MENISCUS SHAPES AND CAPILLARITY EFFECTS IN WIDE TUBES

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[Manuscript received October 11, 1971]

Abstract

Blaisdell's tables for the meniscus shape are extended to wider circular tubes and summarized in forms convenient for precision manometry.

The standard tables of meniscus shape are those of Blaisdell[1] whose notation and conventions are followed here. The origin is at the centre of the vertical cylindrical tube aligned along the y axis; the x axis is coincident with the central part of the meniscus, thus for mercury being below the level of the hypothetical plane surface imagined outside the tube. It is convenient to work with dimensionless coordinates independent of fluid characteristics by setting $X = x/a$, $Y = y/a$, $H_0 = h_0/a$, where $a = \surd[2\sigma/(\rho_1-\rho_2)g]$. Blaisdell integrated Laplace's equation by numerical methods and presented to five significant figures the values of meridian slope

$$\psi = \tan^{-1}(\mathrm{d}Y/\mathrm{d}X)$$

as function of Y and H_0, X as function of Y and H_0, Y as function of X and H_0, and—especially useful in manometry—H_0 as function of Y and X.

These tables cease at $X = 4\cdot5$, corresponding to 12 mm radius for mercury meniscuses and capillary depressions approximately $0\cdot016$ mm for typical meniscus heights, yet tabulated values down to $h_0 = 0\cdot5\ \mu$m can be occasionally useful. Blaisdell provided detailed approximation formulae which are valid for the centre-to-mid-range, and edge of large meniscuses. As these equations do not give H_0 explicitly as a function of the experimentalist's customary variables X and Y, the results of the necessary numerical work to achieve this are next presented.

Table 1 shows the computed X values and ψ values as functions of Y and H_0, together with information on X_{R}, and Y_{R}, the equatorial values. The range of Y values selected was intended to match the most extreme forms of meniscus shape, whether "flat" or "highly curved", likely to be employed in manometry with mercury surfaces.

The manometrist seeks H_0 as a function of Y at a preselected X value, so that a recasting of the data into the form of Table 2 aids convenience. For this purpose Table 1 was combined with similar data at $H_0 = 0\cdot004$–$0\cdot148$ from Blaisdell's work and sixth-order interpolation used. The values are certainly good to 2 in 1000.

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[1] Blaisdell, B. E., *J. Math. Phys.*, 1940, **19**, 186.

Table 2 and Blaisdell's tables taken together cover mercury capillary depressions in tubes of radius 5·32 to 18·62 mm.

TABLE 1

TABLES OF X AND ψ

H_0	Values of X (upright type) and ψ (*italics*, in rad) for Y values of 0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0	X_R	Y_R
0·00015	6·5040	6·7934	6·9896	7·1319	7·2379	7·3164	7·3724	7·4086	7·4261	7·42791	1·04290
	0·26813	*0·40482*	*0·54382*	*0·68590*	*0·83205*	*0·98347*	*1·14179*	*1·30916*	*1·48883*		
0·00025	6·1212	6·4115	6·6083	6·7513	6·8577	6·9366	6·9930	7·0296	7·0476	7·04953	1·04523
	0·26724	*0·40356*	*0·54219*	*0·68391*	*0·82968*	*0·98071*	*1·13859*	*1·30550*	*1·48458*		
0·00035	5·8683	6·1593	6·3566	6·4999	6·6066	6·6859	6·7426	6·7794	6·7977	6·79981	1·04689
	0·26663	*0·40267*	*0·54104*	*0·68250*	*0·82800*	*0·97875*	*1·13633*	*1·30288*	*1·48156*		
0·00050	5·5996	5·8913	6·0891	6·2329	6·3400	6·4196	6·4766	6·5138	6·5324	6·53464	1·04876
	0·26595	*0·40169*	*0·53976*	*0·68093*	*0·82612*	*0·97655*	*1·13378*	*1·29995*	*1·47817*		
0·00070	5·3455	5·6378	5·8362	5·9804	6·0879	6·1678	6·2251	6·2627	6·2816	6·28403	1·05064
	0·26531	*0·40073*	*0·53851*	*0·67937*	*0·82426*	*0·97437*	*1·13125*	*1·29703*	*1·47478*		
0·00100	5·0754	5·3685	5·5674	5·7121	5·8200	5·9003	5·9580	5·9959	6·0152	6·01784	1·05275
	0·26464	*0·39970*	*0·53713*	*0·67765*	*0·82219*	*0·97194*	*1·12842*	*1·29376*	*1·47100*		
0·00140	4·8200	5·1138	5·3132	5·4584	5·5667	5·6474	5·7054	5·7437	5·7633	5·76620	1·05485
	0·26404	*0·39874*	*0·53582*	*0·67599*	*0·82018*	*0·96956*	*1·12564*	*1·29054*	*1·46726*		
0·00200	4·5487	4·8430	5·0431	5·1887	5·2975	5·3786	5·4370	5·4757	5·4957	5·49883	1·05717
	0·26349	*0·39776*	*0·53444*	*0·67423*	*0·81802*	*0·96697*	*1·12261*	*1·28704*	*1·46314*		
0·00280	4·2922	4·5870	4·7876	4·9337	5·0429	5·1244	5·1832	5·2222	5·2426	5·24598	1·05942
	0·26310	*0·39695*	*0·53322*	*0·67260*	*0·81599*	*0·96453*	*1·11972*	*1·28361*	*1·45918*		
0·00400	4·0199	4·3151	4·5161	4·6626	4·7723	4·8541	4·9133	4·9527	4·9734	4·97722	1·06183
	0·26292	*0·39628*	*0·53209*	*0·67102*	*0·81395*	*0·96201*	*1·11670*	*1·28004*	*1·45497*		

