component or two or more solutions intersect. The condition for this latter is that $(\partial f / \partial R) = 0$. It is readily seen that this implies $D = 0$.

IV. DISCUSSION

If we attempt to follow a stable solution curve in its dependence on R , we may do so until either equation (7) is satisfied or a situation of the type previously classified as (2b) arises. We assume that in the first of these cases $\det\{\nabla f(\mathbf{x}_0; R)\}$ has a simple zero. (The case of a multiple root will be extremely rare and in any case adds nothing essentially new to the analysis.) The following situations may arise:

Case 1: A stable solution ceases to be stable with increase (or decrease) of R and the corresponding solution curve is not intersected by any other solution curve.

Case 2: As above, but increase (or decrease) of R leads into a region of the space into which the solution curve does not penetrate.

Case 3: The curve intersects other curves, but these all represent unstable solutions.

Case 4: The curve intersects another curve which represents a stable solution.

Case 5: As in case 4, but two or more stable solutions are intersected.

We may note that cases 1 and 2 contain the behaviour previously classified in (2b) as a special case. They also include cases where $(d\mathbf{x}_0/dR)$ has an infinite component. Cases 3, 4, and 5 deal with the other possibility, that of two or more solution curves intersecting.

V. BIOLOGICAL CONSEQUENCES

Generally R will not be constant but will itself be influenced by selective forces. In particular, if a system is in equilibrium in the sense that equation (4) is satisfied, R will decrease, according to the analysis of Bodmer and Parsons (1962). This does not take into account the effects of other genes on the chromosome, but we may adopt the point of view it suggests for purposes of discussion and expect that where R does in fact decrease (or in fact increase) the results described below will represent phenomena that we should expect to observe in practice. For the purpose of discussion therefore, we shall assume a slow but steady decrease in R .

The system having attained equilibrium or near equilibrium, it will then follow a stable solution curve until such time as this becomes unstable. If such a situation does not occur, the case is without interest from the point of view of this paper. The same can be said of case 4 where only one type of behaviour is possible, i.e. one stable solution curve replaces another.

Cases 1, 2, and 3 are such that if R decreases beyond the critical point in question, the system will no longer be in equilibrium and will therefore adjust itself to another stable equilibrium. In these cases (type I) a sudden shift of values in x_i will be observed. Lewontin's (1964) model 3 exhibits this feature at $R = 0.10$ and 0.375 .

Case 5 (type II) allows two different histories in the subsequent behaviour of the system. Lewontin and Kojima (1960, Table 7) give a numerical example of this behaviour. Further examples occur in Lewontin (1964). The first and second of these,

being asymmetric, are of particular interest. In this latter case, a large and geographically subdivided population might well fragment as some groups could follow one course and others another.

If R continues to decrease, we reach a point where we may neglect it in equations (1). The equilibrium solutions will then maximize \bar{w} (Wright 1955), and hence selection will favour further disjunction of the two subpopulations.

The situation shows some similarity with the case of genetic drift, but small populations are not required to achieve it, nor are sampling errors involved. A chance element determines which of the two or more solutions curves will be followed, but otherwise the model is entirely deterministic.

This provides a possible model for some types of incipient speciation.

VI. CONCLUSION

Taking into account only the interaction of selection and recombination, we may predict cases of sudden shifts in chromosomal frequency and others of incipient speciation. Normally, such phenomena would be attributed to additional factors, such as variation in fitness coefficients, or genetic drift. The analysis presented here shows that these need not always be involved.

VII. REFERENCES

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