TABLE 2

TABLE OF H_0

X	Values of $10^4 H_0$ for Y values of 0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
4·4	24·28	35·78	46·59	56·50	65·26	72·70	78·63	82·87	85·26
4·6	18·68	27·52	35·84	43·45	50·19	55·90	60·46	63·70	65·51
4·8	14·37	21·16	27·55	33·40	38·58	42·97	46·46	48·94	50·32
5·0	11·05	16·26	21·17	25·66	29·64	33·00	35·68	37·58	38·62
5·2	8·489	12·49	16·26	19·71	22·75	25·34	27·38	28·84	29·63
5·4	6·517	9·593	12·48	15·13	17·46	19·44	21·01	22·12	22·72
5·6	5·000	7·361	9·577	11·60	13·39	14·91	16·11	16·95	17·41
5·8	3·833	5·646	7·345	8·898	10·27	11·35	12·34	12·99	13·33
6·0	2·936	4·327	5·630	6·820	7·870	8·757	9·455	9·946	10·21
6·2	2·249	3·315	4·314	5·224	6·028	6·705	7·239	7·612	7·808
6·4	1·723	2·538	3·303	4·000	4·615	5·132	5·540	5·824	5·974
6·6	1·321	1·943	2·528	3·061	3·531	3·926	4·238	4·454	4·566
6·8	1·015	1·487	1·934	2·342	2·701	3·002	3·240	3·405	3·489
7·0	0·783	1·139	1·479	1·791	2·065	2·295	2·477	2·602	2·666

One meniscus property not previously emphasized, though implicit in the approximation formulae for wide tubes and valid throughout the major part of Blaisdell's tables, is the closely linear variation of log H_0 with X at fixed meniscus height Y. Between $X = 2 \cdot 0$ and $X = 7 \cdot 0$ more than three decades in H_0 are covered, yet over all but the smallest tubes, lines of fixed Y are substantially parallel and barely perceptibly curved.

This approximate behaviour is illustrated by the equations

$$\log H_0 = -0\cdot57324X + c \qquad (Y \text{ fixed})$$

with c taking the values at $Y = 0\cdot2$, $0\cdot3$, $0\cdot4$, $0\cdot5$, $0\cdot6$, $0\cdot7$, $0\cdot8$, and $0\cdot9$ of respectively $-0\cdot0916$, $+0\cdot0754$, $+0\cdot1890$, $+0\cdot2709$, $+0\cdot3335$, $+0\cdot3809$, $+0\cdot4118$, and $+0\cdot4335$. The equations are *not* least mean square fits, and were deliberately made parallel to facilitate graphical application: they reproduce H_0 to 2% everywhere in the range $2\cdot2 \leqslant X \leqslant 7\cdot0$. Slightly above the uppermost of these curves is the line representing the vertical edge of the drop with Y values approximately $1\cdot06$.

Taking a to be $2\cdot66$ mm for mercury, analogous equations can be deduced from the set above to give with better than 2% precision the actual capillary depression h_0 in mm in tubes with radius $5\cdot852 < x < 18\cdot62$ mm, for instance:

$$\begin{array}{lll} y = 1\cdot064\ \text{mm} & \log h = -0\cdot215504x + 0\cdot6139 & (30\cdot5°,\ 33°) \\ y = 1\cdot596\ \text{mm} & \log h = -0\cdot215504x + 0\cdot7584 & (46°,\ 50°) \\ y = 2\cdot128\ \text{mm} & \log h = -0\cdot215504x + 0\cdot8367 & (64°,\ 68°) \end{array}$$

There is only a modest variation in the boundary meniscus slope throughout the whole range of validity of each line of fixed meniscus height. The figures in brackets show a mean value for ψ in the range $X = 7\cdot0$ ($x = 18\cdot62$ mm) to $X = 3\cdot0$ ($x = 7\cdot98$ mm), followed by the value reached at the smallest X ($2\cdot2$) or x ($5\cdot852$ mm).

The manometric errors associated with variations of meniscus height for any tube radius can simply be found from log-normal plots of H against X (or h against x). Estimates of H_0 (or h) to 5% will usually suffice, so the empirical equations are more than adequate for general use; otherwise the tables may be treated similarly.

When using the equations or tables to deduce dimensioned capillary depression values (h_0) it is important to appreciate that variations approaching a factor of two in h_0 can follow 10% changes in a. A knowledge of the surface tension of the mercury in the working manometer is therefore essential to deduce precise values of the capillary depression even in wide tubes.

The volume occupied by gas between the mercury meniscus surface of revolution and the plane tangential to the meniscus summit seems only rarely to be of direct concern to experimentalists, though the variation of that volume with meniscus height is pertinent to PVT measurements. Both can be found by direct computation using:

$$V = \pi X^2(Y + H_0) - \pi X \sin\psi \quad \text{and} \quad V = v/a^3$$

so no table need be provided.